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On the Exponential Diophantine Equation $(m^2 + m + 1)^x + m^y = (m + 1)^z$

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Abstract. Let $m \geq 1$ be a positive integer. We show that the exponential Diophantine equation $(m^2 + m + 1)^x + m^y = (m + 1)^z$ has no positive integer solutions other than $(x, y, z) = (1, 1, 2)$ when $m \notin \{1, 2, 3\}$.

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1. Introduction

Let u, v, w be relatively prime positive integers greater than one and assume that the exponential Diophantine equation

$$u^x + v^y = w^z \tag{1.1}$$

in positive integers x, y, z has a solution (x_0, y_0, z_0). Two famous conjectures related to uniqueness of this solution (x_0, y_0, z_0) are due to Jeśmanowicz and Terai with some restriction on (1.1). In 1956, Jeśmanowicz conjectured that if u, v and w are any Pythagorean triples, i.e., positive integers satisfying $u^2 + v^2 = w^2$, then the solution $(x_0, y_0, z_0) = (2, 2, 2)$ is the unique solution of (1.1) [5]. Another similar conjecture is proposed by Terai which states that if u, v, w, p, q, r are fixed positive integers satisfying $u^p + v^q = w^r$ with $u, v, w, p, q, r \geq 2$, then the Eq. (1.1) has unique positive integer solution $(x_0, y_0, z_0) = (p, q, r)$ [19,20]. Exceptional cases are listed explicitly in [24]. Although both conjectures are proved to be true in many special cases, see for example [1,3,4,6,8,10–13,18,21–23,25], they are still remain an unsolved problem yet. We refer to [9,17] for a detailed information on these two conjectures. In this note we study the exponential Diophantine equation

$$(m^2 + m + 1)^x + m^y = (m + 1)^z \tag{1.2}$$

where $m > 1$ is a positive integer, and we prove the following.

Theorem 1.1. *Let $m > 1$ be a positive integer. If $m > 3$ then the Eq. (1.2) has only the positive integer solution $(x, y, z) = (1, 1, 2)$. For $m = 2$ and $m = 3$*

the Eq. (1.1) *has exactly two solutions, namely* $(x,y,z)=(1,1,2),(2,5,4)$ *and* $(x,y,z)=(1,1,2),(1,5,4)$, *respectively.*

In the above theorem, we exclude the case $m=1$ just for preserving the exponent in the expression m^y. In fact, it is easy to see that the equation $3^x+1=2^z$ has only the positive integer solution $(x,z)=(1,2)$ by considering it modulo 8. For the next two values of m, the Eq. (1.2) turns into the equations $7^x+2^y=3^z$ and $13^x+3^y=4^z$, for which both of them have more than one solution [14,26]. So the aim of this study is to give an answer to the question whether or not the Eq. (1.2) has any positive integer solutions other than $(x,y,z)=(1,1,2)$ when $m>3$. The proof depends on elementary congruence considerations and some results on linear forms in two $m-$adic logarithms.

2. Proof of Theorem 1.1

Lemma 2.1. *Let* (x,y,z) *be a positive integer solution of the Eq.* (1.2). *The following conditions hold.*

1. y *is odd.*
2. *There exists an integer* t *such that* $|x-y|=(m+1)t$.

Proof. 1. By reducing Eq. (1.2) modulo $m+1$ we get that $1+(-1)^y\equiv 0 \pmod{(m+1)}$ which implies that y is odd since $m>1$.

2. If $x=y$ then we may take $t=0$. So assume that $|x-y|\geq 1$. It is clear from (1.2) that $z\geq 2$. So we have that

$$\begin{aligned}(m^2+m+1)^x+m^y&\equiv 0 \pmod{(m+1)^2}\\ (-m)^x+m^y&\equiv 0 \pmod{(m+1)^2}\\ m^{|x-y|}+(-1)^x&\equiv 0 \pmod{(m+1)^2}\\ ((m+1)-1)^{|x-y|}+(-1)^x&\equiv 0 \pmod{(m+1)^2}\\ (-1)^{|x-y|}+(-1)^{|x-y|-1}(m+1)|x-y|+(-1)^x&\equiv 0 \pmod{(m+1)^2}.\end{aligned}$$

Taking into account that y is odd we get more precisely

$$|x-y|\equiv 0 \pmod{(m+1)},$$

which means that $|x-y|=(m+1)t$ for some positive integer t.

□

Lemma 2.2. *If* $m\equiv 1 \pmod 4$ *then* $(x,y,z)=(1,1,2)$ *is the only solution of* (1.2).

Proof. If $z\leq 2$ then clearly $(x,y,z)=(1,1,2)$ is the only solution of (1.2). Assume that $z\geq 3$ and $m=4k+1$ for some positive integer k. If x is even, then $(m^2+m+1)^x\equiv 1 \pmod 8$ and $m^{y-1}\equiv 1$, so from (1.2) we get that $1+m\equiv 0 \pmod 8$ which implies $2k+1\equiv 0 \pmod 4$, a contradiction. So x

must be odd and hence $(m^2+m+1)^x \equiv m^2+m+1 \pmod 8$. Then again considering (1.2) modulo 8 we get that

$$\begin{aligned}(m+1)^2 &\equiv 0 \pmod 8\\ (4k+2)^2 &\equiv 0 \pmod 8\\ 4 &\equiv 0 \pmod 8\end{aligned}$$

which is a contradiction. Hence, $(x, y, z) = (1, 1, 2)$ is the only solution of (1.2) when $m \equiv 1 \pmod 4$. □

Lemma 2.3. *Let (x, y, z) be a positive integer solution of the Eq. (1.2). Then x and y are relatively prime integers. In particular, $x \neq y$ for $z > 2$.*

Proof. If $z \le 2$ then $x = y = 1$ and hence the result is clear. So assume that $z \ge 3$ and that there exists an odd prime p such that $x = x_1 p$ and $y = y_1 p$ for some positive integers x_1 and y_1 since y is odd by Lemma 2.1. Let

$$K = (m^2+m+1)^{x_1} + m^{y_1}, \quad L = \frac{(m^2+m+1)^{x_1 p} + m^{y_1 p}}{(m^2+m+1)^{x_1} + m^{y_1}}.$$

So Eq. (1.2) is of the form

$$KL = (m+1)^z \tag{2.1}$$

where $\gcd(K, L) = 1$ or p. Note that $K \equiv 0 \pmod{m+1}$. Hence if $\gcd(K, L) = 1$ then $L = 1$ which is clearly impossible for $p > 1$. Thus, $\gcd(K, L) = p$. Let $m + 1 = p^k q$ for some positive integer k such that $\gcd(p, q) = 1$. From (2.1) we have that either $K = p^{kz-1} q^z$, $L = p$ or $K = pq^z$, $L = p^{kz-1}$. For $p > 1$ it is easy to see that

$$p(m^2+m+1)^{x_1} < (m^2+m+1)^{x_1 p}$$

and

$$pm^{y_1} < m^{y_1 p}.$$

So the case $L = p$ leads to a contradiction. On the other hand it is known that $p^2 \nmid L$, see, for example, [15, P1.2], thus for the case $K = pq^z$, $L = p^{kz-1}$ we have the only possibility $kz - 1 = 1$ which is also a contradiction since $z \ge 3$. So, there do not exist such a prime p and hence x and y are relatively prime integers. □

Let $m' > 1$ be an integer and let $m' = p_1^{t_1} \dots p_k^{t_k}$ be the prime factorization of m' for distinct primes p_i. The proof of Theorem 1.1 mainly depends on a result due to Bugeaud [2] on linear forms in two m'−adic logarithms. Let x_1/y_1 and x_2/y_2 be two non-zero rational numbers with $x_1/y_1 \neq \pm 1$. In [2] Bugeaud provide an upper bound for the m'−adic valuation of

$$\Lambda = (x_1/y_1)^{b_1} - (x_2/y_2)^{b_2}$$

whenever $v_{p_i}(x_1/y_1) = v_{p_i}(x_2/y_2) = 0$ for all $1 \le i \le k$ where b_1 and b_2 are positive integers. Suppose that there exists a positive integer g which is coprime with $p_1, \dots, p_k$ such that for all prime p_i,

$$v_{p_i}\left(\left(\frac{x_1}{y_1}\right)^g - 1\right) \ge t_i, v_{p_i}\left(\left(\frac{x_2}{y_2}\right)^g - 1\right) \ge 1, 1 \le i \le k \tag{2.2}$$

and

$$v_2\left(\left(\frac{x_1}{y_1}\right)^g-1\right)\geq 2, v_2\left(\left(\frac{x_2}{y_2}\right)^g-1\right)\geq 2 \text{ if } 2\mid m'. \tag{2.3}$$

Theorem 2.4. ([[2], Theorem 2]) *Let $A_1>1, A_2>1$ be real numbers such that*

$$\log A_i\geq \max\{\log|x_i|, \log|y_i|, \log m'\},\quad i=1,2$$

and put

$$b'=\frac{b_1}{\log A_2}+\frac{b_2}{\log A_1}.$$

Under the hypotheses (2.2) and (2.3) assume that x_1/y_1 and x_2/y_2 are multiplicatively independent. If m', b_1 and b_2 are relatively prime then we have the upper estimate

$$v_{m'}(\Lambda)\leq \frac{53.6g}{(\log m')^4}\left(\max\{\log b'+\log\log m'+0.64, 4\log m'\}\right)^2\log A_1\log A_2.$$

Now we apply the above theorem to the Eq. (1.2) by considering the $(m+1)-$adic valuation.

Lemma 2.5. *Let $m>7$. If $m\equiv 3 \pmod 4$ or $2\mid m$ then the Eq. (1.2) has only the positive integer solution $(x,y,z)=(1,1,2)$.*

Proof. If $z\leq 2$ then then the assertion is trivially true. Assume that $z\geq 3$. Since y is odd, we rewrite the Eq. (1.2) as

$$(m+1)^z=(m^2+m+1)^x-(-m)^y$$

and consider the $(m+1)-$adic valuation of $(m^2+m+1)^x-(-m)^y$. Since $(m+1)\mid m^2+m$, $(m+1)\mid -m-1$, and also $4\mid m^2+m$, $4\mid(-m-1)$ if $m+1$ is even. So, by Lemma 2.3, the hypotheses of Theorem 2.4 are satisfied for $g=1$ by taking $x_1:=m^2+m+1$ and $x_2:=-m$. Thus, from Theorem 2.4 we have the estimate

$$\begin{aligned} z\leq{}& \frac{53.6}{(\log(m+1))^4}\left(\max\{\log b'+\log\log(m+1)+0.64, 4\log(m+1)\}\right)^2\\ &\times\log(m^2+m+1)\log m \end{aligned}\tag{2.4}$$

where $b'=\dfrac{x}{\log m}+\dfrac{y}{\log m^2+m+1}$.

First assume that $\log b'+\log\log(m+1)+0.64>4\log(m+1)$. We will show that this is not possible. Put $M=\max\{x,y\}$. Then

$$M\left(\frac{1}{\log m}+\frac{1}{\log(m^2+m+1)}\right)\geq b'>\frac{(m+1)^4}{e^{0.64}\log(m+1)}\tag{2.5}$$

and it follows that $M>2205$ since $m\geq 8$. On the other hand, from the Eq. (1.2) we see that

$$x\frac{\log(m^2+m+1)}{\log(m+1)}<z\quad\text{and}\quad y\frac{\log m}{\log(m+1)}<z.$$

Thus $M\dfrac{\log m}{\log(m+1)} < z$. Combining this inequality and (2.4) together with (2.5) we get that

$$M \leq 53.6 \left(\log M + U(m) + 0.64\right)^2 V(m). \tag{2.6}$$

where

$$U(m) = \log\left(\frac{\log(m+1)}{\log m} + \frac{\log(m+1)}{\log(m^2+m+1)}\right)$$

and

$$V(m) = \frac{\log(m^2+m+1)}{\left(\log(m+1)\right)^3}.$$

For $m \geq 8$, both $U(m)$ and $V(m)$ are decreasing and $U(m) \leq U(8) < 0.46$, $V(m) \leq V(8) < 0.41$. Thus from (2.6) we get

$$M < 22\left(\log M + 1.1\right)^2,$$

which implies $M < 1576$, a contradiction. So $\log b' + \log\log(m+1) + 0.64 \leq 4\log(m+1)$. In this case from (2.4) we have that

$$z < 53.6 \times 16 \times W(m),$$

where $W(m) = \dfrac{\log m \log(m^2+m+1)}{(\log(m+1))^2}$. In the above one can see that $W(m) < 2$ for all positive m, and hence we get that $z < 1716$. Therefore, y is also bounded as $0.95y < y\dfrac{\log m}{\log(m+1)} < z < 1716$ and hence $y < 1807$ for $m \geq 8$. Similarly, from the inequality

$$1.96x < x\frac{\log m^2+m+1}{\log(m+1)} < z < 1716,$$

we get $x < 876$ for $m \geq 8$. Thus all x, y and z are bounded. Moreover, from Lemma 2.1 m is also bounded with $m+1 < M < 1807$. As a final step we checked with a short computer program in Maple that the equation (1.2) has no solution other than $(x, y, z) = (1, 1, 2)$ with these restrictions and those of Lemma 2.1 when m is in the range $8 \leq m \leq 1807$. This completes the proof. □

Proof of Theorem 1.1. From Lemmas 2.2 and 2.5 it remains to check the Eq. (1.2) only for $m \in \{2, 3, 4, 6, 7\}$. The results for the equations $7^x + 2^y = 3^z$, $13^x + 3^y = 4^z$, and $57^x + 7^y = 8^z$ which corresponds to the case $m = 2$, $m = 3$ and $m = 7$ in the Eq. (1.2) have already been established by a number of authors, at least [14, 26] and [16, Theorem 6] respectively. For $m = 4$, the equation (1.2) turns into the equation $21^x + 4^y = 5^z$. If x is even then by [7] this equation has no solution in positive integers whereas if x is odd then by [16, Lemma 6] it has only one solution, namely $(x, y, z) = (1, 1, 2)$. Finally we consider the equation $43^x + 6^y = 7^z$ for $m = 6$. If x is odd then the equation has only one solution $(x, y, z) = (1, 1, 2)$ by [16, Lemma 6]. Suppose that x is even, say $x = 2X$. If $y > 1$ then from the congruence $1 \equiv 7^z \pmod 8$ we see that z is also even, say $z = 2Z$. Thus we write

$$2^y 3^y = (7^Z - 43^X)(7^Z + 43^Z).$$

Note that only one of the factors in the right hand side is divisible by 4 and $3 \nmid 7^Z + 43^Z$. So we have two possibilities

$$7^Z - 43^X = 2^{y-1}3^y$$
$$7^Z + 43^Z = 2$$

or

$$7^Z - 43^X = 2 \cdot 3^y$$
$$7^Z + 43^Z = 2^{y-1}$$

Clearly the first one is impossible. From the second one, we get that $7^Z = 2^{y-2} + 3^y$. Reducing this equation modulo 3, we find that y is even, a contradiction. Therefore, we conclude that $y = 1$. Assume that x is even for otherwise the equation $43^x + 6 = 7^z$ has only one solution $(x, y, z) = (1, 1, 2)$ from [16, Lemma 6]. Let $x = 2X$. By reducing modulo 4 we see that z is odd, but it is easy to see that the equation $43^{2X} + 6 = 7^z$ has no solution in positive integers when z is odd by considering it modulo 43. This completes the proof. $\square$

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