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Integral Inequalities for Compact Hypersurfaces with Constant Scalar Curvature in the Euclidean Sphere

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Abstract. We study the rigidity of compact-oriented hypersurfaces with constant scalar curvature isometrically immersed into the unit Euclidean sphere $\mathbb{S}^{n+1}$. In particular, we establish a sharp integral inequality for the behavior of the norm of the total umbilicity tensor, equality characterizing the totally umbilical hypersurfaces, and a certain family of standard tori of the form $\mathbb{S}^1(\sqrt{1-r^2}) \times \mathbb{S}^{n-1}(r)$. Moreover, under an appropriate constraint on the total umbilicity tensor, we are able to extend this result for any integer k, with $2 \leq k \leq n-1$, equality characterizing the totally umbilical hypersurfaces and a certain family of standard product of spheres of the form $\mathbb{S}^k(\sqrt{1-r^2}) \times \mathbb{S}^{n-k}(r)$.

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1. Introduction and Statement of the Main Results

In the seminal paper [7], Cheng and Yau introduced a new operator, denoted here by L, for the study of hypersurfaces with constant scalar curvature in Riemannian space forms. In particular, when the ambient space is the Euclidean sphere $\mathbb{S}^{n+1}$, Cheng and Yau showed, as an application of the operator L, that the only compact hypersurfaces in $\mathbb{S}^{n+1}$ with constant normalized

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scalar curvature $R \geq 1$ and non-negative sectional curvature are either totally umbilical hypersurfaces or isometric to a standard product of spheres $\mathbb{S}^m(\sqrt{1-r^2}) \times \mathbb{S}^{n-m}(r) \subset \mathbb{S}^{n+1}$, with $1 \leq m \leq n-1$ and $0 < r < 1$. After the works of Cheng and Yau and following their approach, there have been many other applications of the operator L, establishing different rigidity results for hypersurfaces with constant scalar curvature (see, for instance, the recent book [3] and the references therein). In particular, in Theorem 2 of [8], Li characterized totally umbilical hypersurfaces and constant scalar curvature tori of the form $\mathbb{S}^1(\sqrt{1-r^2}) \times \mathbb{S}^{n-1}(r) \subset \mathbb{S}^{n+1}$ in terms of a new estimate for the squared norm of the second fundamental form of the hypersurface, extending to the case of constant scalar curvature a previous and well-known result due to Alencar and do Carmo [1] for the case of constant mean curvature. Specifically, as a consequence of Li's result, we can state the following gap result for compact hypersurfaces with constant scalar curvature in $\mathbb{S}^{n+1}$ in terms of the so-called total umbilicity tensor (see also Theorem 1 in [5] and Theorem 1 in [2] for an extension of this gap result to the case of complete hypersurfaces).

Theorem 1.1. *Let Σ^n be a compact-oriented hypersurface isometrically immersed into the unit Euclidean sphere $\mathbb{S}^{n+1}$ $(n \geq 3)$ with constant normalized scalar curvature $R \geq 1$. Let Φ stand for the total umbilicity tensor of the immersion. Assume that:*

$$|\Phi|^2 \leq \alpha_R, \tag{1.1}$$

where

$$\alpha_R = \frac{n(n-1)R^2}{(n-2)(n(R-1)+2)} > 0.$$

Then:

(1) *either $|\Phi| = 0$ and Σ is a totally umbilical hypersurface;*
(2) *or $|\Phi|^2 = \alpha_R > 0$ and Σ is a constant mean curvature torus $\mathbb{S}^1(\sqrt{1-r^2}) \times \mathbb{S}^{n-1}(r) \subset \mathbb{S}^{n+1}$, with $r = \sqrt{(n-2)/nR}$.*

In other words, for every compact-oriented hypersurface Σ isometrically immersed in $\mathbb{S}^{n+1}$ $(n \geq 3)$ with constant normalized scalar curvature $R \geq 1$, it follows that:

(1) either $\max_\Sigma |\Phi| = 0$ and Σ is a totally umbilical hypersurface;
(2) or $\max_\Sigma |\Phi|^2 \geq \alpha_R > 0$, with equality if and only if Σ is a constant mean curvature torus $\mathbb{S}^1(\sqrt{1-r^2}) \times \mathbb{S}^{n-1}(r) \subset \mathbb{S}^{n+1}$, with $r = \sqrt{(n-2)/nR}$.

Motivated by Theorem 1.1, in this paper, we establish an integral version of it, which is given in terms of an integral inequality for the behavior of the norm of the umbilicity tensor. Specifically, we obtain the following integral inequality.

Theorem 1.2. *Let Σ^n be a compact-oriented hypersurface isometrically immersed into the unit Euclidean sphere $\mathbb{S}^{n+1}$ $(n \geq 3)$ with constant normalized scalar curvature R satisfying $R \geq 1$. In the case where $R = 1$, assume further* *that the mean curvature function H does not change sign. Let Φ stand for the total umbilicity tensor of the immersion. Then:*

$$\int_\Sigma |\Phi|^{p+2} Q_R(|\Phi|) \geq 0 \tag{1.2}$$

for every real number $p \geq 2$, where Q_R is the real function:

$$Q_R(x) = (n-2)x^2 + (n-2)x\sqrt{x^2 + n(n-1)(R-1)} - n(n-1)R. \tag{1.3}$$

Moreover, if $R > 1$ the equality holds in (1.2) if and only if:

(1) *either $|\Phi| = 0$ and Σ is a totally umbilical hypersurface,*
(2) *or*

$$|\Phi|^2 = \alpha_R = \frac{n(n-1)R^2}{(n-2)(n(R-1)+2)} > 0$$

and Σ is a torus $\mathbb{S}^1(\sqrt{1-r^2}) \times \mathbb{S}^{n-1}(r) \subset \mathbb{S}^{n+1}$, with $r = \sqrt{(n-2)/nR}$.

As an application of Theorem 1.2, we can give an alternative proof of Theorem 1.1. Actually, since $Q_R(x) \leq 0$ for every $0 \leq x \leq \sqrt{\alpha_R}$ (see Remark 3.5), under assumption (1.1), we have $Q_R(|\Phi|) \leq 0$ on Σ, which gives the equality in (1.2) and allows us to derive Theorem 1.1 as a direct consequence of Theorem 1.2.

Finally, and under an appropriate constraint on the total umbilicity tensor, we are able to extend Theorem 1.2 for any integer k, with $2 \leq k \leq n-1$, equality characterizing the totally umbilical hypersurfaces and a certain family of standard product of spheres of the form $\mathbb{S}^k(\sqrt{1-r^2}) \times \mathbb{S}^{n-k}(r)$ (see Theorem 4.1).

2. Preliminaries

Let $\mathbb{S}^{n+1}$ denote the Euclidean unit sphere:

$$\mathbb{S}^{n+1} = \{x = (x_0, x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+2} : |x|^2 = 1\} \subset \mathbb{R}^{n+2}.$$

Let $\Sigma^n \hookrightarrow \mathbb{S}^{n+1}$ be a connected compact isometrically immersed hypersurface which we assume to be orientable and oriented by a globally defined unit normal vector field N. We set A for the second fundamental tensor of the immersion (with respect to the normal direction N), and let $H = (1/n)\,\mathrm{Tr}(A)$ be the mean curvature function. For our purposes, it will be more appropriate to deal with the traceless part of A, which is given by $\Phi = A - HI$, with I the identity operator on $\mathfrak{X}(\Sigma)$, the $\mathcal{C}^\infty(\Sigma)$-module of smooth vector fields on Σ. Then, $\mathrm{Tr}(\Phi) = 0$ and:

$$|\Phi|^2 = \mathrm{Tr}(\Phi^2) = |A|^2 - nH^2 \geq 0,$$

with equality at $p \in \Sigma$ if and only if p is an umbilical point. Thus, $\Phi \equiv 0$ is equivalent to the fact that the immersion is totally umbilical. For that reason, Φ is also called the total umbilicity tensor of Σ.

The relation between the curvature tensor R of the hypersurface and the curvature of $\mathbb{S}^{n+1}$ is expressed via the Gauss equation, which can be written in terms of A in the form:

$$R(X,Y)Z = \langle X, Z\rangle Y - \langle Y, Z\rangle X + \langle AX, Z\rangle AY - \langle AY, Z\rangle AX, \tag{2.1}$$

for each $X, Y, Z \in \mathfrak{X}(\Sigma)$. In particular, the Ricci and the scalar curvatures of Σ are given, respectively, by:

$$\mathrm{Ric}(X,X) = (n-1)|X|^2 + nH\langle AX, X\rangle - |AX|^2 \tag{2.2}$$

for $X \in \mathfrak{X}(\Sigma)$, and

$$\mathrm{Scal} = n(n-1)R = n(n-1) + n^2H^2 - |A|^2 = n(n-1)(1+H^2) - |\Phi|^2. \tag{2.3}$$

Here, with R, we indicate the *normalized* scalar curvature. From (2.3), we obtain the identities:

$$nH^2 = \frac{1}{n}|A|^2 + (n-1)(R-1), \tag{2.4}$$

and

$$|\Phi|^2 = \frac{n-1}{n}|A|^2 - (n-1)(R-1) = n(n-1)H^2 - n(n-1)(R-1). \tag{2.5}$$

We let P denote the first Newton transformation of A. That is, $P : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ is the operator given by $P = nHI - A$. Observe that P is also a self-adjoint linear operator which commutes with A, and $\mathrm{Tr}(P) = n(n-1)H$. For $u \in \mathcal{C}^2(\Sigma)$, set:

$$L(u) = \mathrm{Tr}(P \circ \mathrm{hess}\ u) = \mathrm{div}(P(\nabla u)). \tag{2.6}$$

Thus, L defines a second-order differential operator which, in general, is not elliptic. It is clear from the definition that L is elliptic if and only if P is positive definite. Note that:

$$L(uv) = uLv + vLu + 2\langle P(\nabla u), \nabla v\rangle \tag{2.7}$$

for every $u, v \in \mathcal{C}^2(\Sigma)$. The operator L arises naturally as the linearized operator of the scalar curvature for normal variations of the hypersurface (see, for instance, [12]).

3. Proof of Theorem 1.2

For the proof of Theorem 1.2, we will need some other preliminary results. One of them is the following result given in Lemma 5 in [2] (see also Lemma 6.6 in [3]).

Lemma 3.1. *Let $\Sigma^n \hookrightarrow \mathbb{S}^{n+1}$ be an oriented isometrically immersed hypersurface. Assume that the mean curvature function H does not change sign, so that, without loss of generality, we may assume $H \geq 0$ on Σ. Let μ_- and*

μ_+ be, respectively, the minimum and the maximum of the eigenvalues of P at every point $p \in \Sigma$. If $R > 1$ on Σ (resp., $R \geq 1$ on Σ), then:

$$\mu_- > 0 (resp., \mu_- \geq 0)$$

and

$$\mu_+ < 2nH (resp., \mu_+ \leq 2nH).$$

Remark 3.2. In particular, Lemma 3.1 implies that if Σ has constant normalized scalar curvature $R > 1$ (resp., $R = 1$), the linear operator P is positive definite (resp., positive semidefinite) and the differential operator L is elliptic (resp., semielliptic).

Observe also that if $R > 1$ on Σ, it follows from (2.4) that H does not vanish. Thus, connectedness of Σ implies that H does not change sign, and without loss of generality, we may assume $H > 0$ on Σ.

We will also need the following auxiliary result, known as Okumura lemma, which can be found in [10] and [1, Lemma 2.6].

Lemma 3.3. *Let $a_1, \ldots, a_n$ be real numbers, such that $\sum_{i=1}^n a_i = 0$. Then:*

$$-\frac{n-2}{\sqrt{n(n-1)}} \left(\sum_{i=1}^n a_i^2 \right)^{3/2} \leq \sum_i a_i^3 \leq \frac{n-2}{\sqrt{n(n-1)}} \left(\sum_{i=1}^n a_i^2 \right)^{3/2}.$$

Moreover, equality holds in the right-hand (resp., left-hand) side if and only if $(n-1)$ of the a_i's are non-positive (resp. non-negative) and equal.

The proof of Theorem 1.2 is based on the following inequality for the operator L acting on the function $|\Phi|^2$.

Lemma 3.4. *Let $\Sigma^n \hookrightarrow \mathbb{S}^{n+1}$ be an oriented isometrically immersed hypersurface with constant normalized scalar curvature $R \geq 1$. In the case where $R > 1$, choose the orientation, such that $H > 0$ on Σ. In the case where $R = 1$, assume further that the mean curvature function H does not change sign, and choose the orientation, such that $H \geq 0$ on Σ. Then:*

$$\frac{1}{2} L(|\Phi|^2) \geq -\frac{1}{\sqrt{n(n-1)}} |\Phi|^2 Q_R(|\Phi|) \sqrt{|\Phi|^2 + n(n-1)(R-1)}, \quad (3.1)$$

where

$$Q_R(x) = (n-2)x^2 + (n-2)x\sqrt{x^2 + n(n-1)(R-1)} - n(n-1)R. \quad (3.2)$$

Moreover, if $R > 1$ and equality holds in (3.1), then Σ is an open piece of a constant mean curvature torus of the form $\mathbb{S}^1(\sqrt{1-r^2}) \times \mathbb{S}^{n-1}(r) \subset \mathbb{S}^{n+1}$, with $r = \sqrt{(n-2)/nR}$.

Remark 3.5. It is direct to see that the function $Q_R(x)$ given in (3.2) is strictly increasing for $x \geq 0$, with $Q_R(0) = -n(n-1)R < 0$ and $Q_R(x) = 0$ has a unique positive root at:

$$x = \sqrt{\alpha_R} = \sqrt{\frac{n(n-1)R^2}{(n-2)(n(R-1)+2)}}.$$

The inequality in the first part of Lemma 3.4 was given in [2, Lemma 7] (see also Lemma 6.8 of [3]). Observe that in this paper, there is a change in the sign of the function $Q_R(x)$ with respect to the definition of $Q_R(x)$ in [2] and [3]. For the sake of completeness, we include here the characterization of the equality when $R > 1$, not given in [2].

Proof of Lemma 3.4. Inequality (3.1) was already proved in [2, Lemma 7]. Moreover, a detailed analysis of its proof shows that if equality holds in (3.1), then all the inequalities in the proof of Lemma 7 in [2] must be equalities. Hence, let us assume that equality holds and $R > 1$. Then, inequality (10) in [2] must be an equality; that is:

$$\frac{n}{2(n-1)} L(|\Phi|^2) = nHL(nH),$$

and therefore:

$$gP(\nabla H)\nabla H = 0. \tag{3.3}$$

Since $R > 1$, P is positive definite and (3.3) implies that H is constant. Besides, (12) in [2] must be also an equality or, equivalently, $|\nabla A|^2 - n^2|\nabla H|^2 = 0$. Since we already know that H is constant, this means that $\nabla A = 0$. That is, the second fundamental form is parallel. Finally, (15) in [2] must be also an equality, so that we obtain the equality in Okumura lemma. This implies that Σ is an isoparametric hypersurface of $\mathbb{S}^{n+1}$ with exactly two constant principal curvatures with multiplicities $(n-1)$ and 1. By the classical results on isoparametric hypersurfaces in $\mathbb{S}^{n+1}$ [6] (see also Chapter 3 in [11] for a more modern reference on the topic), we conclude that Σ must be an open piece of a standard torus $\mathbb{S}^1(\sqrt{1-r^2}) \times \mathbb{S}^{n-1}(r) \subset \mathbb{S}^{n+1}$, with $0 < r < 1$, with principal curvatures:

$$\lambda_1 = -\frac{r}{\sqrt{1-r^2}} \quad \text{and} \quad \lambda_2 = \cdots = \lambda_n = \frac{\sqrt{1-r^2}}{r}.$$

A simple computation gives:

$$|\Phi|^2 = \frac{n-1}{nr^2(1-r^2)} \quad \text{and} \quad R = (n-2)/nr^2 > 0.$$

In particular, $R > 1$ if and only if $r < \sqrt{(n-2)/n}$. Obviously, in this case $r = \sqrt{(n-2)/nR} < \sqrt{(n-2)/n}$, with:

$$|\Phi|^2 = \text{constant} = \alpha_R = \frac{n(n-1)R^2}{(n-2)(n(R-1)+2)},$$

and $Q_R(|\Phi|) = 0$, so that (3.1) holds trivially. □

Now, we are ready to give the proof of our Theorem 1.2.

Proof of Theorem 1.2. Let $u = |\Phi|^2$. By Lemma 3.4, we have:

$$\frac{1}{2} L(u) \geq -\frac{1}{\sqrt{n(n-1)}} u\, Q_R(\sqrt{u}) \sqrt{u + n(n-1)(R-1)},$$

where $Q_R(x)$ is given by (3.2). Taking into account that $R \geq 1$ and $u \geq 0$, we obtain:

$$\frac{\sqrt{n(n-1)}}{2}\frac{u^{p/2}}{\sqrt{u+n(n-1)(R-1)}}L(u) \geq -u^{\frac{p+2}{2}}Q_R(\sqrt{u}),$$

for every $p \geq 2$. In other words:

$$u^{\frac{p+2}{2}}Q_R(\sqrt{u}) \geq -\frac{\sqrt{n(n-1)}}{2}\frac{u^{p/2}}{\sqrt{u+n(n-1)(R-1)}}L(u). \tag{3.4}$$

By the compactness of Σ, we can integrate both sides of (3.4), yielding:

$$\int_\Sigma u^{\frac{p+2}{2}}Q_R(\sqrt{u}) \geq -\frac{\sqrt{n(n-1)}}{2}\int_\Sigma \frac{u^{p/2}}{\sqrt{u+n(n-1)(R-1)}}L(u). \tag{3.5}$$

On the other hand, from the definition of the operator L, we obtain:

$$f(u)L(u) = \operatorname{div}(f(u)P(\nabla u)) - f'(u)\langle P(\nabla u), \nabla u\rangle$$

for every smooth function $f : \mathbb{R} \to \mathbb{R}$. Integrating both sides of the above equality, and using Stokes' theorem and the compactness of Σ, we deduce that:

$$-\int_\Sigma f(u)L(u) = \int_\Sigma f'(u)\langle P(\nabla u), \nabla u\rangle \tag{3.6}$$

for every smooth function f. In our case, choose:

$$f(t) = \frac{t^{\frac{p}{2}}}{\sqrt{t+n(n-1)(R-1)}}, \quad t \geq 0. \tag{3.7}$$

Hence, as $R \geq 1$:

$$f'(t) = \frac{(p-1)t^{\frac{p}{2}} + n(n-1)(R-1)p\,t^{\frac{p-2}{2}}}{2(t+n(n-1)(R-1))^{\frac{3}{2}}} \geq 0 \tag{3.8}$$

for every real number $p \geq 2$. Using (3.6) and (3.8) in (3.5), we can estimate:

$$\int_\Sigma u^{\frac{p+2}{2}}Q_R(\sqrt{u}) \geq \frac{\sqrt{n(n-1)}}{2}\int_\Sigma f'(u)\langle P(\nabla u), \nabla u\rangle \geq 0, \tag{3.9}$$

since we know that the operator P is positive semidefinite (see Remark 3.2). In other words:

$$\int_\Sigma |\Phi|^{p+2}Q_R(|\Phi|) \geq 0. \tag{3.10}$$

This proves the inequality of Theorem 1.2.

Now, let us suppose that $R > 1$ and the equality holds in (3.10). By the inequality (3.9), we obtain:

$$\int_\Sigma f'(u)\langle P(\nabla u), \nabla u\rangle = 0, \tag{3.11}$$

where

$$f'(u) = \frac{(p-1)u^{\frac{p}{2}} + n(n-1)(R-1)p\,u^{\frac{p-2}{2}}}{2(u+n(n-1)(R-1))^{\frac{3}{2}}} \geq 0,$$

with equality if and only if $p > 2$ and $u = 0$. Since $R > 1$ we also know that:

$$\langle P(\nabla u), \nabla u\rangle \geq 0$$

with equality if and only if $\nabla u = 0$. Therefore, it follows from (3.11) that:

$$f'(u)\langle P(\nabla u), \nabla u\rangle = 0 \quad \text{on} \quad \Sigma, \tag{3.12}$$

which implies that the function $u = |\Phi|^2$ must be constant, either $u \equiv 0$ or $u \equiv u_0 > 0$.

The case where $|\Phi|^2 = u \equiv 0$ corresponds to the case where Σ is a totally umbilical hypersurface. In the case where $|\Phi|^2 = u \equiv u_0 > 0$, the equality in (3.10) implies $Q_R(|\Phi|) = 0$, and hence:

$$|\Phi|^2 = \text{constant} = \alpha_R.$$

Thus, (3.1) in Lemma 3.4 becomes trivially an equality:

$$\frac{1}{2}L(|\Phi|^2) = 0 = -\frac{1}{\sqrt{n(n-1)}}|\Phi|^2 Q_R(|\Phi|)\sqrt{|\Phi|^2 + n(n-1)(R-1)},$$

and the proof finishes by applying the characterization of the equality given in the last part of Lemma 3.4. □

4. Some Extensions of Theorem 1.2.

In this section, we introduce an appropriate constraint on the total umbilicity tensor Φ to extend Theorem 1.2, obtaining integral inequalities similar to (1.2) for every integer k, $2 \leq k \leq n-1$. Specifically, we will prove the following result. Notice that, in the notation of Theorem 4.1, Theorem 1.2 corresponds to the case $k = 1$, where the additional hypothesis (4.1) is always satisfied because of Okumura lemma.

Theorem 4.1. *Let Σ^n be a compact-oriented hypersurface isometrically immersed into a unit sphere $\mathbb{S}^{n+1}$ $(n \geq 3)$ with constant normalized scalar curvature R satisfying $R \geq 1$. In the case where $R > 1$, choose the orientation, such that $H > 0$. In the case where $R = 1$, assume further that the mean curvature function H does not change sign, and choose the orientation, such that $H \geq 0$. Let k a integer with $2 \leq k \leq n-1$. Assume that:*

$$\mathrm{Tr}(\Phi^3) \geq -\frac{n-2k}{\sqrt{nk(n-k)}}|\Phi|^3 \tag{4.1}$$

on Σ, where Φ is the total umbilicity tensor of the immersion. Then:

$$\int_\Sigma |\Phi|^{p+2} Q_{R,k}(|\Phi|) \geq 0 \tag{4.2}$$

for every real number $p \geq 2$, where $Q_{R,k}$ is the real function:

$$Q_{R,k}(x) = (n-2)x^2 + \sqrt{\frac{n-1}{k(n-k)}}(n-2k)x\sqrt{x^2 + n(n-1)(R-1)} - n(n-1)R. \tag{4.3}$$

Moreover, if $R > 1$ the equality holds in (4.2) if and only if:

(1) *either $|\Phi| = 0$ and Σ is a totally umbilical hypersurface,*
(2) *or*

$$|\Phi|^2 = \alpha_{R,k} > 0,$$

where $\alpha_{R,k}$ is the square of the only positive root of $Q_{R,k}(x) = 0$ (see Remark 4.2) and Σ is a product of spheres $\mathbb{S}^k(\sqrt{1-r^2}) \times \mathbb{S}^{n-k}(r) \subset \mathbb{S}^{n+1}$, with $r = r(R) < \sqrt{\frac{n-k}{n} - \frac{1}{n}\sqrt{\frac{k(n-k)}{n-1}}}$.

Remark 4.2. Observe that $Q_{R,k}(0) = -n(n-1)R < 0$. When $k = n-1$, a direct computation shows that $Q_{R,n-1}(x)$ has a unique positive root at $x = \sqrt{\alpha_{R,n-1}}$, where:

$$\alpha_{R,n-1} = \frac{n(n-1)R^2}{(n-2)(n(R-1)+2)} = \alpha_R.$$

On the other hand, when $2 \leq k \leq n-2$ (and hence $n > 3$), by Lemma 2.7 and Remark 2.8 in [4], we know that the function $Q_{R,k}(x)$ has a unique positive root at:

$$x = \sqrt{\alpha_{R,k}},$$

where in this case:

$$\alpha_{R,k} = \frac{(n-1)\left[(n-1)(R-1)(n-2k)^2 + 2k(n-k)(n-2)R\right] - (n-2k)\sqrt{\Delta}}{2n(k-1)(n-k-1)} \tag{4.4}$$

with

$$\Delta = (n-1)^4\left[(n(R-1)+2k)^2 - \frac{4nRk(k-1)}{n-1}\right].$$

Observe that, in particular, when $n = 2m$ and $k = n/2 = m$, expression (4.4) reduces to:

$$\alpha_{R,m} = \frac{m(2m-1)R}{m-1} = \frac{n(n-1)R}{n-2}.$$

Remark 4.3. The explicit value of $r = r(R)$ in Theorem 4.1 (and also in Lemma 4.4 and Corollary 4.6 below) can be computed by solving Eq. (4.9) under the constraint (4.10), and it is given by the following expression:

$$r = r(R) = \sqrt{\frac{n-k}{nR} + \frac{R-1}{2R} - \frac{1}{nR}\sqrt{\frac{n^2(R-1)^2}{4} + \frac{k(n-k)(n(R-1)+1)}{n-1}}}.$$

The proof of Theorem 4.1 parallels that of Theorem 1.2, replacing Lemma 3.4 by the following auxiliary result.

Lemma 4.4. *Let $\Sigma^n \hookrightarrow \mathbb{S}^{n+1}$ be an oriented isometrically immersed hypersurface ($n \geq 3$) with constant normalized scalar curvature $R \geq 1$. In the case where $R > 1$, choose the orientation, such that $H > 0$ on Σ. In the case where $R = 1$, assume further that the mean curvature function H does not change* *sign, and choose the orientation such that* $H \geq 0$ *on* Σ. *Let* $2 \leq k \leq n-1$ *and assume that:*

$$\mathrm{Tr}(\Phi^3) \geq -\frac{n-2k}{\sqrt{nk(n-k)}}|\Phi|^3 \tag{4.5}$$

on Σ, *where* $\Phi = A - HI$. *Then:*

$$\frac{1}{2}L(|\Phi|^2) \geq -\frac{1}{\sqrt{n(n-1)}}|\Phi|^2 Q_{R,k}(|\Phi|)\sqrt{|\Phi|^2 + n(n-1)(R-1)}. \tag{4.6}$$

Moreover, if $R > 1$ *and equality holds in* (4.6)*, then* Σ *is an open piece of a standard product of spheres of the form* $\mathbb{S}^k(\sqrt{1-r^2}) \times \mathbb{S}^{n-k}(r) \subset \mathbb{S}^{n+1}$, *with* $r = r(R) < \sqrt{\frac{n-k}{n} - \frac{1}{n}\sqrt{\frac{k(n-k)}{n-1}}}$.

For the proof of Lemma 4.4, we need the following extension of Okumura lemma for the case $2 \leq k \leq n-1$ (see Lemma 2.2 in [9]).

Lemma 4.5. *Let* $a_1, \cdots, a_n$ *be real numbers, such that* $\sum_{i=1}^n a_i = 0$. *Then, the equation:*

$$\sum_{i=1}^n a_i^3 = -\frac{n-2k}{\sqrt{nk(n-k)}}\left(\sum_{i=1}^n a_i^2\right)^{3/2}$$

holds if and only if $(n-k)$ *of the* a_i*s are non-negative and equal and the rest* k *of the* a_i*'s are non-positive and equal.*

Proof of Lemma 4.4. Inequality (4.6) was already proved in [4, Lemma 2.6]. It is worth pointing out two important facts. First, there is a change in the sign of our function $Q_{R,k}(x)$ with respect to the definition of the function denoted by $Q_R(x)$ in equation (2.11) of [4]. Second, although the statement of Lemma 2.6 in [4] assumes $2 \leq k \leq n-2$, the proof works also for $k = n-1$. There is no difference at all.

On the other hand, a detailed analysis of the proof of Lemma 2.6 in [4] shows that if equality holds in (4.6), then all the inequalities in the proof must be equalities. Hence, let us assume that equality holds and $R > 1$. Then, inequality (2.12) in [4] must be an equality; that is:

$$\frac{n}{2(n-1)}L(|\Phi|^2) = nHL(nH),$$

and therefore:

$$\langle P(\nabla H), \nabla H\rangle = 0. \tag{4.7}$$

Since $R > 1$, P is positive definite and (4.7) implies that H is constant. Besides, (2.13) in [4] must be also an equality or, equivalently, $|\nabla A|^2 - n^2|\nabla H|^2 = 0$. Since we already know that H is constant, this means that $\nabla A = 0$. That is, the second fundamental form is parallel. Finally, (2.9) in [4] [that is, (4.5)] must be also an equality, and using Lemma 4.5, we know that Σ is an isoparametric hypersurface of $\mathbb{S}^{n+1}$ with exactly two constant principal curvatures with multiplicities $(n-k)$ and k. By the classical results on isoparametric hypersurfaces in $\mathbb{S}^{n+1}$, we know that Σ must be an open piece of a standard product of spheres $\mathbb{S}^k(\sqrt{1-r^2}) \times \mathbb{S}^{n-k}(r) \subset \mathbb{S}^{n+1}$, with $0 < r < 1$, with principal curvatures given by:

$$\lambda_1 = \cdots = \lambda_k = -\frac{r}{\sqrt{1-r^2}} \quad \text{and} \quad \lambda_{k+1} = \cdots = \lambda_n = \frac{\sqrt{1-r^2}}{r}.$$

A direct computation gives:

$$nH = \frac{(n-k) - nr^2}{r\sqrt{1-r^2}} \quad \text{and} \quad |\Phi|^2 = \frac{k(n-k)}{nr^2(1-r^2)}. \tag{4.8}$$

Observe that we always have the equality:

$$\mathrm{Tr}(\Phi^3) = -\frac{n-2k}{\sqrt{nk(n-k)}}|\Phi|^3$$

for all the values of r but $H > 0$ if and only if $r < \sqrt{(n-k)/n}$. Moreover, the constant scalar curvature, which is given by (2.3), is:

$$R = \frac{(n-k)(n-k-1) - (n-1)(n-2k)r^2}{n(n-1)r^2(1-r^2)}. \tag{4.9}$$

Therefore, since $r < \sqrt{(n-k)/n}$, we have $R > 1$ if and only if:

$$r^2 < \frac{n-k}{n} - \frac{1}{n}\sqrt{\frac{k(n-k)}{n-1}}. \tag{4.10}$$

Substituting the second equation of (4.8) and (4.9) into (4.3) and using $r^2 < (n-k)/n$ it follows that $Q_{R,k}(|\Phi|) = 0$, so that equality holds trivially in (4.6). This finishes the proof. See Remark 4.3 for the explicit value of $r = r(R)$. □

Once we have established Lemma 4.4, the proof of Theorem 4.1 follows exactly as the proof of Theorem 1.2, replacing Lemma 4.4, Q_R and α_R in the proof of Theorem 1.2 by Lemma 4.4, $Q_{R,k}$ and $\alpha_{R,k}$, respectively, in the proof of Theorem 4.1.

As an application, we get the next result whose proof is similar to that of Theorem 1.1, using Theorem 4.1.

Corollary 4.6. *Let Σ^n be a compact-oriented hypersurface isometrically immersed into the unit Euclidean sphere $\mathbb{S}^{n+1}$ ($n \geq 3$) with constant normalized scalar curvature $R > 1$. Let Φ stand for the total umbilicity tensor of the immersion. Assume that:*

$$\mathrm{Tr}(\Phi^3) \geq -\frac{n-2k}{\sqrt{nk(n-k)}}|\Phi|^3$$

for an integer $2 \leq k \leq n-1$ and:

$$|\Phi|^2 \leq \alpha_{R,k}. \tag{4.11}$$

Then:

1. *either $|\Phi| = 0$ and Σ is a totally umbilical hypersurface,*
2. *or $|\Phi|^2 = \alpha_{R,k} > 0$ and Σ is a product of spheres $\mathbb{S}^k(\sqrt{1-r^2}) \times \mathbb{S}^{n-k}(r) \subset \mathbb{S}^{n+1}$, with $r = r(R) < \sqrt{\frac{n-k}{n} - \frac{1}{n}\sqrt{\frac{k(n-k)}{n-1}}}$.*

Finally, in the particular case where n is even, say $n = 2m$, and $k = n/2 = m$, the real function $Q_{R,m}(x)$ simplifies to:

$$Q_{R,m}(x) = 2(m-1)\left(x^2 - \frac{m(2m-1)R}{m-1}\right),$$

so that we can compute explicitly the value of:

$$\alpha_{R,m} = \frac{m(2m-1)R}{m-1}.$$

Besides, (4.9) simplifies to:

$$R = \frac{m-1}{2(2m-1)r^2(1-r^2)},$$

which yields:

$$r = \sqrt{\frac{1}{2} - \frac{1}{2}\sqrt{1 - \frac{2(m-1)}{(2m-1)R}}}.$$

Therefore, in this case, Theorem 4.1 becomes as follows.

Corollary 4.7. *Let Σ^{2m} be a compact-oriented hypersurface isometrically immersed into a unit sphere $\mathbb{S}^{2m+1}$ $(m \geq 2)$ with constant normalized scalar curvature R satisfying $R > 1$. In the case where $R = 1$, assume further that the mean curvature function H does not change sign. Assume that:*

$$\mathrm{Tr}(\Phi^3) \geq 0$$

on Σ, where Φ is the total umbilicity tensor of the immersion. Then:

$$\int_\Sigma |\Phi|^{p+2}\left(|\Phi|^2 - \frac{m(2m-1)R}{m-1}\right) \geq 0 \tag{4.12}$$

for every real number $p \geq 2$. Moreover, if $R > 1$, the equality holds in (4.12) if and only if:

(1) *either $|\Phi| = 0$ and Σ is a totally umbilical hypersurface,*

(2) *or*

$$|\Phi|^2 = \frac{m(2m-1)R}{m-1} > 0,$$

and Σ is a product of spheres $\mathbb{S}^m(\sqrt{1-r^2}) \times \mathbb{S}^m(r) \subset \mathbb{S}^{2m+1}$ with:

$$r = \sqrt{\frac{1}{2} - \frac{1}{2}\sqrt{1 - \frac{2(m-1)}{(2m-1)R}}} < \sqrt{\frac{1}{2} - \frac{1}{2}\sqrt{\frac{1}{2m-1}}}.$$

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References

[1] Alencar, H., do Carmo, M.: Hypersurfaces with constant mean curvature in spheres. Proc. Am. Math. Soc. **120**, 1223–1229 (1994)

[2] Alías, L.J., García-Martínez, S.C., Rigoli, M.: A maximum principle for hypersurfaces with constant scalar curvature and applications. Ann. Glob. Anal. Geom. **41**, 307–320 (2012)

[3] Alías, L.J., Mastrolia, P., Rigoli, M.: *Maximum principles and geometric applications*, Springer Monographs in Mathematics. Springer, Cham (2016). xvii+570 pp. ISBN: 978-3-319-24335-1; 978-3-319-24337-5

[4] Alías, L.J., Meléndez, J., Palmas, O.: Hypersurfaces with constant scalar curvature in space forms. Differ. Geom. Appl. **58**, 65–82 (2018)

[5] Brasil Jr., A., Colares, A.G., Palmas, O.: Complete hypersurfaces with constant scalar curvature in spheres. Monatsh. Math. **161**, 369–380 (2010)

[6] Cartan, E.: Familles de surfaces isoparamétriques dans les espaces à courbure constante. Ann. Mat. Pura Appl. **17**, 177–191 (1938)

[7] Cheng, S.Y., Yau, S.T.: Hypersurfaces with constant scalar curvature. Math. Ann. **225**, 195–204 (1977)

[8] Li, H.: Hypersurfaces with constant scalar curvature in space forms. Math. Ann. **305**, 665–672 (1996)

[9] Meléndez, J.: Rigidity theorems for hypersurfaces with constant mean curvature. Bull. Braz. Math. Soc. (N.S.) **45**, 385–404 (2014)

[10] Okumura, M.: Hypersurfaces and a pinching problem on the second fundamental tensor. Am. J. Math. **96**, 207–213 (1974)

[11] Cecil, T., Ryan, P.J.: Geometry of hypersurfaces. Springer Monographs in Mathematics. Springer, New York (2015). xi+596 pp. ISBN: 978-1-4939-3245-0; 978-1-4939-3246-7

[12] Reilly, R.C.: Variational properties of functions of the mean curvature for hypersurfaces in space forms. J. Differ. Geom. **8**, 465–477 (1973)

[13] Guoxin, W.: J. Simons' type integral formula for hypersurfaces in a unit sphere. J. Math. Anal. Appl. **340**(2), 1371–1379 (2008)

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