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A Complete Asymptotic Expansion for Bernstein–Chlodovsky Polynomials for Functions on $\mathbb{R}$

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Abstract. We consider a variant of the Bernstein–Chlodovsky polynomials approximating continuous functions on the entire real line and study its rate of convergence. The main result is a complete asymptotic expansion. As a special case we obtain a Voronovskaja-type formula previously derived by Karsli [11].

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1. Introduction

Let f be a real function on $\mathbb{R}$ which is bounded on each finite interval. For $a, b \in \mathbb{R}$ with $a < b$, define the function $f_{a,b}$ on $[0,1]$ by $f_{a,b}(t) = f(a + (b-a)t)$. Furthermore, put

$$\|f\|_{a,b} = \sup_{a \le t \le b} |f(t)|.$$

Obviously, $f_{0,1}$ is the restriction of f to $[0,1]$ and we have $\|f\|_{a,b} = \|f_{a,b}\|_{0,1}$.

The Bernstein–Chlodovsky operators applied to the function f described above are defined by

$$(C_{n,a,b}f)(x) = (B_n f_{a,b})\left(\frac{x-a}{b-a}\right), \qquad a \le x \le b,$$

where B_n denote the Bernstein operators defined by

$$(B_n f)(x) = \sum_{\nu=0}^{n} p_{n,\nu}(x) f\left(\frac{\nu}{n}\right), \qquad 0 \le x \le 1,$$

with Bernstein basis polynomials

$$p_{n,\nu}(x) = \binom{n}{\nu} x^{\nu} (1-x)^{n-\nu}, \quad 0 \le \nu \le n.$$

In the special case $[a,b]=[0,1]$, we have $C_{n,0,1}\equiv B_n$.

The symmetric version

$$(C_{n,-c,c}f)(x)=(B_nf_{-c,c})\left(\frac{x+c}{2c}\right)$$

with $-a=b=c>0$ was recently introduced by Kilgore [12, Eq. (7.1)]. Using this Chlodovsky generalization of the Bernstein operators he [12, Theorems 1 and 2] gave a constructive proof for the Weierstrass approximation theorem in weighted spaces of continuous functions defined on $[0,\infty)$ or on $(-\infty,\infty)$. See also [13].

In the following we suppose that the parameters a,b are coupled with n, i.e., $a=a_n$ and $b=b_n$. Because the difference between two nodes of $C_{n,a,b}$ is at least $(b-a)/n$ it is clear that the condition $b_n-a_n=o(n)$ as $n\to\infty$ is necessary for having convergence of $(C_{n,a_n,b_n}f)(x)$ to $f(x)$.

In the special case $a_n=0<b_n$, for $n\in\mathbb{N}$, these polynomials were introduced by I. Chlodovsky [7] in 1937 in order to approximate functions on infinite intervals. He showed that under the condition (1.3), if a function f satisfies

$$\lim_{n\to\infty}\exp\left(-\frac{\sigma n}{b_n}\right)\|f\|_{b_n}=0, \tag{1.1}$$

for every $\sigma>0$, then

$$\lim_{n\to\infty}(C_{n,0,b_n}f)(x)=f(x)$$

at each point x of continuity of f. Moreover, he proved convergence in each continuity point for the large class of functions f satisfying the growth condition $f(t)=O(\exp(t^p))$ as $t\to+\infty$, if the sequence (b_n) satisfies the condition

$$b_n=O\left(n^{1/(p+1+\eta)}\right)\qquad(n\to\infty), \tag{1.2}$$

for an arbitrary small $\eta>0$. For more results on Chlodovsky operators see the survey article [9] by Karsli.

Explicit expressions of the coefficients $c_k^{[b_n]}(f,x)$ in terms of Stirling numbers were given by Karsli [10]. He derived the asymptotic expansion if the function f satisfies condition (1.1) for every $\sigma>0$.

Throughout the paper we assume that the sequences (a_n) and (b_n) satisfy

$$b_n-a_n>0,\quad \lim_{n\to\infty}(-a_n)=\lim_{n\to\infty}b_n=+\infty,\quad\text{and}\quad \lim_{n\to\infty}\frac{-a_nb_n}{n}=0. \tag{1.3}$$

The purpose of this note is a pointwise complete asymptotic expansion for the sequence of Bernstein–Chlodovsky operators in the form:

$$(C_{n,a_n,b_n}f)(x)\sim f(x)+\sum_{k=1}^{\infty}c_k^{[a_n,b_n]}(f,x)\left(\frac{-a_nb_n}{n}\right)^k+o\left(\left(\frac{-a_nb_n}{n}\right)^q\right) \tag{1.4}$$

as $n\to\infty$, for sufficiently smooth functions f satisfying $f(t)=O(\exp(\alpha t^p))$ as $t\to+\infty$, provided that the sequences (a_n), (b_n) satisfy $(-a_nb_n)=$

$o\left(n^{1/(p+1)}\right)$ as $n \to \infty$. The coefficients $c_k^{[a_n,b_n]}(f,x)$, which depend on f and a_n, b_n, are bounded with respect to n.

The latter formula means that, for each fixed $x > 0$ and for all positive integers q:

$$(C_{n,a_n,b_n} f)(x) = f(x) + \sum_{k=1}^{q} c_k^{[a_n,b_n]}(f,x) \left(\frac{-a_n b_n}{n}\right)^k + o\left(\left(\frac{-a_n b_n}{n}\right)^q\right)$$

as $n \to \infty$.

2. Main Result

For real constants $\alpha \geq 0$ and $p \geq 0$, let $W_{\alpha,p}$ denote the class of functions $f \in C(\mathbb{R})$ satisfying the growth condition:

$$f(t) = O\left(\exp\left(\alpha |t|^p\right)\right) \qquad (|t| \to +\infty).$$

Note that in the special instance $p = 0$ the class $W_{\alpha,0}$ consists of the bounded continuous functions on $\mathbb{R}$. Since $W_{0,p}$ and $W_{\alpha,0}$ coincide we consider only the case $\alpha > 0$.

Recall that the Stirling numbers $s(n,k)$ and $S(n,k)$ of first and second kind, respectively, are defined by the relations:

$$z^{\underline{n}} = \sum_{k=0}^{n} s(n,k) z^k \qquad \text{and} \qquad z^n = \sum_{k=0}^{n} S(n,k) z^{\underline{k}} \qquad (z \in \mathbb{C}),$$

where $z^{\underline{0}} = 1$ and $z^{\underline{n}} = z(z-1)\cdots(z-n+1)$, for $n \in \mathbb{N}$, denote the falling factorials.

The following theorem is the main result.

Theorem 2.1. *Let $\alpha, p \geq 0$. Suppose that the function $f \in W_{\alpha,p}$ is $2q$ times differentiable in the point $x > 0$. Let $(-a_n)$ and (b_n) be sequences of reals tending to infinity and satisfying the growth condition:*

$$-a_n b_n = o\left(n^{1/(p+1)}\right) \qquad (n \to \infty). \tag{2.1}$$

Then, for any positive integer q, the Bernstein–Chlodovsky operators C_{n,a_n,b_n} possess the asymptotic expansion:

$$(C_{n,a_n,b_n} f)(x) = f(x) + \sum_{k=1}^{q} c_k^{[a_n,b_n]}(f,x) \left(\frac{-a_n b_n}{n}\right)^k + o\left(\left(\frac{-a_n b_n}{n}\right)^q\right)$$

as $n \to \infty$, where

$$c_k^{[a_n,b_n]}(f,x) = O(1) \qquad (n \to \infty). \tag{2.2}$$

The coefficients $c_k^{[a,b]}(f,x)$ have the explicit representation:

$$c_k^{[a,b]}(f,x) = \sum_{s=k}^{2k} \frac{f^{(s)}(x)}{s!} \sum_{j=0}^{s} A(k,s,j) \frac{(b-a)^j (x-a)^{s-j}}{(-ab)^k} \qquad (a < b)$$

with numbers

$$A\left(k,s,j\right)=\sum_{r=\max\{j,k\}}^{s}\left(-1\right)^{s-r}\binom{s}{r}s\left(r-j,r-k\right)S\left(r,r-j\right). \qquad (2.3)$$

Remark 2.2. Note that the coefficients $c_k^{[a_n,b_n]}\left(f,x\right)$ depend on n but are bounded with respect to n.

Remark 2.3. Our assumption (2.1) on the sequence $(-a_nb_n)$ corresponds to Chlodovsky's condition (1.2). Furthermore, it is related to the assumption in the case $a_n=0$ (see, [3, Theorem 1, Eq. (4)]).

In the special case $q=1$ Theorem 2.1 implies the following Voronovskaja-type result.

Corollary 2.4. *Let $\alpha,p\geq 0$. Suppose that the function $f\in W_{\alpha,p}$ admits a second derivative at the point $x>0$. Let $(-a_n)$ and (b_n) be sequences of reals tending to infinity and satisfying the growth condition (2.1) . Then, the Bernstein–Chlodovsky operators C_{n,a_n,b_n} satisfy the asymptotic relation:*

$$\lim_{n\to\infty}\frac{n}{-a_nb_n}\left(\left(C_{n,a_n,b_n}f\right)\left(x\right)-f\left(x\right)\right)=\frac{1}{2}f^{(2)}\left(x\right) \qquad (2.4)$$

Remark 2.5. The expansion in Theorem 2.1 is completely different to the (pointwise) complete asymptotic expansion:

$$\left(B_nf\right)\left(x\right)\sim f\left(x\right)+\sum_{k=1}^{\infty}c_k^{[0,1]}\left(f,x\right)n^{-k} \qquad \left(n\to\infty\right),$$

for the classical Bernstein polynomials B_n, which is valid for all bounded functions $f:[0,1]\to\mathbb{R}$ being sufficiently smooth in $x\in[0,1]$. The Voronovskaja formula states that

$$\lim_{n\to\infty}n\left(\left(B_nf\right)\left(x\right)-f\left(x\right)\right)=\frac{1}{2}x\left(1-x\right)f^{(2)}\left(x\right).$$

The same is true in the case of the classical Bernstein–Chlodovsky operators $C_{n,0,b_n}$. Their Voronovskaja-type formula:

$$\lim_{n\to\infty}\frac{n}{b_n}\left(\left(C_{n,0,b_n}f\right)\left(x\right)-f\left(x\right)\right)=\frac{1}{2}xf^{(2)}\left(x\right)$$

was derived in 1960 by Albrycht and Radecki [5]. For further history consult the survey article [9].

3. Auxiliary Results and Proof of the Main Theorem

Our starting-point is an explicit representation of the central moments of the Bernstein polynomials in terms of Stirling numbers of the first and second kind. In the following we write $e_m\left(x\right)=x^m$, $m\in\mathbb{N}_0$, for the m-th monomial and $\psi_x\left(t\right)=t-x$ for $x\in\mathbb{R}$.

Lemma 3.1. *The central moments of the Bernstein polynomials possess the representation*

$$\left(B_n\psi_x^s\right)(x)=\sum_{k=\left\lfloor\frac{s+1}{2}\right\rfloor}^{s} n^{-k}\sum_{j=0}^{s}A\left(k,s,j\right)x^{s-j}$$

$(s=0,1,2,\ldots)$, *where the coefficients are given by Eq. (2.3).*

For a proof see, e.g., [2].

Lemma 3.2. *The central moments of the Bernstein–Chlodovsky operators possess the representation:*

$$\left(C_{n,a,b}\psi_x^s\right)(x)=\sum_{k=\left\lfloor\frac{s+1}{2}\right\rfloor}^{s} n^{-k}\sum_{j=0}^{s}A\left(k,s,j\right)\left(b-a\right)^j\left(x-a\right)^{s-j}, \tag{3.1}$$

where the coefficients $A\left(k,s,j\right)$ are given by Eq. (2.3) .

Proof. We have

$$\begin{aligned}\left(C_{n,a,b}\psi_x^s\right)(x)&=\sum_{\nu=0}^{n}p_{n,\nu}\left(\frac{x-a}{b-a}\right)\left(a+\left(b-a\right)\frac{\nu}{n}-x\right)^s\\&=\left(b-a\right)^s\sum_{\nu=0}^{n}p_{n,\nu}\left(\frac{x-a}{b-a}\right)\left(\frac{\nu}{n}-\frac{x-a}{b-a}\right)^s\\&=\left(b-a\right)^s\left(B_n\psi_{\frac{x-a}{b-a}}^s\right)\left(\frac{x-a}{b-a}\right)\end{aligned}$$

and the lemma follows by Lemma 3.1. □

As we have seen in the proof of Lemma 3.2 the central moments of the Bernstein–Chlodovsky operators can be expressed in terms of Bernstein polynomials:

$$\left(C_{n,a,b}\psi_x^s\right)(x)=\left(b-a\right)^s\left(B_n\psi_{\frac{x-a}{b-a}}^s\right)\left(\frac{x-a}{b-a}\right).$$

As a consequence from well-known properties of the Bernstein polynomials we obtain the following result:

Lemma 3.3. *For $a\le x\le b$, it holds*

$$\begin{aligned}\left(C_{n,a,b}\psi_x^{2s}\right)(x)&=\sum_{k=s}^{2s}\left(\frac{\left(x-a\right)\left(b-x\right)}{n}\right)^k\\&\quad\times\sum_{i=2s-k}^{s}d_{2s,i}\left(\frac{\left(x-a\right)\left(b-x\right)}{\left(b-a\right)^2}\right)^{i-k}\left(b-a\right)^{2s-2k},\\\left(C_{n,a,b}\psi_x^{2s+1}\right)(x)&=\left(b+a-2x\right)\sum_{k=s+1}^{2s+1}\left(\frac{\left(x-a\right)\left(b-x\right)}{n}\right)^k\\&\quad\times\sum_{i=2s+1-k}^{s}d_{2s+1,i}\left(\frac{\left(x-a\right)\left(b-x\right)}{\left(b-a\right)^2}\right)^{i-k}\left(b-a\right)^{2s-2k},\end{aligned}$$

where $d_{s,i}$ are certain real numbers.

Proof. Taking advantage of the well-known formulas for the central moments of the Bernstein polynomials (see [8, Chapt. 10, Theorem 1.1]):

$$\left(B_n\psi_x^{2s}\right)(x) = \sum_{j=0}^{s} n^{j-2s} \sum_{i=j}^{s} d_{2s,i}\left(x\left(1-x\right)\right)^i,$$

$$\left(B_n\psi_x^{2s+1}\right)(x) = (1-2x)\sum_{j=0}^{s} n^{j-(2s+1)} \sum_{i=j}^{s} d_{2s+1,i}\left(x\left(1-x\right)\right)^i,$$

where $d_{s,i}$ are certain real numbers, we obtain

$$\left(C_{n,a,b}\psi_x^{2s}\right)(x) = (b-a)^{2s}\sum_{k=s}^{2s}\frac{1}{n^k}\sum_{i=2s-k}^{s} d_{2s,i}\left(\frac{(x-a)(b-x)}{(b-a)^2}\right)^i,$$

$$\left(C_{n,a,b}\psi_x^{2s+1}\right)(x) = (b-a)^{2s}(b+a-2x)\sum_{k=s+1}^{2s+1}\frac{1}{n^k}$$
$$\times \sum_{i=2s+1-k}^{s} d_{2s+1,i}\left(\frac{(x-a)(b-x)}{(b-a)^2}\right)^i.$$

This can be rewritten in the form as stated in the lemma. □

For the sake of brevity, in the following, we write

$$\left(C_{n,a,b}\psi_x^{s}\right)(x) = \sum_{k=\left\lfloor\frac{s+1}{2}\right\rfloor}^{s}\left(\frac{-ab}{n}\right)^k Q_{k,s}\left(a,b,s;x\right). \tag{3.2}$$

Note that, for $a \le x \le b$, with $a < 0 < b$, we have

$$\left|\frac{(x-a)(b-x)}{n}\right| \le \left|1-\frac{x}{a}\right|\left|1-\frac{x}{b}\right|\cdot\frac{|ab|}{n},$$
$$\left|\frac{(x-a)(b-x)}{(b-a)^2}\right| \le \frac{1}{4},$$
$$\left|\frac{b+a-2x}{b-a}\right| \le 1,$$
$$(b-a)^m \le 1, \quad \text{for } m \le 0, \quad \text{if } b-a \ge 1.$$

This immediately implies the following estimate for the central moment of the Bernstein–Chlodovsky operators.

Lemma 3.4. *Let $(-a_n)$ and (b_n) be sequences of reals tending to infinity and satisfying the condition $-a_nb_n = o(n)$ as $n \to \infty$. Then, for $s = 0,1,2,\ldots$, the quantities $Q_{k,s}(a_n,b_n,s;x)$ $\left(\left\lfloor\frac{s+1}{2}\right\rfloor \le k \le s\right)$ are bounded with respect to n, and*

$$\left(C_{n,a_n,b_n}\psi_x^{s}\right)(x) = O\left(\left(\frac{-a_nb_n}{n}\right)^{\left\lfloor\frac{s+1}{2}\right\rfloor}\right) \qquad (n\to\infty).$$

A crucial tool is the following estimate due to Bernstein (see [14, Theorem 1.5.3, p. 18f]).

Lemma 3.5. (Bernstein)*For* $0 \leq t \leq 1$*, the inequality*

$$0 \leq z \leq \frac{3}{2}\sqrt{nt\,(1-t)}$$

implies

$$\sum_{\substack{\nu \\ |\nu - nt| \geq 2z\sqrt{nt(1-t)}}} p_{n\nu}\,(t) \leq 2\exp\left(-z^2\right).$$

The next lemma presents a form of Lemma 3.5 which is more useful for application to Chlodovsky operators on the real line. It follows the idea of Albrycht and Radecki [5] who proved a similar result for the classical Chlodovsky operators.

Lemma 3.6. *Let* $a < x < b$*. If* $0 < \delta \leq 3\,(x-a)\,(b-x)\,/\,(b-a)$ *it holds*

$$\sum_{\substack{\nu \\ \left|a+(b-a)\frac{\nu}{n}\right| \geq \delta}} p_{n\nu}\left(\frac{x-a}{b-a}\right) \leq 2\exp\left(-\frac{n\delta^2}{4\,(x-a)\,(b-x)}\right)$$

Proof of Lemma 3.6. Putting $t = (x-a)\,/\,(b-a)$ in Lemma 3.5, we have

$$\sum_{\substack{\nu \\ \left|\nu - n\frac{x-a}{b-a}\right| \geq 2z\sqrt{n\frac{x-a}{b-a}\frac{b-x}{b-a}}}} p_{n\nu}\left(\frac{x-a}{b-a}\right)$$

$$= \sum_{\substack{\nu \\ \left|a-x+(b-a)\frac{\nu}{n}\right| \geq 2z\sqrt{n^{-1}(x-a)(b-x)}}} p_{n\nu}\left(\frac{x-a}{b-a}\right) \leq 2\exp\left(-z^2\right)$$

if $0 \leq z \leq \frac{3}{2}\sqrt{n\frac{x-a}{b-a}\frac{b-x}{b-a}}$. Choose $\delta = 2z\sqrt{n^{-1}\,(x-a)\,(b-x)}$. Then

$$\sum_{\substack{\nu \\ \left|a-x+(b-a)\frac{\nu}{n}\right| \geq \delta}} p_{n\nu}\left(\frac{x-a}{b-a}\right) \leq 2\exp\left(-\frac{n\delta^2}{4\,(x-a)\,(b-x)}\right)$$

if $0 \leq \delta/\left(2\sqrt{n^{-1}\,(x-a)\,(b-x)}\right) \leq \frac{3}{2}\sqrt{n\frac{x-a}{b-a}\frac{b-x}{b-a}}$. The latter inequality is equivalent to

$$\delta \leq 3\frac{(x-a)\,(b-x)}{b-a}$$

which is a condition of the lemma. □

Lemma 3.7. *Let* $a < x < b$ *and* $0 < \delta \leq 3\,(x-a)\,(b-x)\,/\,(b-a)$*. If a bounded function* $f : [a,b] \to \mathbb{R}$ *satisfies* $f\,(t) = 0$*, for all* $t \in (x-\delta, x+\delta) \cap [a,b]$*, it follows the estimate*

$$|(C_{n,a,b}f)\,(x)| \leq 2\exp\left(-\frac{n\delta^2}{4\,(x-a)\,(b-x)}\right)\|f\|_{a,b}\,.$$

Proof. Because of $f\left(a+(b-a)\frac{\nu}{n}\right)=0$ for all $\nu\in\{0,\ldots,n\}$ with $\left|a+(b-a)\frac{\nu}{n}-x\right|<\delta$ we have

$$\begin{aligned}\left|\left(C_{n,a,b}f\right)(x)\right| &= \left|\sum_{\substack{\nu\\ \left|a+(b-a)\frac{\nu}{n}\right|\geq\delta}} p_{n,\nu}\left(\frac{x-a}{b-a}\right) f\left(a+(b-a)\frac{\nu}{n}\right)\right| \\ &\leq \|f\|_{a,b}\sum_{\substack{\nu\\ \left|a+(b-a)\frac{\nu}{n}\right|\geq\delta}} p_{n,\nu}\left(\frac{x-a}{b-a}\right)\end{aligned}$$

and the assertion follows by an application of Lemma 3.6. □

A direct consequence is the following localization result for Bernstein–Chlodovsky polynomials which is interesting in itself.

Proposition 3.8 (Localization theorem). *Let $\alpha,p\geq 0$ be fixed constants and propose that $f\in W_{\alpha,p}$ satisfies the estimate*

$$|f(t)|\leq K\exp\left(\alpha|t|^{p}\right)\qquad(t\in\mathbb{R}).$$

Furthermore, fix the real number $x\in(a,b)$ and let $\delta>0$. Then $f(t)=0$, for all $t\in(x-\delta,x+\delta)$, implies

$$\left|\left(C_{n,a,b}f\right)(x)\right|\leq 2K\exp\left(-\frac{n\delta^{2}}{4(x-a)(b-x)}\right)\exp\left(\alpha\max\{|a|,|b|\}^{p}\right).$$

Suppose that $\lim_{n\to\infty}(-a_n)=\lim_{n\to\infty}b_n=+\infty$. Then, for sufficiently large values of n, we can assume that $a_n<-1<1<b_n$, such that $\max\{|a_n|,|b_n|\}\leq -a_nb_n$. Since $(x-a_n)(b_n-x)=O(-a_nb_n)$ as $n\to\infty$, there is as positive constant $M(x)$ (independent of n), such that

$$\frac{1}{4(x-a_n)(b_n-x)}\geq\frac{M(x)}{-a_nb_n},$$

for sufficiently large values of n. Hence, we have

$$\exp\left(-\frac{n\delta^{2}}{4(x-a)(b-x)}\right)\leq\exp\left(-M(x)\frac{n\delta^{2}}{-a_nb_n}\right).$$

We conclude that, for large n,

$$\begin{aligned}\left|\left(C_{n,a_n,b_n}f\right)(x)\right| &\leq 2K\exp\left(-M(x)\frac{n\delta^{2}}{-a_nb_n}+\alpha(-a_nb_n)^{p}\right)\\ &= 2K\exp\left(-M(x)\frac{n}{-a_nb_n}\left(\delta^{2}-\frac{\alpha}{M(x)}\cdot\frac{(-a_nb_n)^{p+1}}{n}\right)\right).\end{aligned}$$

Proof of Theorem 2.1. Suppose that f is continuous on $\mathbb{R}$ being $2q$ times differentiable at the point $x\in\mathbb{R}$. Define the function h_x by

$$f=\sum_{s=0}^{2q}\frac{f^{(s)}(x)}{s!}\psi_x^{s}+h_x\psi_x^{2q}\tag{3.3}$$

and $h_x(x) = 0$. It is a consequence of Taylor's theorem that h_x is continuous at x. Hence, $h_x \in C(\mathbb{R})$. Applying the operator $C_{n,a,b}$ to both sides of Eq. (3.3) we obtain

$$\left(C_{n,a,b}f\right)(x) = \sum_{s=0}^{2q} \frac{f^{(s)}(x)}{s!} \left(C_{n,a,b}\psi_x^s\right)(x) + \left(C_{n,a,b}\left(h_x\psi_x^{2q}\right)\right)(x).$$

The first sum is equal to

$$\begin{aligned}\sum_{s=0}^{2q} \frac{f^{(s)}(x)}{s!} \left(C_{n,a,b}\psi_x^s\right)(x) &= \sum_{s=0}^{2q} \frac{f^{(s)}(x)}{s!} \sum_{k=\left\lfloor \frac{s+1}{2}\right\rfloor}^{s} \left(\frac{-ab}{n}\right)^k Q_{k,s}(a,b,s;x) \\ &= \sum_{k=0}^{2q} \left(\frac{-ab}{n}\right)^k \sum_{s=k}^{2k} \frac{f^{(s)}(x)}{s!} Q_{k,s}(a,b,s;x) \\ &= \sum_{k=0}^{2q} c_k^{[a,b]}(f,x) \left(\frac{-ab}{n}\right)^k.\end{aligned}$$

Note that $c_0^{[a,b]}(f,x) = 1$. Eq. (2.2) is a consequence of Lemma 3.4. We conclude that

$$\sum_{s=0}^{2q} \frac{f^{(s)}(x)}{s!} \left(C_{n,a_n,b_n}\psi_x^s\right)(x) = \sum_{k=0}^{q} c_k^{[a_n,b_n]}(f,x) \left(\frac{-a_nb_n}{n}\right)^k + o\left(\left(\frac{-a_nb_n}{n}\right)^q\right)$$

as $n \to \infty$. In order to complete the proof we have to show that the remainder can be estimated by

$$\left(C_{n,a_n,b_n}\left(h_x\psi_x^{2q}\right)\right)(x) = o\left(\left(\frac{-a_nb_n}{n}\right)^q\right) \qquad (n \to \infty).$$

To this end let (δ_n) be a sequence of positive numbers such that

$$\delta_n^2 = \frac{-a_nb_n}{nM(x)} \left(\alpha\left(-a_nb_n\right)^p - q\log\frac{-a_nb_n}{n} + \sqrt{\frac{n}{-a_nb_n}}\right) \qquad (n \in \mathbb{N}). \tag{3.4}$$

Note that the conditions (1.3) and (2.1) imply that $\delta_n = o(1)$ as $n \to \infty$. Define

$$\varepsilon_n = \sup\left\{\left|h_x(t)\right| : t \in (x-\delta_n, x+\delta_n)\right\}.$$

Because h_x is continuous with $h_x(x) = 0$ we have $\varepsilon_n = o(1)$ as $n \to \infty$. We split the remainder into two parts

$$\begin{aligned}&\left(C_{n,a_n,b_n}\left(h_x\psi_x^{2q}\right)\right)(x) \\ &= \sum_{\substack{\nu \\ \left|a_n+(b_n-a_n)\frac{\nu}{n}-x\right|<\delta_n}} p_{n,\nu}\left(\frac{x-a_n}{b_n-a_n}\right)\left(h_x\psi_x^{2q}\right)\left(a_n+(b_n-a_n)\frac{\nu}{n}\right) \\ &\quad + \sum_{\substack{\nu \\ \left|a_n+(b_n-a_n)\frac{\nu}{n}-x\right|\geq\delta_n}} p_{n,\nu}\left(\frac{x-a_n}{b_n-a_n}\right)\left(h_x\psi_x^{2q}\right)\left(a_n+(b_n-a_n)\frac{\nu}{n}\right) \\ &= \sum\nolimits_1 + \sum\nolimits_2,\end{aligned}$$

say. Let us start with the estimate of the first sum:

$$\left|\sum\nolimits_1\right| \leq \varepsilon_n \sum_{\substack{\nu \\ \left|a_n+(b_n-a_n)\frac{\nu}{n}-x\right|<\delta_n}} p_{n,\nu}\left(\frac{x-a_n}{b_n-a_n}\right)\psi_x^{2q}\left(a_n+(b_n-a_n)\frac{\nu}{n}\right)$$

$$\leq \varepsilon_n\left(C_{n,a_n,b_n}\psi_x^{2q}\right)(x) = \varepsilon_n O\left(\left(\frac{-a_nb_n}{n}\right)^q\right) = o\left(\left(\frac{-a_nb_n}{n}\right)^q\right)$$

as $n \to \infty$, where we used Lemma 3.4. By the Taylor formula (3.3), the second sum can be rewritten as

$$\sum\nolimits_2 = \sum_{\substack{\nu \\ \left|a_n+(b_n-a_n)\frac{\nu}{n}-x\right|\geq\delta_n}} p_{n,\nu}\left(\frac{x-a_n}{b_n-a_n}\right)$$

$$\times\left(f\left(a_n+(b_n-a_n)\frac{\nu}{n}\right) - \sum_{s=0}^{2q}\frac{f^{(s)}(x)}{s!}\psi_x^s\left(a_n+(b_n-a_n)\frac{\nu}{n}\right)\right)$$

and we obtain

$$\left|\sum\nolimits_2\right| \leq 2\exp\left(-M(x)\frac{n\delta_n^2}{-a_nb_n}\right)$$

$$\times\left(\|f\|_{a_n,b_n} + \sum_{s=0}^{2q}\frac{\left|f^{(s)}(x)\right|}{s!}\max\left\{|x-a_n|^s, |b_n-x|^s\right\}\right),$$

where in the last step Lemma 3.6 was applied. Note that

$$\sum_{s=0}^{2q}\frac{\left|f^{(s)}(x)\right|}{s!}\max\left\{|x-a_n|^s, |b_n-x|^s\right\} = O\left((-a_nb_n)^{2q}\right) \qquad (n\to\infty).$$

Hence,

$$\sum\nolimits_2 = O\left(\exp\left(\alpha(-a_nb_n)^p - M(x)\frac{n\delta_n^2}{-a_nb_n}\right)\right)$$

$$+ O\left(\exp\left(2q\log(-a_nb_n) - M(x)\frac{n\delta_n^2}{-a_nb_n}\right)\right)$$

as $n \to \infty$. In the case $p = 0$, i.e., f is bounded on $\mathbb{R}$, we have

$$\sum\nolimits_2 = O\left(\exp\left(2q\log(-a_nb_n) - M(x)\frac{n\delta_n^2}{-a_nb_n}\right)\right) \qquad (n\to\infty).$$

We can assume that $\alpha > 0$. Therefore, in the case $p > 0$, we have

$$\sum\nolimits_2 = O\left(\exp\left(\alpha(-a_nb_n)^p - M(x)\frac{n\delta_n^2}{-a_nb_n}\right)\right) \qquad (n\to\infty).$$

Obviously, it is sufficient to estimate the latter relation. By Eq. (3.4), we infer that

$$\sum\nolimits_2 = O\left(\exp\left(q\log\frac{-a_nb_n}{n} - \sqrt{\frac{n}{-a_nb_n}}\right)\right)$$

$$= O\left(\left(\frac{-a_nb_n}{n}\right)^q e^{-\sqrt{n/(-a_nb_n)}}\right) \qquad (n\to\infty).$$

Finally, we conclude that the remainder can be estimated by

$$\left(C_{n,a_n,b_n}\left(h_x\psi_x^{2q}\right)\right)(x)=o\left(\left(\frac{-a_nb_n}{n}\right)^q\right)\qquad(n\to\infty)$$

which completes the proof of the theorem. □

Proof of Corollary 2.4. In the special case $q=1$, Theorem 2.1 states that

$$(C_{n,a_n,b_n}f)(x)=f(x)+c_1^{[a_n,b_n]}(f,x)\frac{-a_nb_n}{n}+o\left(\frac{-a_nb_n}{n}\right)\qquad(n\to\infty)$$

which can be rewritten in the form

$$\frac{n}{-a_nb_n}\left((C_{n,a_n,b_n}f)(x)-f(x)\right)=c_1^{[a_n,b_n]}(f,x)+o(1)\qquad(n\to\infty).$$

We have

$$c_1^{[a,b]}(f,x)=\frac{n}{-ab}\cdot\frac{f^{(2)}(x)}{2!}\left(C_{n,a,b}\psi_x^2\right)(x)=\frac{1}{2}f^{(2)}(x)\frac{(x-a)(b-x)}{-ab}$$

and the desired formula follows because

$$\lim_{n\to\infty}c_1^{[a_n,b_n]}(f,x)=\frac{1}{2}f^{(2)}(x)\lim_{n\to\infty}\frac{(x-a_n)(b_n-x)}{-a_nb_n}=\frac{1}{2}f^{(2)}(x).$$

This completes the proof. □

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