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Undominated Sequences of Integrable Functions

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Abstract. In this paper, we investigate to what extent the conclusion of the Lebesgue dominated convergence theorem holds if the assumption of dominance is dropped. Specifically, we study both topological and algebraic genericity of the family of all null sequences of functions that, being continuous on a locally compact space and integrable with respect to a given Borel measure in it, are not controlled by an integrable function.

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1. Introduction

Lebesgue's Dominated Convergence Theorem (LDCT) is probably the most useful tool to interchange limits and integrals of a sequence of functions. In its most common version (see, e.g., [22, Chapter 1]), it asserts that if $(X, \mathcal{M}, \mu)$ is a measure space and $f, f_1, f_2, \ldots$ are extended real-valued measurable functions on X, such that $f_n(x) \longrightarrow f(x)$ $(n \to \infty)$ for μ-almost every $x \in X$ and there is an integrable function $g : X \to [-\infty, +\infty]$ with $|f_n(x)| \leq g(x)$ for μ-almost every $x \in X$ and all $n \geq 1$, then f is integrable on X and $\|f_n - f\|_1 \to 0$ as $n \to \infty$ (where $\|h\|_1$ denotes the 1-norm $\int_X |h| \,\mathrm{d}\mu$), so that, in particular, $\lim_{n\to\infty} \int_X f_n \,\mathrm{d}\mu = \int_X f \,\mathrm{d}\mu$. The result can be generalized to extended complex-valued functions, to orders of integration $p \geq 1$ and to other kinds of convergence, such as convergence in measure or μ-almost uniform convergence (see, e.g., [19, Chapter 21]), but we will focus on the former version.

Since measurability of the f_ns and almost everywhere pointwise convergence $f_n \longrightarrow f$ seem to be "natural" conditions in order that 1-norm convergence can take place, the following question arises:

Is it feasible to expect $\|f_n - f\|_1 \to 0$ *without assuming the existence*
of some dominating integrable function g?

In turn, since $|f_n| \le g$ implies automatically integrability for the f_ns and f, then, after replacing f_n by $f_n - f$, the problem can be reduced to get $\|f_n\|_1 \to 0$ by assuming $f_n(x) \longrightarrow 0$ almost everywhere but not dominance.

The aim of this paper is to provide an affirmative answer to the above question, in both topological and algebraic senses. The preliminary background and terminology is collected in Sect. 2. Our assertions, together with motivating related results in the literature, are presented in Sect. 3. Finally, the proof of our results will be provided in Sects. 4 to 6.

2. Notation and Preliminaries

Those readers who are familiar with Borel measures, lineability, prevalency, and Baire categories can skip this section. As usual, we will denote by $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{R}$, $\mathbb{Q}$ and $\mathfrak{c}$, respectively, the set of natural numbers, the set $\mathbb{N} \cup \{0\}$, the real line, the field of rational numbers, and the cardinality of the continuum.

Assume that X is a Hausdorff topological space. Then, the family $\mathcal{B}$ of Borel sets of X is the least σ-algebra on X containing all open sets. Then, any continuous function $X \to \mathbb{R}$ is measurable if X is endowed with a measurable space structure defined by a σ-algebra $\mathcal{M} \supset \mathcal{B}$. The symbols $C(X)$, $C_c(X)$, $C_0(X)$ will represent, respectively, the set of all continuous functions $X \to \mathbb{R}$, the subset of those $f \in C(X)$ having compact support, and the subset of those $f \in C(X)$ vanishing at infinity. Recall that the support of an $f : X \to \mathbb{R}$ is the set $\overline{\{x \in X : f(x) \neq 0\}}$ ($\overline{A}$ denotes closure of A), and that f is said to vanish at infinity provided that, given $\varepsilon > 0$, there exists a compact $K \subset X$, such that $|f(x)| < \varepsilon$ for all $x \in X \setminus K$ (with the agreement $C_0(X) := C(X)$ if X itself is compact). The functional $\|f\|_\infty := \sup_{x \in X} |f(x)|$ is a norm both in $C_c(X)$ and $C_0(X)$, and if X is a locally compact Hausdorff space, then $C_0(X)$ is the completion of $C_c(X)$ (see [22, Chapter 3]), so that, in particular, $C_0(X)$ becomes a Banach space under the last norm with $C_c(X)$ being dense in it.

For any measure space $(X, \mathcal{M}, \mu)$, the vector space $L^1(\mu)$ of measurable functions $f : X \to [-\infty, +\infty]$, such that $\|f\|_1 < +\infty$ is a Banach space under the norm $\|\cdot\|_1$ (see, e.g., [19, Chapter 14]). Recall that, in $L^1(\mu)$, two functions are identified whenever they are equal μ-almost everywhere (μ-a.e.).

Suppose that X is a locally compact Hausdorff space. A *Borel measure* μ on X is a positive measure defined on some σ-algebra $\mathcal{M} \supset \mathcal{B}$. If this is the case, then μ is called *regular* provided it satisfies, for all $A \in \mathcal{M}$, that $\mu(A) = \sup\{\mu(K) : K \text{ compact}, K \subset A\}$ (inner regularity) and $\mu(A) = \inf\{\mu(G) : G \text{ open}, G \supset A\}$ (outer regularity). Observe that if $f \in C(X)$ and μ is a Borel measure on X, then f is measurable. Then, expressions as $Y \cap L^1(\mu)$, where

$Y \subset C(X)$, make sense, meaning the set of all $f : X \to \mathbb{R}$ that are in Y, such that $\|f\|_1 < +\infty$.

A subset A of a topological space Z is said to be of first category whenever there are countably many nowhere dense sets F_n $(n \in \mathbb{N})$, such that $A = \bigcup_{n=1}^{\infty} F_n$. Recall that a subset $B \subset Z$ is called nowhere dense if its closure has empty interior. A set $S \subset Z$ is said to be *residual* whenever $X \backslash S$ is of first category. Baire's category theorem (see, e.g., [20]) asserts that if Z is completely metrizable, then any countable intersection of dense open subsets is still dense. If this is the case, a set $S \subset Z$ is residual if and only if it contains a dense G_δ subset. In a topological sense, residual sets are "very large" in such spaces Z. Moreover, recall that a topological space Z is called *second-countable* if it possesses a countable open basis, and σ-*compact* if Z is the union of countable many compact subsets.

Another way to assess the largeness of a property is by means of the modern theory of lineability (see [2–5,8,14,15,23] for terminology and background), which focusses on the algebraic genericity of a family within a vector space. Assume that Z is a vector space and $A \subset Z$. Then, A is said to be *lineable* if there is an infinite-dimensional vector space M, such that $M \backslash \{0\} \subset A$; and *maximal-lineable* if, moreover, $\dim(M) = \dim(Z)$. If, in addition, Z is a topological vector space, then A is called *spaceable* (*dense-lineable*, *maximal dense-lineable*, resp.) in Z whenever there is a closed infinite-dimensional (a dense, a dense $\dim(Z)$-dimensional, resp.) vector subspace M of Z, such that $M \backslash \{0\} \subset A$. Now, assume that Z is a vector space contained in some (linear) algebra. Then, the subset A is called *algebrable* if there is an infinitely generated algebra M—that is, the cardinality of any system of generators of M is infinite—so that $M \backslash \{0\} \subset A$; and, if α is a cardinal number, then A is said to be *strongly* α-*algebrable* if there exists an α-generated *free* algebra M with $M \backslash \{0\} \subset A$. Recall that if Z is contained in a commutative algebra, then a set $B \subset Z$ is a generating set of some free algebra contained in A if and only if, for any $N \in \mathbb{N}$, any nonzero polynomial P in N variables without constant term, and any distinct $f_1, \ldots, f_N \in B$, we have $P(f_1, \ldots, f_N) \neq 0$ and $P(f_1, \ldots, f_N) \in A$. The reader can easily check that many implications among these properties hold; for instance, spaceability implies lineability, dense-lineability (if $\dim(X) = \infty$) implies lineability, strong α-algebrability (if α is infinite) implies algebrability, and others.

3. Statement of the Results

There are in the literature a number of results related to the topic we are concerned with, see, for instance, [9,21]. Unless otherwise stated, the measure considered on an interval of $\mathbb{R}$ will be always the Lebesgue measure m.

In [7], it is proved that, in the vector space of sequences:

$$CBL_s := \{(f_k)_k \in (\mathbb{R}^{\mathbb{R}})^{\mathbb{N}} : \text{each } f_k \text{ is continuous, bounded and integrable,} \\ \|f_k\|_\infty \underset{k\to\infty}{\longrightarrow} 0 \quad \text{and} \quad \sup_{k\geq 1} \|f_k\|_1 < +\infty\}$$

(which becomes a non-separable Banach space when endowed with the norm $\|(f_k)_k\| = \sup_k \|f_k\|_\infty + \sup_k \|f_k\|_1$), the subset $\{(f_k)_k \in CBL_s : \|f_k\|_1 \not\to 0\}$ is spaceable. Note that what does not hold for the sequences of this subset is the conclusion of LDCT. As a complementary statement, it is shown in [10] that in the F-space:

$$Y := \Big\{ (f_k)_k \in (\mathbb{R}^{\mathbb{R}})^{\mathbb{N}} : \text{ each } f_k \text{ is continuous and integrable, } \|f_k\|_1 \underset{k\to\infty}{\longrightarrow} 0 \text{ and } f_k \underset{k\to\infty}{\longrightarrow} 0 \text{ uniformly on compacta in } \mathbb{R} \Big\}$$

(under the topology of both compact and 1-norm convergence), the subset $\{(f_k)_k \in Y :$ each f_k is *unbounded*$\}$ is maximal dense-lineable in Y (the result is formulated for functions $[0, +\infty) \to \mathbb{R}$, but minor changes in the proof yields that it holds for functions $\mathbb{R} \to \mathbb{R}$). This time, the conclusion of LDCT holds for the sequence of the subset, but each member of each sequence is unbounded (which, incidentally, is not an obstacle for dominance). For results dealing with lineability of families of sequences of measurable functions $[0, 1] \to \mathbb{R}$ or $[0, +\infty) \to \mathbb{R}$—where several kinds of convergence are considered—see [1, Section 7] and [11]. See also [6] and [12] for lineability facts related to expect values of sequences of random variables defined on a probability space.

In this paper, we focus on the effect of dropping the dominance hypothesis in LDCT, so as to complement the results from the previous paragraph. We shall show that, under a topological or algebraic point of view, such effect is almost imperceptible, in the sense that the conclusion of LDCT still holds for "many" sequences, even *uniformly bounded* sequences. Moreover, this will be carried out into a rather general setting.

To state our assertions, we adopt the following notation and conventions:

- X is a fixed locally compact Hausdorff space.
- μ is a Borel measure on X, so that we have a measure space $(X, \mathcal{M}, \mu)$ with $\mathcal{M} \supset \mathcal{B}$.
- μ is a Baire measure (that is, $\mu(K) < +\infty$ for all compact subsets $K \subset X$), regular, and non-finite (that is, $\mu(X) = +\infty$).
- We say that a sequence $(f_k)_k \subset L^1(\nu)$ is L^1-*undominated* if there is no $g \in L^1(\nu)$, such that $|f_k| \le g$ on X for all $k \in \mathbb{N}$ or, equivalently, if $\sup_k |f_k| \notin L^1(\nu)$ (note that the function $\sup_k |f_k| : X \to [0, +\infty]$ is always measurable).

In what follows, we define the space we are going to deal with:

Definition 3.1. The space $c_{0,1,\infty}(C_0, L^1)$ will denote the vector space of all sequences $(f_k)_k \subset C_0(X) \cap L^1(\mu)$, such that $\|f_k\|_1 \longrightarrow 0$ and $\|f_k\|_\infty \longrightarrow 0$ as $k \to \infty$ and $\mathcal{F}$ will stand for *the class of* L^1-*undominated sequences* $(f_k)_k \in c_{0,1,\infty}(C_0, L^1)$.

Besides the linear structure, we endow $c_{0,1,\infty}(C_0, L^1)$ with the natural structure of (linear) *algebra* by completing sum and scalar multiplication with the coordinatewise product $\mathbf{fg} = (f_k g_k)_k$, where $\mathbf{f} = (f_k)_k$ and $\mathbf{g} = (g_k)_k$.

This makes sense, because the product of two functions from $C_0(X) \cap L^1(\mu)$ is still in $C_0(X) \cap L^1(\mu)$ (use that the members of $C_0(X)$ are bounded) and, for $\mathbf{f}$ and $\mathbf{g}$ as above, we have $\|f_k g_k\|_\infty \to 0 \leftarrow \|f_k g_k\|_1$ as $k \to \infty$, which in turn follows from the facts $\|f_k g_k\|_\infty \leq \|f_k\|_\infty \cdot \sup_n \|g_n\|_\infty$ and $\|f_k g_k\|_1 \leq \|f_k\|_1 \cdot \sup_n \|g_n\|_\infty$.

Remark 3.2. 1. A number of assumptions are sometimes redundant. For instance, in the case $\mathcal{M} = \mathcal{B}$, if X satisfies, in addition, that every open subset is σ-compact, then the sole condition of finiteness of μ on compacta implies regularity for μ (see [22, Chapter 1]).
2. The assumption $\mu(X) = +\infty$ is necessary if we demand uniform convergence $f_n \longrightarrow 0$. Indeed, uniform convergence implies the existence of $m \in \mathbb{N}$, such that $F := \sup_{n>m} |f_n|$ is bounded. It is clear that F is measurable. If μ were finite, the function $\sup_n |f_n| = \max\{|f_1|, \ldots, |f_m|, F\}$ would be integrable, which is the non-desired property. In particular, X cannot be compact.

We first introduce the following auxiliary statement. Note that none of the assumptions on μ of being Baire, non-finite, or regular is needed this time.

Lemma 3.3. *The vector space* $c_{0,1,\infty}(C_0, L^1)$ *becomes a Banach space when endowed with the norm:*

$$\|\mathbf{f}\| := \sup_{k\geq 1} \|f_k\|_1 + \sup_{k\geq 1} \|f_k\|_\infty,$$

where $\mathbf{f} = (f_k)_k$. *In particular, it is a Baire space.*

Proof. That $\|\cdot\|$ makes sense on $Z := c_{0,1,\infty}(C_0, L^1)$ and is a norm on it is an easy exercise. Regarding completeness, assume that $(\mathbf{f}^j)_j \subset Z$ is a Cauchy sequence for $\|\cdot\|$. Let $\mathbf{f}^j = (f_k^j)_k$ $(j \in \mathbb{N})$. Fix $\varepsilon > 0$. Then, there is $j_0 \in \mathbb{N}$ satisfying:

$$\sup_{k\geq 1} \|f_k^j - f_k^l\|_\infty + \sup_{k\geq 1} \|f_k^j - f_k^l\|_1 < \varepsilon \quad \text{for all } j, l \geq j_0. \tag{1}$$

It follows at once that each sequence $(f_k^l)_k$ $(l \in \mathbb{N})$ is Cauchy both in $(C_0(X), \|\cdot\|_\infty)$ and $(L^1(\mu), \|\cdot\|_1)$, which are complete metric spaces. Consequently, there are functions $f_k \in C_0(X)$, $g_k \in L^1(\mu)$ $(k \in \mathbb{N})$, such that, in their respective topologies, $f_k^l \longrightarrow f_k$ and $f_k^l \longrightarrow g_k$ as $l \to \infty$. The latter property implies (see, e.g., [19, Theorems 21.4 and 21.9]) the existence of subsequence $(f_1^{l(1,s)})_s$ of (f_1^l) satisfying $f_1^{l(1,s)}(x) \longrightarrow g_1(x)$ $(s \to \infty)$ for all $x \in X \backslash Z_1$, where $\mu(Z_1) = 0$. However, $f_2^{l(1,s)} \longrightarrow g_2$ $(s \to \infty)$ for $\|\cdot\|_1$. Hence, there is a subsequence $(f_2^{l(2,s)})_s$ of $(f_2^{l(1,s)})_s$ satisfying $f_2^{l(2,s)}(x) \longrightarrow g_2(x)$ $(s \to \infty)$ for all $x \in X \backslash Z_2$, where $\mu(Z_2) = 0$. Following this procedure, the diagonal subsequence $(l(s,s))_s$ possesses the property that $f_k^{l(s,s)}(x) \longrightarrow g_k(x)$ $(s \to \infty)$ for all $k \in \mathbb{N}$ and all $x \in X \backslash \bigcup_{k\in\mathbb{N}} Z_k$. Trivially, $f_k^{l(s,s)}(x) \longrightarrow f_k(x)$ $(s \to \infty)$ for all $k \in \mathbb{N}$ and all $x \in X$. By uniqueness of pointwise limit, we get $f_k = g_k$ μ-a.e. for all $k \in \mathbb{N}$, because $\mu(\bigcup_{k\in\mathbb{N}} Z_k) = 0$. Hence $f_k^l \longrightarrow f_k$ $(l \to \infty)$ both in maximum norm and

1-norm for all $k \in \mathbb{N}$. Finally, a standard reasoning using (1) yields that $\mathbf{f} := (f_k)_k \in Z$ and $\mathbf{f}^j \longrightarrow \mathbf{f}$ as $j \to \infty$ in $\|\cdot\|$. □

We can now state or main results that will be proved in the forthcoming sections:

Theorem 3.4. *The set $\mathcal{F}$ is a residual subset of $(c_{0,1,\infty}(C_0, L^1), \|\cdot\|)$.*

Theorem 3.5. *The set $\mathcal{F}$ is spaceable in $(c_{0,1,\infty}(C_0, L^1), \|\cdot\|)$.*

Theorem 3.6. *If X is second-countable and every open subset of X is σ-compact, then $\mathcal{F}$ is maximal dense-lineable in $(c_{0,1,\infty}(C_0, L^1), \|\cdot\|)$.*

Theorem 3.7. *Assume that μ satisfies the following condition:*

(C) *There exist $\alpha, \beta \in (0, +\infty)$, such that every open set having infinite*
measure contains a measurable set M with $\alpha < \mu(M) < \beta$.

Then, the set $\mathcal{F}$ is strongly $\mathfrak{c}$-algebrable.

Concerning condition (C) above, see several remarks in Sect. 7 below.

4. Topological Genericity of Unbounded Convergence: Proof of Theorem 3.4

Let us abridge $Z := c_{0,1,\infty}(C_0, L^1)$. It is enough to show that $\mathcal{F}$ is dense in Z and that the set

$$\mathcal{A} := Z \backslash \mathcal{F} = \left\{ \mathbf{f} = (f_k)_k \in Z : \ \| \sup_{k \geq 1} |f_k| \|_1 < +\infty \right\}$$

is F_σ in Z; that is, a union of countably many closed sets. With this aim, note that we can write $\mathcal{A} = \bigcup_{n \in \mathbb{N}} F_n$, where:

$$F_n := \left\{ \mathbf{f} \in Z : \ \left\| \sup_{k \geq 1} |f_k| \right\|_1 \leq n \right\}.$$

That $\mathcal{A}$ is an F_σ will be proved by showing that each F_n is closed.
1. The set $\mathcal{F}$ is dense in Z. Observe that $\mathcal{A}$ is a vector subspace of Z. Indeed, if $\mathbf{f} = (f_k)_k, \mathbf{g} = (g_k)_k \in \mathcal{A}$ and $\alpha, \beta \in \mathbb{R}$, then:

$$\left\| \sup_{k \geq 1} |\alpha f_k + \beta g_k| \right\|_1 \leq |\alpha| \left\| \sup_{k \geq 1} |f_k| \right\|_1 + |\beta| \left\| \sup_{k \geq 1} |g_k| \right\|_1 < +\infty.$$

If we were able to prove that $\mathcal{F} \neq \varnothing$ then we would have $\mathcal{A} \neq Z$, and it is an elementary fact that any proper vector subspace of a topological vector space has empty interior. Hence, its complement $\mathcal{F}$ would be dense in Z, as required. Therefore, it is enough to exhibit an element $\mathbf{f} \in \mathcal{F}$.

With this aim, note that, by regularity and the fact $\mu(X) = +\infty > 1$, there is a compact set K_1, such that $\mu(K_1) \geq 1$. Since X is locally compact and Hausdorff, we can find an open set V_1 with compact closure, such that

$$K_1 \subset V_1 \subset \overline{V_1} \subset X$$

(see, e.g., [22, Theorem 2.7]). Then, $\mu(\overline{V_1}) < +\infty$, so $\mu(X\backslash\overline{V_1}) = +\infty > 1$. Again, by regularity, there is a compact set $K_2 \subset X\backslash\overline{V_1}$ with $\mu(K_2) \geq 1$. Now, take an open set V_2 with compact closure, such that

$$K_2 \subset V_2 \subset \overline{V_2} \subset X\backslash\overline{V_1}.$$

Then, $\mu(\overline{V_1} \cup \overline{V_2}) < +\infty$, so $\mu(X\backslash\overline{V_1 \cup V_2}) = +\infty > 1$. Thus, there is a compact set $K_3 \subset X\backslash\overline{V_1 \cup V_2}$ with $\mu(K_3) \geq 1$ and, subsequently, there is an open set V_3 with compact closure, such that

$$K_3 \subset V_3 \subset \overline{V_3} \subset X\backslash\overline{V_1 \cup V_2}.$$

By following this procedure, we can built a sequence $(K_n)_n$ of compact sets as well as a sequence $(V_n)_n$ of open sets satisfying:

$$K_n \subset V_n,\ V_m \cap V_n = \varnothing\ (m \neq n) \text{ and } 1 \leq \mu(K_n) < +\infty \quad \text{for all } n \in \mathbb{N}.$$

In addition, regularity allows us to select the V_ns, so that:

$$\mu(V_n) < 2\mu(K_n) \quad \text{for every } n \in \mathbb{N}.$$

According to a result due to Urysohn (see [22, Theorem 2.12]), there exists a function $\varphi_n \in C_c(X)$, such that $0 \leq \varphi_n \leq 1$ on X, $\varphi_n = 1$ on K_n and $\varphi_n = 0$ outside V_n. Let us define:

$$f_n := \frac{1}{n \cdot \mu(K_n)} \cdot \varphi_n \quad (n \in \mathbb{N}) \quad \text{and} \quad \mathbf{f} := (f_n)_n.$$

Clearly, $\|f_n\|_\infty \leq \frac{1}{n} \longrightarrow 0$ as $n \to \infty$, whereas:

$$\|f_n\|_1 = \int_{V_n} \frac{\varphi_n}{n \cdot \mu(K_n)}\, \mathrm{d}\mu \leq \int_{V_n} \frac{1}{n \cdot \mu(K_n)}\, \mathrm{d}\mu = \frac{\mu(V_n)}{n \cdot \mu(K_n)} \leq \frac{2}{n} \longrightarrow 0$$

as $n \to \infty$. Hence, $\mathbf{f} \in Z$. Finally, since the supports of the f_ns are mutually disjoint, we get:

$$\begin{aligned}\Big\| \sup_{n\geq 1} |f_n| \Big\|_1 &= \sum_{n=1}^{\infty} \|f_n\|_1 = \sum_{n=1}^{\infty} \int_{V_n} \frac{\varphi_n}{n \cdot \mu(K_n)}\, \mathrm{d}\mu \geq \sum_{n=1}^{\infty} \int_{K_n} \frac{\varphi_n}{n \cdot \mu(K_n)}\, \mathrm{d}\mu \\ &= \sum_{n=1}^{\infty} \frac{1}{n \cdot \mu(K_n)} \cdot \mu(K_n) = \sum_{n=1}^{\infty} \frac{1}{n} = +\infty.\end{aligned}$$

To summarize, $\sup_n |f_n| \notin L^1(\mu)$ and $\mathbf{f} \in \mathcal{F}$.

2. For each $n \in \mathbb{N}$, the set F_n is closed. Assume that $\{\mathbf{f}^j : j \geq 1\} \subset F_n$ and $\mathbf{f}^j \longrightarrow \mathbf{f} \in Z$ as $j \to \infty$. It should be shown that $\mathbf{f} \in F_n$. Let $\mathbf{f} = (f_k)_k$ and $\mathbf{f}^j = (f_k^j)_k$ $(j \in \mathbb{N})$. Let $g_j := \sup_k |f_k^j|$ $(j \in \mathbb{N})$ and $g := \sup_k |f_k|$. Then, $\|g_j\|_1 \leq n$ for all $j \in \mathbb{N}$. Our goal is to prove that $\|g\|_1 \leq n$. By assumption, $\|\mathbf{f}^j - \mathbf{f}\| \longrightarrow 0$ as $j \to \infty$. Then, $\sup_k \|f_k^j - f_k\|_\infty \longrightarrow 0$. In particular, given $x \in X$, we obtain from the reverse triangle inequality that:

$$\lim_{j\to\infty} \sup_{k\geq 1} \Big| |f_k^j(x)| - |f_k(x)| \Big| = 0. \tag{2}$$

Now, it is easy to see that, if $(a_k)_k \cup (b_k)_k \subset [0, +\infty)$ and $\alpha := \sup_k a_k$, $\beta := \sup_k b_k$ (so that α, $\beta \in [0, +\infty]$), then (under the convention $|(+\infty) - (+\infty)| = 0$) we have $|\alpha - \beta| \leq \sup_k |a_k - b_k|$. It follows from (2) that $g_j(x) \longrightarrow g(x)$ as $j \to \infty$ for all $x \in X$. Then, the extended real-valued functions g_js are non-negative and measurable, and $g = \liminf_{j\to\infty} g_j$. From Fatou's Lemma (see, e.g., [19, p. 201]), we get:

$$\|g\|_1 = \int_X g\,\mathrm{d}\mu \le \liminf_{j\to\infty} \int_X g_j\,\mathrm{d}\mu = \liminf_{j\to\infty} \|g_j\|_1 \le \liminf_{j\to\infty} n = n.$$

Consequently, $\|g\|_1 \le n$, as required. The proof is finished.

5. Spaceability and Lineability of Unbounded Convergence: Proofs of Theorems 3.5 and 3.6

First, we prove that $\mathcal{F}$ is spaceable in $Z := c_{0,1,\infty}(C_0, L^1)$. To this aim, we are going to construct an infinite-dimensional closed subspace M with $M\setminus\{0\} \subset \mathcal{F}$.

As in the previous section, we can find a sequence $(K_n)_n$ of compact sets, a sequence $(V_n)_n$ of open sets, and a sequence $(\varphi_n)_n \subset C_c(X)$ satisfying:

$$K_n \subset V_n,\ 1 \le \mu(K_n) \le \mu(V_n) < 2\mu(K_n) < +\infty \quad (n \in \mathbb{N}),$$
$$V_m \cap V_n = \varnothing \quad (m, n \in \mathbb{N};\ m \ne n),$$
$$0 \le \varphi_n \le 1 \text{ on } X,\ \varphi_n = 1 \text{ on } K_n, \text{ and } \varphi_n = 0 \text{ on } X\setminus V_n \quad (n \in \mathbb{N}).$$
$$N_j = \{n(j,1) < n(j,2) < n(j,3) < \cdots < n(j,k) < \cdots\} \quad (j \in \mathbb{N}).$$

For each $j \in \mathbb{N}$, define the sequence $\mathbf{f}^j = (f_{j,k})_k$ by:

$$f_{j,k} := \frac{1}{k \cdot \mu(K_{n(j,k)})} \cdot \varphi_{n(j,k)} \quad (k \in \mathbb{N}).$$

As in the last section, it is easy to see that each $\mathbf{f}^j$ belongs to Z. Let us show that $(\mathbf{f}^j)_j$ is a basic sequence in Z. Plainly, no $\mathbf{f}^j$ is zero. Now, assume that $(c_j)_{j\in\mathbb{N}} \subset \mathbb{R}$. By taking into account that the V_ns are pairwise disjoint, that $0 \le \varphi_n \le 1$, and that $\varphi_n = 1$ on K_n, we obtain:

$$\left\|\sum_{j=1}^N c_j \mathbf{f}^j\right\| = \sup_{k\ge1}\sup_{1\le j\le N} \frac{|c_j|}{k\cdot\mu(K_{n(j,k)})} + \sup_{k\ge1}\sum_{j=1}^N \frac{|c_j|}{k\cdot\mu(K_{n(j,k)})}\int_{V_{n(j,k)}} \varphi_{n(j,k)}\,\mathrm{d}\mu$$

for all $N \in \mathbb{N}$. Therefore:

$$\left\|\sum_{j=1}^p c_j \mathbf{f}^j\right\| \le \left\|\sum_{j=1}^q c_j \mathbf{f}^j\right\|$$

whenever $p, q \in \mathbb{N}$ with $p < q$. Consequently, Nikolskii's theorem (see, e.g., [13, pp. 36–38]) guarantees that $(\mathbf{f}^j)_j$ is a basic sequence in the Banach space Z (with basic constant 1).

Now, we define:

$$M := \overline{\mathrm{span}}\,\{\mathbf{f}^j\}_{j\in\mathbb{N}}.$$

Since $(\mathbf{f}^j)_j$ is a basic sequence, we have that M is an infinite-dimensional closed vector subspace of Z. It must be shown that $M\setminus\{0\} \subset \mathcal{F}$. Take

$\mathbf{f} = (f_k)_k \in M \setminus \{0\}$. Then, there is a unique sequence $(c_j)_j \in \mathbb{R}^{\mathbb{N}}$, such that:

$$\mathbf{f} = \sum_{j=1}^{\infty} c_j \mathbf{f}^j \text{ in } (Z, \|\cdot\|).$$

Moreover, there is at least one j with $c_j \neq 0$. Let N be the first among such js. For each $k \in \mathbb{N}$, we have $f_k = \sum_{j=N}^{\infty} c_j f_{j,k} = \sum_{j=N}^{\infty} \frac{c_j}{k \cdot \mu(K_{n(j,k)})} \cdot \varphi_{n(j,k)}$. Since the V_n's are pairwise disjoint and the φ_n's are non-negative, we get:

$$|f_k| = \sum_{j=N}^{\infty} \frac{|c_j|}{k \cdot \mu(K_{n(j,k)})} \cdot \varphi_{n(j,k)} \geq \frac{|c_N|}{k \cdot \mu(K_{n(N,k)})} \cdot \varphi_{n(N,k)}.$$

Again, the disjointness of the V_ns yields:

$$\sup_{k \geq 1} |f_k| \geq \sum_{k=1}^{\infty} \frac{|c_N|}{k \cdot \mu(K_{n(N,k)})} \cdot \varphi_{n(N,k)}.$$

Recall that $\varphi_n = 1$ on K_n and $K_n \subset V_n$. We conclude that:

$$\begin{aligned} \left\| \sup_{k \geq 1} |f_k| \right\|_1 &\geq |c_N| \cdot \sum_{k=1}^{\infty} \int_{V(N,k)} \frac{1}{k \cdot \mu(K_{n(N,k)})} \cdot \varphi_{n(N,k)} \, \mathrm{d}\mu \\ &\geq |c_N| \cdot \sum_{k=1}^{\infty} \int_{K(N,k)} \frac{1}{k \cdot \mu(K_{n(N,k)})} \, \mathrm{d}\mu \\ &= |c_N| \cdot \sum_{k=1}^{\infty} \frac{1}{k} = +\infty. \end{aligned}$$

Consequently, $\sup_k |f_k|$ is not μ-integrable; that is, $\mathbf{f} \in \mathcal{F}$. This finishes the proof Theorem 3.5.

To face dense-lineability, the following lemmas will be invoked. The first one might be well known, but since we have not been able to find an exact reference, we provide a proof. The content of the second one is taken from [2, Theorem 7.3.1].

Lemma 5.1. (a) *The set*

$$\mathcal{D} := \big\{ \mathbf{f} = (f_k)_k \subset C_c(X) : \text{ there exists } k_0 = k_0(\mathbf{f}) \in \mathbb{N} \text{ such that } f_k = 0 \text{ for all } k > k_0 \big\}$$

is a dense subset of Z.
(b) *If* X *is second-countable and every open subset of* X *is* σ*-compact, then* Z *is separable.*

Proof. (a) First of all, every $h \in C_c(X)$ belongs to $C_0(X)$, and is integrable, because $\|h\|_1 \leq \|h\|_\infty \cdot \mu(K)$, where K is the support of h, which is compact. Then, $\mu(K) < +\infty$, whence $\|h\|_1 < +\infty$. This implies that $\mathcal{D} \subset Z$.

As for the density, fix an $\varepsilon > 0$ and a vector $\mathbf{f} = (f_k)_k \in Z$. Then, $\|f_k\|_1 + \|f_k\|_\infty \to 0$ as $k \to \infty$. Take $k_0 \in \mathbb{N}$, such that $\|f_k\|_1 < \frac{\varepsilon}{3}$ and $\|f_k\|_\infty < \frac{\varepsilon}{3}$ for all $k > k_0$. Define $g_k := 0$ for all $k > k_0$. Trivially:

$$\|f_k - g_k\|_1 < \frac{\varepsilon}{3} \text{ and } \|f_k - g_k\|_\infty < \frac{\varepsilon}{3} \text{ for all } k > k_0.$$

Fix $k \in \{1, 2, \ldots, k_0\}$. On one hand, since $h := f_k \in L^1(\mu)$, there is $A \in \mathcal{M}$ with $\mu(A) < +\infty$, such that $\int_{X\setminus A} |h| < \frac{\varepsilon}{4}$; and since $h \in C_0(X)$, there is a compact set L_1, such that $|h| < \min\{\frac{\varepsilon}{3}, 1\}$ outside L_1. On the other hand, the regularity of μ entails the existence of a compact set $L_2 \subset A$ with $\mu(A\setminus L_2) < \frac{\varepsilon}{4}$. Let us define $K := L_1 \cup L_2$, which is a compact set. Then, $|h| < \frac{\varepsilon}{3}, 1$ on $X\setminus K$ and $\mu(A\setminus K) < \frac{\varepsilon}{4}$. Moreover, as $X\setminus K \subset (X\setminus A) \cup (A\setminus K)$, we get:

$$\int_{X\setminus K} |h| \, \mathrm{d}\mu \leq \int_{X\setminus A} |h| \, \mathrm{d}\mu + \int_{A\setminus K} |h| \, \mathrm{d}\mu < \frac{\varepsilon}{4} + \mu(A\setminus K) \cdot 1 < \frac{\varepsilon}{2}.$$

Again by Urysohn's result used in the previous section (see [22, Theorem 2.12]), there exists a function $\varphi \in C_c(X)$, such that $0 \leq \varphi \leq 1$ on X and $\varphi = 1$ on K. Then, the function $g_k := \varphi \cdot h$ belongs to $C_c(X)$ and satisfies for $k \in \{1, \ldots, k_0\}$ the following:

- $\|f_k - g_k\|_\infty = \|h(1-\varphi)\|_\infty = \sup_{X\setminus K} |h| \leq \frac{\varepsilon}{3} < \frac{\varepsilon}{2}$, and
- $\|f_k - g_k\|_1 = \|h(1-\varphi)\|_1 = \int_{X\setminus K} |h| \, \mathrm{d}\mu < \frac{\varepsilon}{2}$.

It follows easily that the vector $\mathbf{g} := (g_k)_k$ belongs to $\mathcal{D}$ and satisfies $\|\mathbf{g} - \mathbf{f}\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$, which proves the density of $\mathcal{D}$.
(b) Note that in the proof of (a), we have in fact shown that $C_c(X)$ is dense in $C_0(X) \cap L^1(\mu)$ in the topology generated by the norms $\|\cdot\|_\infty$ and $\|\cdot\|_1$. Since $\mathcal{D}$ is dense in Z and the members of $\mathcal{D}$ are essentially N-tuples $(f_1, \ldots, f_N)$ of functions from $C_c(X)$ $(N \in \mathbb{N})$, it is enough to prove the existence of a countable set $\mathcal{C} \subset C_c(X)$, such that every member of $C_c(X)$ can be approximated (in both cited norms) by members of $\mathcal{C}$.

Fix a nonempty open subset $O \subset X$. From the assumption, O is σ-compact. However, it is also second-countable, because this property is inherited by every topological subspace. Then, there is a countable open basis $\{W_m : m \in \mathbb{N}\}$ for the restriction of the topology of X to O. By σ-compactness and local compactness (which is also inherited by O because O is open), there is a sequence $(U_k)_k$ of open sets in O with compact closures contained in O such that $O = \bigcup_{k\in\mathbb{N}} U_k$ and $\overline{U_n} \subset U_{n+1}$ for all $n \in \mathbb{N}$ (see, e.g., [16, pp. 325–326]). From this, it follows easily that the countable collection $(G_n)_{n\in\mathbb{N}}$ of all nonempty intersections of the form $W_m \cap U_k$ is still an open basis for the topology of O and its members have compact closures contained in O.

Now, local compactness implies the existence, for each $n \in \mathbb{N}$, of an open subset V_n with compact closure, such that $G_n \subset \overline{G_n} \subset V_n \subset \overline{V_n} \subset O$ (see Remark 4 in Sect. 7 below). Note that every $X\setminus V_n$ is a closed subset of X that is not empty because X is not compact. Since X is Hausdorff, locally compact and second-countable, it is metrizable (see, e.g., [16, p. 342]). Fix a metric d generating the topology of X. For every $n \in \mathbb{N}$, define the function $\varphi_n : X \to \mathbb{R}$ by:

$$\varphi_n(x) = \frac{d(x, X\setminus V_n)}{d(x, \overline{G_n}) + d(x, X\setminus V_n)}.$$

This function is well defined, because both sets $\overline{G_n}$ and $X \setminus V_n$ are closed and have empty intersection. Trivially, $\varphi_n \in C_c(X)$ and $0 \leq \varphi_n \leq 1$. However, by considering its restriction to O, we also get $\varphi_n \in C_c(O)$, because its support is contained in $\overline{V_n}$, which is a compact subset of O. It is important the fact that $\varphi_n(x) = 1$ *if and only if* $x \in \overline{G_n}$. The family Φ of the restrictions of the φ_n's $(n \in \mathbb{N})$ to O enjoys the following properties:

- It is nonvanishing; that is, given $x_0 \in O$, there is $\varphi \in \Phi$ with $\varphi(x_0) \neq 0$. This is evident, because there exists $n \in \mathbb{N}$, such that $x_0 \in G_n$, so $x_0 \in \overline{G_n}$. Then, $\varphi_n(x_0) = 1 \neq 0$.
- It is separating; that is, given distinct points $x, y \in O$, there is $\varphi \in \Phi$ with $\varphi(x) \neq \varphi(y)$. Indeed, there are open sets G, $S \subset O$, such that $x \in G$, $y \in S$ and $G \cap S = \varnothing$ (Hausdorff property is inherited by any subspace). From local compactness, one derives the existence of open sets U, V with compact closures, such that $x \in U \subset \overline{U} \subset G$ and $y \in V \subset \overline{V} \subset S$. Since $(G_n)_n$ is an open basis, there exist $m, n \in \mathbb{N}$ with $x \in G_m \subset U$ and $y \in G_n \subset V$. However $\overline{G_m} \subset G$, $\overline{G_n} \subset S$ and $G \cap S = \varnothing$, so $x \in \overline{G_m} \not\ni y$. Then, the function $\varphi := \varphi_m \in \Phi$ satisfies $\varphi(x) = 1 \neq \varphi(y)$.

According to the Stone–Weierstrass theorem in its version for completely regular spaces (see [18, Theorem 16.5.7]; recall that any Hausdorff locally compact space is completely regular, see [24, p. 136]), the algebra $\mathcal{B}$ generated by Φ is dense in $C(O)$ under the compact-open topology. The members of $\mathcal{B}$ are finite linear combinations, with coefficients in $\mathbb{R}$, of finite products of powers of elements of Φ. A simple argument invoking the continuity of the scalar multiplication on a topological vector space shows that the collection $\mathcal{C}_O$ of all finite linear combinations, with coefficients in $\mathbb{Q}$ of the above products is a (countable) dense subset of $C(O)$. Note that $\mathcal{C}_O \subset C_c(O)$. Again by σ-compactness and local compactness, there is a sequence $(O_k)_k$ of open sets in X with compact closures, such that $O = \bigcup_{k \in \mathbb{N}} O_k$ and $\overline{O_n} \subset O_{n+1}$ for all $n \in \mathbb{N}$. Let us define:

$$\mathcal{C} := \bigcup_{n \in \mathbb{N}} \mathcal{C}_{O_n}.$$

Then, $\mathcal{C}$ is a countable subset of $C_c(X)$.

Fix $f \in C_c(X)$ and $\varepsilon > 0$. Then, the support of f is contained in some O_m. Note that $\mu(O_m) < +\infty$, because $\overline{O_m}$ is compact. Since $f \in C(O_m)$, the above proved denseness yields the existence of $\varphi \in \mathcal{C}_{O_m}$ with $|f(x) - \varphi(x)| < \frac{\varepsilon}{1+\mu(O_m)}$ for all $x \in O_m$. Since f and φ vanish outside O_m, we get:

$$\|f - \varphi\|_\infty < \varepsilon \text{ and } \|f - \varphi\|_1 = \int_{O_m} |f - \varphi| \, \mathrm{d}\mu \leq \frac{\varepsilon \cdot \mu(O_m)}{1 + \mu(O_m)} < \varepsilon,$$

as required. □

Lemma 5.2. *Assume that E is a metrizable topological vector space. Suppose that A and B are subsets of E satisfying the following:*

(i) *A is maximal-lineable,*

(ii) B *is dense-lineable,*
(i) $A + B \subset A$, *and*
(iv) $A \cap B = \varnothing$.

Then, A *is maximal dense-lineable in* E.

Under the assumptions of Theorem 3.6 and thanks to Lemma 5.1(b), Z is a separable infinite-dimensional Banach space. Then, a standard application of Baire's category theorem yields that $\dim(Z) = \mathfrak{c}$. Recall that we have denoted $\mathcal{A} := Z \setminus \mathcal{F}$ and that $\mathcal{A}$ is a vector space. We have already proved that $\mathcal{F}$ is spaceable, which, together with a new application of Baire's theorem, gives that $\mathcal{F}$ is maximal-lineable. On one hand, $\mathcal{D}$ is dense in Z by Lemma 5.1(a). However, $\mathcal{D}$ itself is a vector space, so it is dense-lineable. As $\mathcal{D} \subset \mathcal{A}$, we get $\mathcal{D} \cap Z = \varnothing$ and $\mathcal{F} + \mathcal{D} \subset \mathcal{F} + \mathcal{A} \subset \mathcal{F}$. Therefore, we can apply Lemma 5.2 with $E := Z$, $A := \mathcal{F}$ and $B := \mathcal{D}$. This finishes the proof of Theorem 3.6.

6. Algebrability of Unbounded Convergence: Proof of Theorem 3.7

Our next goal is to prove Theorem 3.7. Consider the constants α, β furnished by condition (C). By following a procedure similar to the one given in the proof of Theorem 3.4 (see Sect. 4), and using (C), we can inductively produce a sequence $(K_n)_n$ of compact sets as well as a sequence $(V_n)_n$ of mutually disjoint, relatively compact, open sets, and a sequence $\varphi_n \in C_c(X)$, such that, for all $n \in \mathbb{N}$, we have:

$$K_n \subset V_n,\ \alpha < \mu(K_n) < \beta,\ \mu(V_n) < 2\mu(K_n) < 2\beta,\ 0 \le \varphi_n \le 1,$$
$$\varphi_n(x) = 1 \text{ for all } x \in K_n, \text{ and } \varphi_n(x) = 0 \text{ for all } x \in X \setminus V_n.$$

This time, the existence of K_n in the nth step is guaranteed by (C) and the fact that the open set $X \setminus \overline{V_1 \cup \cdots \cup V_{n-1}}$ (defined as X if $n = 1$) has infinite measure, so that K_n is extracted by regularity from a measurable set $M \subset X \setminus \overline{V_1 \cup \cdots \cup V_{n-1}}$ satisfying $\alpha < \mu(M) < \beta$.

Take a linearly $\mathbb{Q}$-independent set $H \subset (0, +\infty)$ with $\mathrm{card}(H) = \mathfrak{c}$. For each $t \in H$, define the function sequence $\mathbf{f}^t = (f_{t,n})_n$ by:

$$f_{t,n} = \frac{1}{(\log(n+1))^t \cdot \mu(K_n)} \cdot \varphi_n.$$

As in Sect. 4, it is easy to see that $\mathbf{f}^t \in Z$ (the facts $\mu(K_n) > \alpha$, $\mu(V_n) < 2\beta$ are crucial). Now, we denote by $\mathcal{B}$ the linear algebra generated by the family $\{\mathbf{f}^t : t \in H\}$. We are going to show that $\mathcal{B}$ is freely generated by $\{\mathbf{f}^t : t \in H\}$ and is contained in $\mathcal{F} \cup \{0\}$.

With this aim, fix $N \in \mathbb{N}$, a nonzero polynomial P of N real variables without constant term and pairwise distinct numbers $t_1, \ldots, t_N \in H$. It suffices to prove that $\mathbf{g} = (g_n)_n := P(\mathbf{f}^{t_1}, \ldots, \mathbf{f}^{t_N}) \in \mathcal{F}$. Note first that $\mathbf{g} \in Z$, because Z is an algebra (under the coordinatewise product). Since

$P \neq 0$, there are a nonempty finite set $F \subset \mathbb{N}_0^N \backslash \{(0,0,\ldots,0)\}$ and constants $\alpha_{\mathbf{m}} \in \mathbb{R}\backslash\{0\}$ ($\mathbf{m} = (m_1,\ldots,m_N) \in F$), such that $P(x_1,\ldots,x_N) = \sum_{\mathbf{m}\in F} \alpha_{\mathbf{m}} x_1^{m_1} \cdots x_N^{m_N}$. Then, each component g_n has the expression:

$$g_n = \sum_{\mathbf{m}\in F} \alpha_{\mathbf{m}} \frac{\varphi_n^{\mathbf{m}\,\mathbf{t}}}{(\log(n+1))^{\mathbf{m}\,\mathbf{t}} \cdot \mu(K_n)^{|\mathbf{m}|}},$$

where $\mathbf{m}\,\mathbf{t} := m_1 t_1 + \cdots + m_N t_N$ and $|\mathbf{m}| := m_1 + \cdots + m_N$. Our unique task is to show that:

$$G := \sup_{n\geq 1} |g_n| = \sup_{n\geq 1} \left| \sum_{\mathbf{m}\in F} \alpha_{\mathbf{m}} \frac{\varphi_n^{\mathbf{m}\,\mathbf{t}}}{(\log(n+1))^{\mathbf{mt}} \cdot \mu(K_n)^{|\mathbf{m}|}} \right| \notin L^1(\mu).$$

Notice the numbers $\mathbf{m}\,\mathbf{t}$ ($\mathbf{m} \in F$) are pairwise distinct due to the $\mathbb{Q}$-independence of $t_1,\ldots,t_N$. Then, there is a unique $\mathbf{n} \in F$, such that $\mathbf{n}\,\mathbf{t} < \mathbf{m}\,\mathbf{t}$ for all $\mathbf{m} \in F\backslash\{\mathbf{n}\}$. Since the supports of the g_ns are mutually disjoint, we obtain:

$$\begin{aligned} G &= \sum_{n=1}^{\infty} |g_n| = \sum_{n=1}^{\infty} \left| \sum_{\mathbf{m}\in F} \alpha_{\mathbf{m}} \frac{\varphi_n^{\mathbf{m}\,\mathbf{t}}}{(\log(n+1))^{\mathbf{m}\,\mathbf{t}} \cdot \mu(K_n)^{|\mathbf{m}|}} \right| \\ &= \sum_{n=1}^{\infty} \left| \frac{\alpha_{\mathbf{n}}}{(\log(n+1))^{\mathbf{n}\,\mathbf{t}} \cdot \mu(K_n)^{|\mathbf{n}|}} \cdot \left(\varphi_n^{\mathbf{n}\,\mathbf{t}} + \Phi_n\right) \right|, \end{aligned}$$

where

$$\Phi_n := \sum_{\mathbf{m}\in F\backslash\{\mathbf{n}\}} \frac{\alpha_{\mathbf{m}}\, \varphi_n^{\mathbf{m}\,\mathbf{t}}\, \mu(K_n)^{|\mathbf{n}|-|\mathbf{m}|}}{\alpha_{\mathbf{n}}\, (\log(n+1))^{\mathbf{m}\,\mathbf{t}-\mathbf{n}\,\mathbf{t}}}.$$

Now, observe that for each $\mathbf{m} \in F\backslash\{\mathbf{n}\}$ the sequence $\left\{ \frac{\alpha_{\mathbf{m}}\, \varphi_n^{\mathbf{m}\,\mathbf{t}}\, \mu(K_n)^{|\mathbf{n}|-|\mathbf{m}|}}{\alpha_{\mathbf{n}}} \right\}_n$ is uniformly bounded on X, and that $\frac{1}{(\log(n+1))^r} \to 0$ as $n \to \infty$, for every $r > 0$. Since $F\backslash\{\mathbf{n}\}$ is finite, it follows that $\Phi_n \longrightarrow 0$ uniformly on X. Therefore, there exists $n_0 \in \mathbb{N}$, such that $|\Phi_n(x)| < \frac{1}{2}$ for all $n \in \mathbb{N}, n \geq n_0$ and all $x \in X$. Hence, by the reverse triangle inequality, we obtain:

$$G(x) \geq \sum_{n=n_0}^{\infty} \left| \frac{\alpha_{\mathbf{n}}}{(\log(n+1))^{\mathbf{n}\,\mathbf{t}} \cdot \mu(K_n)^{|\mathbf{n}|}} \right| \cdot \left| \varphi_n^{\mathbf{n}\,\mathbf{t}}(x) - \frac{1}{2} \right| \text{ for all } x \in X.$$

Consequently:

$$\begin{aligned} \int_X G(x)\,\mathrm{d}\mu &\geq \sum_{n=n_0}^{\infty} \int_{K_n} \left| \frac{\alpha_{\mathbf{n}}}{(\log(n+1))^{\mathbf{n}\,\mathbf{t}} \cdot \mu(K_n)^{|\mathbf{n}|}} \right| \cdot \left| \varphi_n^{\mathbf{n}\,\mathbf{t}}(x) - \frac{1}{2} \right| \mathrm{d}\mu \\ &= \sum_{n=n_0}^{\infty} \int_{K_n} \frac{|\alpha_{\mathbf{n}}|}{2(\log(n+1))^{\mathbf{n}\,\mathbf{t}} \cdot \mu(K_n)^{|\mathbf{n}|}}\,\mathrm{d}\mu \\ &= \sum_{n=n_0}^{\infty} \frac{|\alpha_{\mathbf{n}}|}{2(\log(n+1))^{\mathbf{n}\,\mathbf{t}} \cdot \mu(K_n)^{|\mathbf{n}|-1}} \\ &\geq \frac{|\alpha_{\mathbf{n}}|}{2 \cdot \beta^{|\mathbf{n}|-1}} \cdot \sum_{n=n_0}^{\infty} \frac{1}{(\log(n+1))^{\mathbf{n}\,\mathbf{t}}} = +\infty, \end{aligned}$$

where the facts $\mu(K_n) < \beta$, $\varphi_n|_{K_n} = 1$, and $\mathbf{n}\,\mathbf{t} > 0$ have been used. To summarize, $G \notin L^1(\mu)$, as required.

7. Final Remarks

1. As the most evident example, all preceding Theorems 3.4–3.7 can be applied to $X = \mathbb{R}^N$ ($N \in \mathbb{N}$) (or to a rectangle $X = I_1 \times \cdots \times I_N$, with the I_j's intervals of $\mathbb{R}$, being unbounded at least one of them) and $\mu = m =$ the Lebesgue N-dimensional measure. Indeed, $\mathbb{R}^N$ is a second-countable locally compact Hausdorff space all of whose open subsets are sigma-compact, and m is a regular Borel non-finite measure that is finite on compacta and satisfies condition (C) in Theorem 3.7 (any pair $0 < \alpha < \beta < +\infty$ works).
2. In fact, we can formulate a more general situation in which condition (C) is satisfied; namely, (C) is fulfilled by a nonatomic Borel measure ν satisfying the assumptions in Sect. 3. Recall that a measure ν defined on a measurable space (Ω, Σ) is *nonatomic* if Ω lacks atoms, and a set $A \subset \Omega$ is called an atom if $\nu(A) > 0$ and, given $B \in \Sigma$, one has either $\nu(B) = 0$ or $\nu(A \backslash B) = 0$. The measure ν is called *semifinite* provided that $\nu(A) = \sup\{\nu(B) : B \in \Sigma, B \subset A \text{ and } \nu(B) < +\infty\}$. If μ is as in Sect. 3, then finiteness at compacta together with regularity implies semifiniteness. Now, it is known (see [19, Theorem 11.27]) that, if ν is semifinite and nonatomic, then $[0, \nu(A)] = \{\nu(B) : B \in \Sigma \text{ and } B \subset A\}$. This proves our claim, because, again, any pair $0 < \alpha < \beta < +\infty$ does the job.
3. Nevertheless, being nonatomic is not necessary for (C) to hold. For instance, if $X = \mathbb{N}$ under the discrete topology and μ is the cardinal measure on the set $\mathcal{M} = \mathcal{P}(\mathbb{N})$ of all parts of $\mathbb{N}$, then μ satisfies all axioms given in Sect. 3, including (C) (with $\alpha = 1/2$ and $\beta = 2$, say), and each singleton $\{m\}$ is an atom. However, the measure $\mu(A) := \sum_{n=1}^{\infty} n \cdot \text{card}(A \cap \{n\})$ satisfies all axioms given in Sect. 3 prior to Theorems 3.4–3.7 and each $\{m\}$ is an atom for it, but (C) is *not* fulfilled.
4. In the proof of Lemma 5.1(b), the following facts have been tacitly used. Assume that (X, τ) is a Hausdorff topological space and that $A \subset B \subset X$. Then, the closure of A with respect to the induced topology τ_B of τ in B, denoted $\overline{A}^B$, can be computed as $\overline{A}^B = \overline{A} \cap B$. Moreover, A is τ-compact if and only if it is τ_B-compact. It follows that if $\overline{A}^B$ is τ_B-compact, then $\overline{A} = \overline{A}^B$ (which implies $\overline{A} \subset B$): indeed, $\overline{A}^B$ is τ-compact, so τ-closed, because X is Hausdorff; then, $\overline{A}^B$ is a τ-closed set containing A, so $\overline{A}^B \supset \overline{A}$; but $\overline{A}^B = \overline{A} \cap B \subset \overline{A}$, which yields the identity.
5. We can also consider the size of a subset of a vector space from a measure-theoretical point of view. In this direction, Hunt, Sauer, and Yorke [17] coined in 1992 the following concept of prevalence. Let Z be a metrizable topological vector space over $\mathbb{R}$ or $\mathbb{C}$. A subset $A \subset Z$ is

called *prevalent* in Z provided that there exist a Borel set $S \subset Z$ and a Borel measure μ on Z satisfying the following conditions:

(i) $A \supset Z \backslash S$,

(ii) $\mu(S + v) = 0$ for every $v \in Z$,

(iii) $0 < \mu(K) < \infty$ for some compact subset $K \subset Z$.

In [17, p. 222], it is shown that if Z is infinite-dimensional, then the complement of a proper vector subspace is always prevalent. However, in Sect. 4, it is proved that $\mathcal{A} := (c_{0,1,\infty}(C_0, L^1) \backslash \mathcal{F}$ is a proper vector subspace of $c_{0,1,\infty}(C_0, L^1)$. Thus, we can conclude: *The set* $\mathcal{F}$ *is a prevalent subset of* $(c_{0,1,\infty}(C_0, L^1), \| \cdot \|)$.

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