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Golden GCR-Lightlike Submanifolds of Golden Semi-Riemannian Manifolds

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Abstract. We introduce golden GCR-lightlike submanifolds of golden semi-Riemannian manifolds. We investigate several properties of such submanifolds. Moreover, we find some necessary and sufficient conditions for minimal golden GCR-lightlike submanifolds of golden semi-Riemannian manifolds.

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1. Introduction

It is well known that in case the induced metric on the submanifold of semi-Riemannian manifold is degenerate, the study becomes more different from the study of non-degenerate submanifolds. The primary difference between the lightlike submanifolds and non-degenerate submanifolds arises due to the fact in the first case that the normal vector bundle has non-trivial intersection with the tangent vector bundle and moreover in a lightlike hypersurface the normal vector bundle is contained in the tangent vector bundle. The lightlike submanifolds were introduced by Duggal–Bejancu [4] . Later, they were developed by Duggal and Şahin [9].

Duggal and Bejancu [4] introduced CR-lightlike submanifolds of indefinite Kaehler manifolds. But CR-lightlike submanifolds exclude the complex and totally real submanifolds as subcases. Then, Duggal and Şahin introduced screen Cauchy–Riemann (SCR)-lightlike submanifolds of indefinite Kaehler manifolds [6]. But there is no inclusion relation between CR and SCR submanifolds, so Duggal and Şahin introduced a new class called GCR-lightlike submanifolds of indefinite Kaehler manifolds which is an umbrella for all these types of submanifolds [7] and then of indefinite Sasakian manifolds in [8]. These types of submanifolds have been studied in various manifolds by many authors [13,14,16,17].

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Manifolds which are determined differential-geometric structures have an important role in differential geometry. Really, almost complex manifolds and almost product manifolds and maps between such manifolds which are given by a (1,1)-tensor field such that the square of $\tilde{P}$ satisfies certain conditions, like $\tilde{P}^2 = -I$ or $\tilde{P}^2 = I$, have been studied extensively by many authors. As a generalization of almost complex and almost contact structures, Yano introduced the notion of an $f-$structure which is a (1,1)-tensor field of constant rank on $\tilde{M}$ and satisfies the equality $f^3 + f = 0$ [22]. It has been generalized by Goldberg and Yano in [11]. As a particular case of polynomial structure, Crasmareanu and Hreţcanu studied the golden structures and defined golden Riemannian manifold in [3]. They also investigated the geometry of the golden structure on a manifold by using the corresponding almost product structure. In [21], Şahin and Akyol introduced golden maps between golden Riemannian manifolds and showed that such maps are harmonic maps. Gök, Keleş and Kılıç studied some characterizations for any submanifold of a golden Riemannian manifold to be semi-invariant in terms of canonical structures on the submanifold, induced by the golden structure of the ambient manifold [12]. Poyraz and Yaşar introduced lightlike submanifolds of golden semi-Riemannian manifolds [19]. Acet intoduced lightlike hypersurfaces of metallic semi-Riemannian manifolds [1]. Erdoğan studied transversal lightlike submanifolds of metallic semi-Riemannian manifolds [10].

In this paper, we introduce golden GCR-lightlike submanifolds of golden semi-Riemannian manifolds. We find some equivalent conditions for integrability of distributions and investigate the geometry of the leaves of distributions. Moreover, we find some necessary and sufficient conditions for minimal golden GCR-lightlike submanifolds of golden semi-Riemannian manifolds.

2. Preliminaries

Let $\tilde{M}$ be a $C^\infty-$differentiable manifold. If a tensor field $\tilde{P}$ of type $(1,1)$ satisfies the following equation

$$\tilde{P}^2 = \tilde{P} + I, \tag{2.1}$$

then $\tilde{P}$ is named a golden structure on $\tilde{M}$, where I is the identity transformation [15].

Let $(\tilde{M}, \tilde{g})$ be a semi-Riemannian manifold and $\tilde{P}$ be a golden structure on $\tilde{M}$. If $\tilde{P}$ holds the following equation

$$\tilde{g}(\tilde{P}X, Y) = \tilde{g}(X, \tilde{P}Y), \tag{2.2}$$

then $(\tilde{M}, \tilde{g}, \tilde{P})$ is named a golden semi-Riemannian manifold [20].

If $\tilde{P}$ is a golden structure, then the Eq. (2.2) is equivalent to

$$\tilde{g}(\tilde{P}X, \tilde{P}Y) = \tilde{g}(\tilde{P}X, Y) + \tilde{g}(X, Y), \tag{2.3}$$

for any $X, Y \in \Gamma(T\tilde{M})$.

Let $(\tilde{M}, \tilde{g})$ be a real $(m+n)-$dimensional semi-Riemannian manifold of constant index q, such that $m, n \geq 1$, $1 \leq q \leq m+n-1$ and (M, g) be an $m-$dimensional submanifold of $\tilde{M}$, where g is the induced metric of $\tilde{g}$ on

M. If $\tilde{g}$ is degenerate on the tangent bundle TM of M, then M is called a lightlike submanifold of $\tilde{M}$. For a degenerate metric g on M,

$$TM^{\perp} = \cup \left\{ u \in T_x\tilde{M} : \tilde{g}(u,v) = 0, \forall v \in T_xM, x \in M \right\} \tag{2.4}$$

is a degenerate $n-$dimensional subspace of $T_x\tilde{M}$. Thus, both T_xM and $T_xM^{\perp}$ are degenerate orthogonal subspaces, but no longer complementary. In this case, there exists a subspace $Rad(T_xM) = T_xM \cap T_xM^{\perp}$ which is known as radical (null) space. If the mapping $Rad(TM) : x \in M \longrightarrow Rad(T_xM)$ defines a smooth distribution, called radical distribution on M of rank $r > 0$, then the submanifold M of $\tilde{M}$ is called an $r-$lightlike submanifold.

Let $S\,(TM)$ be a screen distribution which is a semi-Riemannian complementary distribution of $Rad(TM)$ in TM. This means that

$$TM = S\,(TM) \perp Rad\,(TM) \tag{2.5}$$

and $S\left(TM^{\perp}\right)$ is a complementary vector subbundle to $Rad(TM)$ in $TM^{\perp}$. Let $tr\,(TM)$ and $ltr\,(TM)$ be complementary (but not orthogonal) vector bundles to TM in $T\tilde{M}_{|_M}$ and $Rad(TM)$ in $S\left(TM^{\perp}\right)^{\perp}$, respectively. Then, we have

$$tr\,(TM) = ltr\,(TM) \perp S(TM^{\perp}), \tag{2.6}$$

$$\begin{aligned} T\tilde{M}\mid_M &= TM \oplus tr\,(TM) \\ &= \{Rad(TM) \oplus ltr\,(TM)\} \perp S\,(TM) \perp S(TM^{\perp})\,. \end{aligned} \tag{2.7}$$

Theorem 1. *[4] Let $(M, g, S(TM), S\left(TM^{\perp}\right))$ be an $r-$lightlike submanifold of a semi-Riemannian manifold $(\tilde{M}, \tilde{g})$. Suppose U is a coordinate neighbourhood of M and $\{\xi_i\}$, $i \in \{1,..,r\}$ is a basis of $\Gamma(Rad\left(TM_{|_U}\right))$. Then, there exist a complementary vector subbundle $ltr(TM)$ of $Rad(TM)$ in $S\left(TM^{\perp}\right)^{\perp}_{|_U}$ and a basis $\{N_i\}$, $i \in \{1,..,r\}$ of $\Gamma(ltr\,(TM)_{|_U})$ such that*

$$\tilde{g}(N_i, \xi_j) = \delta_{ij}, \quad \tilde{g}(N_i, N_j) = 0 \tag{2.8}$$

for any $i, j \in \{1,..,r\}$.

We say that a submanifold $(M, g, S(TM), S\left(TM^{\perp}\right))$ of $\tilde{M}$ is
Case 1: $r-$lightlike if $r < min\,\{m, n\}$,
Case 2: coisotropic if $r = n < m$; $S\left(TM^{\perp}\right) = \{0\}$,
Case 3: isotropic if $r = m < n$; $S\,(TM) = \{0\}$,
Case 4: totally lightlike if $r = m = n$; $S\,(TM) = \{0\} = S\left(TM^{\perp}\right)$.

Let $\tilde{\nabla}$ be the Levi–Civita connection on $\tilde{M}$. Then, using (2.7), the Gauss and Weingarten formulas are given by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.9}$$

$$\tilde{\nabla}_X U = -A_U X + \nabla^t_X U, \tag{2.10}$$

for any $X, Y \in \Gamma(TM)$ and $U \in \Gamma(tr\,(TM))$, where $\{\nabla_X Y, A_U X\}$ and $\{h(X, Y), \nabla^t_X U\}$ belong to $\Gamma(TM)$ and $\Gamma(tr\,(TM))$, respectively. ∇ and ∇^t are linear connections on M and on the vector bundle $tr\,(TM)$, respectively.

According to (2.6), considering the projection morphisms L and S of $tr\,(TM)$ on $ltr\,(TM)$ and $S\left(TM^{\perp}\right)$, respectively, (2.9) and (2.10) become

$$\tilde{\nabla}_X Y = \nabla_X Y + h^l(X,Y) + h^s(X,Y), \tag{2.11}$$

$$\tilde{\nabla}_X N = -A_N X + \nabla^l_X N + D^s(X,N), \tag{2.12}$$

$$\tilde{\nabla}_X W = -A_W X + \nabla^s_X W + D^l(X,W), \tag{2.13}$$

for any $X,Y \in \Gamma(TM)$, $N \in \Gamma(ltr\,(TM))$ and $W \in \Gamma(S\left(TM^{\perp}\right))$, where $h^l(X,Y) = Lh(X,Y)$, $h^s(X,Y) = Sh(X,Y)$, $\nabla_X Y, A_N X, A_W X \in \Gamma(TM)$, $\nabla^l_X N, D^l(X,W) \in \Gamma(ltr\,(TM))$ and $\nabla^s_X W, D^s(X,N) \in \Gamma(S\left(TM^{\perp}\right))$. Then, by using (2.11)–(2.13) and taking into account that $\tilde{\nabla}$ is a metric connection we obtain

$$g(h^s(X,Y),W) + g(Y,D^l(X,W)) = g(A_W X,Y), \tag{2.14}$$

$$g(D^s(X,N),W) = g(A_W X,N). \tag{2.15}$$

Let Q be a projection of TM on $S(TM)$. Then, using (2.5) we can write

$$\nabla_X QY = \nabla^*_X QY + h^*(X,QY), \tag{2.16}$$

$$\nabla_X \xi = -A^*_\xi X + \nabla^{*t}_X \xi, \tag{2.17}$$

for any $X,Y \in \Gamma(TM)$ and $\xi \in \Gamma(Rad(TM))$, where $\{\nabla^*_X QY, A^*_\xi X\}$ and $\{h^*(X,QY), \nabla^{*t}_X \xi\}$ belong to $\Gamma(S\,(TM))$ and $\Gamma(Rad\,(TM))$, respectively.

Using the equations given above, we derive

$$g(h^l(X,QY),\xi) = g(A^*_\xi X, QY), \tag{2.18}$$

$$g(h^*(X,QY),N) = g(A_N X, QY), \tag{2.19}$$

$$g(h^l(X,\xi),\xi) = 0, \quad A^*_\xi \xi = 0. \tag{2.20}$$

Generally, the induced connection ∇ on M is not metric connection. Since $\tilde{\nabla}$ is a metric connection, from (2.11), we obtain

$$(\nabla_X g)(Y,Z) = \tilde{g}(h^l(X,Y),Z) + \tilde{g}(h^l(X,Z),Y). \tag{2.21}$$

But, ∇^* is a metric connection on $S(TM)$.

Definition 2. A lightlike submanifold (M,g) of a semi-Riemannian manifold $(\tilde{M},\tilde{g})$ is said to be an irrotational submanifold if $\tilde{\nabla}_X \xi \in \Gamma(TM)$ for any $X \in \Gamma(TM)$ and $\xi \in \Gamma(Rad(TM))$ [18]. Thus, M is an irrotational lightlike submanifold iff $h^l(X,\xi) = 0$, $h^s(X,\xi) = 0$.

Theorem 3. *Let M be an r-lightlike submanifold of a semi-Riemannian manifold $\tilde{M}$. Then the induced connection ∇ is a metric connection iff $Rad(TM)$ is a parallel distribution with respect to ∇ [4].*

Definition 4. A lightlike submanifold (M,g) of a semi-Riemannian manifold $(\tilde{M},\tilde{g})$, is said to be totally umbilical in $\tilde{M}$ if there is a smooth transversal vector field $H \in \Gamma(tr(TM))$ on M, called the transversal curvature vector field of M, such that

$$h(X,Y) = Hg(X,Y) \tag{2.22}$$

for any $X,Y \in \Gamma(TM)$. In case $H = 0$, M is called totally geodesic [5].

Using (2.11) and (2.22), it is easy to see that M is totally umbilical iff on each coordinate neighborhood U there exists smooth vector fields $H^l \in \Gamma(ltr\,(TM))$ and $H^s \in \Gamma(S\left(TM^{\perp}\right))$ such that

$$h^l(X,Y) = g(X,Y)H^l,\ h^s(X,Y) = g(X,Y)H^s \text{ and } D^l(X,W) = 0 \quad (2.23)$$

for any $X, Y \in \Gamma(TM)$ and $W \in \Gamma(S\left(TM^{\perp}\right))$.

3. Golden Generalized Cauchy–Riemann (GCR)-Lightlike Submanifolds

Definition 5. Let $(M, g, S(TM))$ be a real lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$. Then we say that M is a golden generalized Cauchy–Riemann (GCR)-lightlike submanifold if the following conditions are satisfied:

(A) There exist two subbundles D_1 and D_2 of $Rad(TM)$ such that

$$Rad(TM) = D_1 \oplus D_2, \quad \tilde{P}(D_1) = D_1,\ \tilde{P}(D_2) \subset S(TM). \quad (3.1)$$

(B) There exist two subbundles D_0 and D' of $S\,(TM)$ such that

$$S(TM) = \{\tilde{P}D_2 \oplus D'\} \perp D_0,\ \tilde{P}(D_0) = D_0,\ \tilde{P}(L_1 \perp L_2) = D^{'}, \quad (3.2)$$

where D_0 is a non-degenerate distribution on M, L_1 and L_2 are vector subbundles of $ltr\,(TM)$ and $S\left(TM^{\perp}\right)$, respectively.

Let $\tilde{P}(L_1) = M_1$ and $\tilde{P}(L_2) = M_2$. Then we have

$$D^{'} = \tilde{P}(L_1) \perp \tilde{P}(L_2) = M_1 \perp M_2. \quad (3.3)$$

Thus, we have the following decomposition:

$$TM = D \oplus D',\ D = Rad(TM) \perp D_0 \perp \tilde{P}(D_2). \quad (3.4)$$

We say that M is a proper golden GCR-lightlike submanifold of a golden semi-Riemannian manifold if $D_0 \neq \{0\}$, $D_1 \neq \{0\}$, $D_2 \neq \{0\}$ and $L_2 \neq \{0\}$.

Definition 6. Let M be a lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$. We say that M is a semi-invariant lightlike submanifold of $\tilde{M}$, if the following conditions are satisfied:

1) $\tilde{P}(Rad(TM))$ is a distribution on $S(TM)$.
2) $\tilde{P}(L_1 \perp L_2)$ is a distribution on $S(TM)$, where $L_1 = ltr(TM)$ and L_2 is a vector subbundle of $S(TM^{\perp})$.

Proposition 7. *Let M be a golden GCR-lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$. Then, M is a semi-invariant lightlike submanifold iff $D_1 = \{0\}$.*

Proof. Let M be a semi-invariant lightlike submanifold of a golden semi-Riemannian manifold $\tilde{M}$. Then $\tilde{P}(Rad(TM))$ is a distribution on M. Thus, we obtain $D_2 = Rad(TM)$ and $D_1 = \{0\}$. Then it follows that $\tilde{P}(ltr(TM)) \subset S(TM)$. Conversely, suppose that M is a golden GCR-lightlike submanifold such that $D_1 = \{0\}$. Then, we have $D_2 = Rad(TM)$. Thus, M is a semi-invariant lightlike submanifold, which completes the proof. □

Proposition 8. *There exist no coisotropic, isotropic or totally lightlike proper golden GCR-lightlike submanifolds M of a golden semi-Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$.*

Proof. If M is isotropic or totally lightlike, then $S(TM) = \{0\}$ and if M is coisotropic then $S(TM^{\perp}) = \{0\}$. Hence, conditions (A) and (B) of Definition 5 are not satisfied. □

Let M be a golden GCR-lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$. Thus, for any $X \in \Gamma(TM)$, we derive

$$\tilde{P}X = PX + wX, \tag{3.5}$$

where PX and wX are tangential and transversal parts of $\tilde{P}X$.

For $V \in \Gamma(tr(TM))$, we write

$$\tilde{P}V = BV + CV, \tag{3.6}$$

where BV and CV are tangential and transversal parts of $\tilde{P}V$.

Lemma 9. *Let M be a golden GCR-lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$. Then, one has*

$$P^2X = PX + X - BwX, \tag{3.7}$$

$$wPX = wX - CwX, \tag{3.8}$$

$$PBV = BV - BCV, \tag{3.9}$$

$$C^2V = CV + V - wBV, \tag{3.10}$$

$$g(PX, Y) - g(X, PY) = g(X, wY) - g(wX, Y), \tag{3.11}$$

$$\begin{aligned} g(PX, PY) &= g(PX, Y) + g(X, Y) + g(wX, Y) - g(PX, wY) \\ &\quad - g(wX, PY) - g(wX, wY) \end{aligned} \tag{3.12}$$

for any $X, Y \in \Gamma(TM)$.

Proof. Applying $\tilde{P}$ to (3.5), using (2.1) and taking tangential and transversal parts of the resulting equation, we obtain (3.7) and (3.8). Similarly, applying $\tilde{P}$ in (3.6), using (2.1), we get (3.9) and (3.10). Using (2.2), (2.3) and (3.5), we obtain (3.11) and (3.12). □

Theorem 10. *Let M be a golden GCR-lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$. Then, P is a golden structure on D.*

Proof. In the definition of golden GCR-lightlike submanifold, we have $wX = 0$, for any $X \in \Gamma(D)$. From (3.7), we have $P^2X = PX + X$. Thus, P is golden structure on D. □

Example 11. Let $(\tilde{M} = \mathbb{R}^{12}_4, \tilde{g})$ be a 12-dimensional semi-Euclidean space with signature $(-,-,+,+,-,-,+,+,+,+,+,+)$ and $(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12})$ be the standard coordinate system of $\mathbb{R}^{12}_4$. If we define a mapping $\tilde{P}$ by

$$\begin{aligned} \tilde{P}(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}) &= (x_1 + x_2, x_1, x_3 + x_4, x_3, \\ & x_5 + x_6, x_5, x_7 + x_8, x_7, x_9 + x_{10}, x_9, x_{11} + x_{12}, x_{11}), \end{aligned}$$

then $\tilde{P}^2 = \tilde{P} + I$ and $\tilde{P}$ is a golden structure on $\mathbb{R}^{12}_4$. Let M be a submanifold of $\tilde{M}$ given by

$$\begin{aligned} &x_1 = u_2, x_2 = u_1, x_3 = u_2, x_4 = u_1, x_5 = u_4 - \frac{1}{2}u_5, \\ &x_6 = u_3, x_7 = u_4 + \frac{1}{2}u_5, x_8 = u_3, \\ &x_9 = -u_7 + u_8, x_{10} = -u_6, x_{11} = u_7 + u_8, x_{12} = u_6, \end{aligned}$$

where u_i, $1 \leq i \leq 8$, are real parameters. Thus, $TM = Span\{U_1, U_2, U_3, U_4, U_5, U_6, U_7, U_8\}$, where

$$\begin{aligned} U_1 &= \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_4}, U_2 = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3}, U_3 = \frac{\partial}{\partial x_6} + \frac{\partial}{\partial x_8}, \\ U_4 &= \frac{\partial}{\partial x_5} + \frac{\partial}{\partial x_7}, U_5 = \frac{1}{2}(-\frac{\partial}{\partial x_5} + \frac{\partial}{\partial x_7}), U_6 = -\frac{\partial}{\partial x_{10}} + \frac{\partial}{\partial x_{12}}, \\ U_7 &= -\frac{\partial}{\partial x_9} + \frac{\partial}{\partial x_{11}}, U_8 = \frac{\partial}{\partial x_9} + \frac{\partial}{\partial x_{11}}. \end{aligned}$$

Then, M is a $3-$lightlike submanifold with $Rad(TM) = Span\{U_1, U_2, U_3\}$ and $\tilde{P}U_1 = U_2$. Thus, $D_1 = Span\{U_1, U_2\}$. On the other hand, $\tilde{P}U_3 = U_4 \in \Gamma(S(TM))$ implies that $D_2 = Span\{U_3\}$. Moreover, $\tilde{P}U_6 = U_7$ thus $D_0 = Span\{U_6, U_7\}$. We can easily obtain

$$\begin{aligned} ltr(TM) =&Span\{N_1 = \frac{1}{2}(-\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_4}), N_2 = \frac{1}{2}(-\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3}), \\ N_3 =&\frac{1}{2}(-\frac{\partial}{\partial x_6} + \frac{\partial}{\partial x_8})\} \end{aligned}$$

and

$$S(TM^{\perp}) = Span\{W = \frac{\partial}{\partial x_{10}} + \frac{\partial}{\partial x_{12}}\}.$$

Moreover, $Span\{N_1, N_2\}$ is invariant with respect to $\tilde{P}$. Since $\tilde{P}N_3 = U_5$ and $\tilde{P}W = U_8$, then $L_1 = Span\{N_3\}$, $L_2 = Span\{W\}$, $M_1 = Span\{U_5\}$ and $M_2 = Span\{U_8\}$. Thus, M is a proper golden GCR-lightlike submanifold of $\tilde{M}$.

Theorem 12. *Let M be a golden GCR-lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$. Then the induced connection is a metric connection iff for any $X \in \Gamma(TM)$, the following hold*

$$P(-A^*_{\tilde{P}Y}X + \nabla^{*t}_X \tilde{P}Y) + A^*_{\tilde{P}Y}X - \nabla^{*t}_X \tilde{P}Y \in \Gamma(Rad(TM)),\ Y \in \Gamma(D_1), \tag{3.13}$$

$$\begin{aligned} &P(\nabla^*_X \tilde{P}Y + h^*(X, \tilde{P}Y)) - \nabla^*_X \tilde{P}Y \\ &\quad -h^*(X, \tilde{P}Y) \in \Gamma(Rad(TM)),\ Y \in \Gamma(D_2), \end{aligned} \tag{3.14}$$

$$Bh(X, \tilde{P}Y) = 0, \quad Y \in \Gamma(Rad(TM)). \tag{3.15}$$

Proof. Since $\tilde{P}$ is the golden structure of $\tilde{M}$, we have

$$\tilde{\nabla}_X Y = \tilde{P}\tilde{\nabla}_X \tilde{P}Y - \tilde{\nabla}_X \tilde{P}Y \tag{3.16}$$

for any $X \in \Gamma(TM)$ and $Y \in \Gamma(Rad(TM))$. Since $Rad(TM) = D_1 \oplus D_2$, using (2.9), (2.17), (3.5), (3.6) and taking the tangential part of the resulting equation, we obtain

$$\nabla_X Y = P(-A^*_{\tilde{P}Y}X + \nabla^{*t}_X \tilde{P}Y) + A^*_{\tilde{P}Y}X - \nabla^{*t}_X \tilde{P}Y + Bh(X, \tilde{P}Y) \quad (3.17)$$

for any $X \in \Gamma(TM)$ and $Y \in \Gamma(D_1)$. Thus from (3.17), we obtain $\nabla_X Y \in \Gamma(Rad(TM))$ iff

$$P(-A^*_{\tilde{P}Y}X + \nabla^{*t}_X \tilde{P}Y) + A^*_{\tilde{P}Y}X - \nabla^{*t}_X \tilde{P}Y \in \Gamma(Rad(TM)) \quad (3.18)$$

and

$$Bh(X, \tilde{P}Y) = 0 \quad (3.19)$$

for any $X \in \Gamma(TM)$ and $Y \in \Gamma(D_1)$. Using (2.9), (2.16), (3.5) and (3.6) we derive

$$\begin{aligned} \nabla_X Y = {} & P(\nabla^*_X \tilde{P}Y + h^*(X, \tilde{P}Y)) - \nabla^*_X \tilde{P}Y \\ & - h^*(X, \tilde{P}Y) + Bh(X, \tilde{P}Y) \end{aligned} \quad (3.20)$$

for any $X \in \Gamma(TM)$ and $Y \in \Gamma(D_2)$. From (3.20) we get $\nabla_X Y \in \Gamma(Rad(TM))$ iff

$$P(\nabla^*_X \tilde{P}Y + h^*(X, \tilde{P}Y)) - \nabla^*_X \tilde{P}Y - h^*(X, \tilde{P}Y) \in \Gamma(Rad(TM)) \quad (3.21)$$

and

$$Bh(X, \tilde{P}Y) = 0 \quad (3.22)$$

for any $X \in \Gamma(TM)$ and $Y \in \Gamma(D_2)$. Then considering Theorem 3, the proof follows from (3.18), (3.19), (3.21) and (3.22). □

Theorem 13. *Let M be a golden GCR-lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$. Then, the distribution D is integrable iff*

(i) $\tilde{g}(h^l(X, \tilde{P}Y), \xi) = \tilde{g}(h^l(Y, \tilde{P}X), \xi)$,
(ii) $\tilde{g}(h^s(X, \tilde{P}Y), W) = \tilde{g}(h^s(Y, \tilde{P}X), W)$, *for any $X, Y \in \Gamma(D)$, $\xi \in \Gamma(D_2)$ and $W \in \Gamma(L_2)$.*

Proof. Using the definition of GCR-lightlike submanifolds, D is integrable iff

$$g([X, Y], \tilde{P}\xi) = g([X, Y], \tilde{P}W) = 0$$

for any $X, Y \in \Gamma(D)$, $\xi \in \Gamma(D_2)$ and $W \in \Gamma(L_2)$. Then from (2.2) and (2.11), we obtain

$$\begin{aligned} g([X, Y], \tilde{P}\xi) &= \tilde{g}(\tilde{\nabla}_X Y - \tilde{\nabla}_Y X, \tilde{P}\xi) = \tilde{g}(\tilde{\nabla}_X \tilde{P}Y - \tilde{\nabla}_Y \tilde{P}X, \xi) \\ &= \tilde{g}(h^l(X, \tilde{P}Y) - h^l(Y, \tilde{P}X), \xi), \end{aligned} \quad (3.23)$$

$$\begin{aligned} g([X, Y], \tilde{P}W) &= \tilde{g}(\tilde{\nabla}_X Y - \tilde{\nabla}_Y X, \tilde{P}W) = \tilde{g}(\tilde{\nabla}_X \tilde{P}Y - \tilde{\nabla}_Y \tilde{P}X, W) \\ &= \tilde{g}(h^s(X, \tilde{P}Y) - h^s(Y, \tilde{P}X), W). \end{aligned} \quad (3.24)$$

From (3.23) and (3.24), we derive our theorem. □

Theorem 14. *Let M be a golden GCR-lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$. Then, the distribution D' is integrable iff*

(i) $g(A_{V_1}W, \tilde{P}X) = g(A_{V_2}Z, \tilde{P}X)$,
(ii) $\tilde{g}(h^*(Z, W) - h^*(W, Z), N') = g(A_{V_2}Z - A_{V_1}W, N')$,

(iii) $g(A_N Z, W) = g(A_N W, Z)$,
for any $Z, W \in \Gamma(D')$, $X \in \Gamma(D_o)$, $N' \in \Gamma(L_1)$, $V_1, V_2 \in \Gamma(L_1 \perp L_2)$ *and* $N \in \Gamma(ltr(TM))$.

Proof. Using the definition of GCR-lightlike submanifolds, D' is integrable iff

$$g([Z,W], X) = g([Z,W], \tilde{P}N') = \tilde{g}([Z,W], N) = 0$$

for any $Z, W \in \Gamma(D')$, $X \in \Gamma(D_o)$, $N' \in \Gamma(L_1)$ and $N \in \Gamma(ltr(TM))$. Choosing $Z, W \in \Gamma(D')$, there is a vector field $V_1, V_2 \in \Gamma(L_1 \perp L_2)$ such that $Z = \tilde{P}V_1$ and $W = \tilde{P}V_2$. Since $\tilde{\nabla}$ is a metric connection, from (2.2), (2.3), (2.10), (2.12) and (2.16) we obtain

$$\begin{aligned} g([Z,W], X) &= \tilde{g}(\tilde{\nabla}_Z W - \tilde{\nabla}_W Z, X) = \tilde{g}(\tilde{\nabla}_Z \tilde{P}V_2 - \tilde{\nabla}_W \tilde{P}V_1, X) \\ &= \tilde{g}(\tilde{\nabla}_Z V_2 - \tilde{\nabla}_W V_1, \tilde{P}X) = g(A_{V_1} W - A_{V_2} Z, \tilde{P}X), \end{aligned} \tag{3.25}$$

$$\begin{aligned} g([Z,W], \tilde{P}N') &= g(\tilde{\nabla}_Z W - \tilde{\nabla}_W Z, \tilde{P}N') = g(\tilde{\nabla}_Z W, \tilde{P}N') - g(\tilde{\nabla}_W Z, \tilde{P}N') \\ &= g(\tilde{\nabla}_Z \tilde{P}V_2, \tilde{P}N') - g(\tilde{\nabla}_W \tilde{P}V_1, \tilde{P}N') \\ &= \tilde{g}(\tilde{\nabla}_Z \tilde{P}V_2, N') + \tilde{g}(\tilde{\nabla}_Z V_2, N') - \tilde{g}(\tilde{\nabla}_W \tilde{P}V_1, N') - \tilde{g}(\tilde{\nabla}_W V_1, N') \\ &= \tilde{g}(h^*(Z,W) - h^*(W,Z), N') - g(A_{V_2} Z - A_{V_1} W, N'), \end{aligned} \tag{3.26}$$

$$\begin{aligned} \tilde{g}([Z,W], N) &= \tilde{g}(\tilde{\nabla}_Z W - \tilde{\nabla}_W Z, N) = \tilde{g}(\tilde{\nabla}_Z W, N) - \tilde{g}(\tilde{\nabla}_W Z, N) \\ &= -\tilde{g}(W, \tilde{\nabla}_Z N) + \tilde{g}(Z, \tilde{\nabla}_W N) = g(A_N Z, W) - g(A_N W, Z). \end{aligned} \tag{3.27}$$

Thus, the proof follows from (3.25)-(3.27). □

Theorem 15. *Let M be a golden GCR-lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$. Then, the distribution D defines a totally geodesic foliation in M iff $Bh(X, \tilde{P}Y) = 0$ for any $X, Y \in \Gamma(D)$.*

Proof. Using the definition of golden GCR-lightlike submanifolds, D defines a totally geodesic foliation in M iff

$$g(\nabla_X Y, \tilde{P}\xi) = g(\nabla_X Y, \tilde{P}W) = 0$$

for any $X, Y \in \Gamma(D)$, $\xi \in \Gamma(D_2)$ and $W \in \Gamma(L_2)$. Using (2.2) and (2.11), we derive

$$g(\nabla_X Y, \tilde{P}\xi) = \tilde{g}(\tilde{\nabla}_X Y, \tilde{P}\xi) = \tilde{g}(h^l(X, \tilde{P}Y), \xi), \tag{3.28}$$

$$g(\nabla_X Y, \tilde{P}W) = \tilde{g}(\tilde{\nabla}_X Y, \tilde{P}W) = \tilde{g}(h^s(X, \tilde{P}Y), W). \tag{3.29}$$

It follows from (3.28) and (3.29) that D defines a totally geodesic foliation in M iff $h^l(X, \tilde{P}Y)$ has no components in L_1 and $h^s(X, \tilde{P}Y)$ has no components in L_2 for any $X, Y \in \Gamma(D)$, that is, using (3.6), $Bh(X, \tilde{P}Y) = 0$ for any $X, Y \in \Gamma(D)$. □

Theorem 16. *Let M be a golden GCR-lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$. Then, the distribution D' defines a totally geodesic foliation in M iff*

(i) *$A_V Z$ has no components in D_o,*
(ii) $\tilde{g}(h^*(Z,W), N') = \tilde{g}(A_V Z, N')$,

(iii) *$A_N Z$ has no components in $\tilde{P}(D_2) \perp M_2$,*
for any $Z, W \in \Gamma(D')$, $N' \in \Gamma(L_1)$, $V \in \Gamma(L_1 \perp L_2)$ and $N \in \Gamma(ltr(TM))$.

Proof. From definition of GCR-lightlike submanifolds, D' defines a totally geodesic foliation iff

$$g(\nabla_Z W, X) = g(\nabla_Z W, \tilde{P}N') = \tilde{g}(\nabla_Z W, N) = 0$$

for any $Z, W \in \Gamma(D')$, $X \in \Gamma(D_o)$, $N' \in \Gamma(L_1)$ and $N \in \Gamma(ltr(TM))$. Choosing $W \in \Gamma(D')$, there is a vector field $V \in \Gamma(L_1 \perp L_2)$ such that $W = \tilde{P}V$. From (2.2), (2.9) and (2.10), we get

$$g(\nabla_Z W, X) = \tilde{g}(\tilde{\nabla}_Z W, X) = \tilde{g}(\tilde{\nabla}_Z \tilde{P}V, X) = -g(A_V Z, \tilde{P}X). \quad (3.30)$$

Similarly, using (2.3), (2.9), (2.10) and (2.16), we have

$$\begin{aligned} g(\nabla_Z W, \tilde{P}N') &= \tilde{g}(\tilde{\nabla}_Z W, \tilde{P}N') = \tilde{g}(\tilde{\nabla}_Z \tilde{P}V, \tilde{P}N') \\ &= \tilde{g}(\tilde{\nabla}_Z \tilde{P}V, N') + \tilde{g}(\tilde{\nabla}_Z V, N') \\ &= \tilde{g}(\tilde{\nabla}_Z W, N') - g(A_V Z, N') \\ &= \tilde{g}(h^*(Z, W), N') - \tilde{g}(A_V Z, N'). \end{aligned} \quad (3.31)$$

Since $\tilde{\nabla}$ is a metric connection, from (2.9) and (2.12) we obtain

$$\tilde{g}(\nabla_Z W, N) = g(A_N Z, W). \quad (3.32)$$

From (3.30)–(3.32), we derive our theorem. □

Definition 17. A golden GCR-lightlike submanifold of a golden semi-Riema nnian manifold is called $D-$geodesic golden GCR-lightlike submanifold if its second fundamental form h satisfies $h(X, Y) = 0$ for any $X, Y \in \Gamma(D)$.

Theorem 18. *Let M be a golden GCR-lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$. Then the distribution D defines a totally geodesic foliation in $\tilde{M}$ if and only if M is $D-$geodesic.*

Proof. Suppose that D defines a totally geodesic foliation in $\tilde{M}$, then $\tilde{\nabla}_X Y \in \Gamma(D)$ for any $X, Y \in \Gamma(D)$. Then from (2.11), we get

$$\tilde{g}(h^l(X, Y), \xi) = \tilde{g}(\tilde{\nabla}_X Y, \xi) = 0, \; \tilde{g}(h^s(X, Y), W) = \tilde{g}(\tilde{\nabla}_X Y, W) = 0 \quad (3.33)$$

for any $X, Y \in \Gamma(D)$, $\xi \in \Gamma(Rad(TM))$ and $W \in \Gamma(S(TM^\perp))$. Hence $h^l(X, Y) = h^s(X, Y) = 0$, which implies that M is $D-$geodesic.

Conversely, assume that M is $D-$geodesic. Using (2.2) and (2.11), we have

$$\tilde{g}(\tilde{\nabla}_X Y, \tilde{P}\xi) = \tilde{g}(\tilde{\nabla}_X \tilde{P}Y, \xi) = \tilde{g}(h^l(X, \tilde{P}Y), \xi) = 0 \quad (3.34)$$

and

$$\tilde{g}(\tilde{\nabla}_X Y, \tilde{P}W) = \tilde{g}(\tilde{\nabla}_X \tilde{P}Y, W) = \tilde{g}(h^s(X, \tilde{P}Y), W) = 0 \quad (3.35)$$

for any $X, Y \in \Gamma(D)$, $\xi \in \Gamma(D_2)$ and $W \in \Gamma(L_2)$. Hence $\tilde{\nabla}_X Y \in \Gamma(D)$, which completes the proof. □

4. Minimal Golden GCR-Lightlike Submanifolds

Definition 19. We say that a lightlike submanifold $(M, g, S(TM))$ isometrically immersed in a semi-Riemannian manifold $(\tilde{M}, \tilde{g})$ is minimal if:

(i) $h^s = 0$ on $Rad(TM)$ and
(ii) $traceh = 0$, where trace is written with respect to g restricted to $S(TM)$.

It has been shown in [2] that the above definition is independent of $S(TM)$ and $S(TM^\perp)$, but it depends on $tr(TM)$.

Example 20. Let $(\tilde{M} = \mathbb{R}^{16}_4, \tilde{g})$ be a 16-dimensional semi-Euclidean space with signature $(+,+,+,+,-,-,-,-,+,+,+,+,+,+,+,+)$ and $(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16})$ be the standard coordinate system of $\mathbb{R}^{16}_4$. If we define a mapping $\tilde{P}$ by

$$\begin{aligned}
&\tilde{P}(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}) \\
&= (x_1 + x_2, x_1, x_3 + x_4, x_3, x_5 + x_6, x_5, x_7 + x_8, x_7, x_9 + x_{10}, x_9, \\
&\qquad x_{11} + x_{12}, x_{11}, x_{13} + x_{14}, x_{13}, x_{15} + x_{16}, x_{15}),
\end{aligned}$$

then $\tilde{P}^2 = \tilde{P} + I$ and $\tilde{P}$ is a golden structure on $\mathbb{R}^{16}_4$. Let M be a submanifold of $\tilde{M}$ given by

$$\begin{aligned}
x_1 &= u_2 \sin\alpha + u_7 \cos\alpha, x_2 = u_1 \sin\alpha + u_6 \cos\alpha, \\
x_3 &= u_2 \cos\alpha - u_7 \sin\alpha, x_4 = u_1 \cos\alpha - u_6 \sin\alpha, \\
x_5 &= u_2, x_6 = u_1, x_7 = u_4 - \frac{1}{2}u_5, x_8 = u_3, \\
x_9 &= u_4 + \frac{1}{2}u_5, x_{10} = u_3, \\
x_{11} &= \sin u_8 \sinh u_9, x_{12} = 0, \\
x_{13} &= \sin u_8 \cosh u_9, x_{14} = 0, x_{15} = \sqrt{2} \cos u_8 \cosh u_9, x_{16} = 0,
\end{aligned}$$

where u_i, $1 \leq i \leq 9$, are real parameters. Thus, $TM = Span\{U_1, U_2, U_3, U_4, U_5, U_6, U_7, U_8, U_9\}$, where

$$\begin{aligned}
U_1 &= \sin\alpha \frac{\partial}{\partial x_2} + \cos\alpha \frac{\partial}{\partial x_4} + \frac{\partial}{\partial x_6}, U_2 = \sin\alpha \frac{\partial}{\partial x_1} + \cos\alpha \frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_5}, \\
U_3 &= \frac{\partial}{\partial x_8} + \frac{\partial}{\partial x_{10}}, U_4 = \frac{\partial}{\partial x_7} + \frac{\partial}{\partial x_9}, U_5 = \frac{1}{2}(-\frac{\partial}{\partial x_7} + \frac{\partial}{\partial x_9}), \\
U_6 &= \cos\alpha \frac{\partial}{\partial x_2} - \sin\alpha \frac{\partial}{\partial x_4}, U_7 = \cos\alpha \frac{\partial}{\partial x_1} - \sin\alpha \frac{\partial}{\partial x_3}. \\
U_8 &= \cos u_8 \sinh u_9 \frac{\partial}{\partial x_{11}} + \cos u_8 \cosh u_9 \frac{\partial}{\partial x_{13}} - \sqrt{2} \sin u_8 \cosh u_9 \frac{\partial}{\partial x_{15}}, \\
U_9 &= \sin u_8 \cosh u_9 \frac{\partial}{\partial x_{11}} + \sin u_8 \sinh u_9 \frac{\partial}{\partial x_{13}} + \sqrt{2} \cos u_8 \sinh u_9 \frac{\partial}{\partial x_{15}}.
\end{aligned}$$

Then M is a $3-$lightlike submanifold with $Rad(TM) = Span\{U_1, U_2, U_3\}$ and $\tilde{P}U_1 = U_2$. Thus, $D_1 = Span\{U_1, U_2\}$. On the other hand, $\tilde{P}U_3 = U_4 \in \Gamma(S(TM))$ implies that $D_2 = Span\{U_3\}$. Moreover, $\tilde{P}U_6 = U_7$, thus

$D_0 = Span\{U_6, U_7\}$. We can easily obtain

$$ltr(TM) = Span\{N_1 = \frac{1}{2}(\sin\alpha\frac{\partial}{\partial x_2} + \cos\alpha\frac{\partial}{\partial x_4} - \frac{\partial}{\partial x_6}),$$
$$N_2 = \frac{1}{2}(\sin\alpha\frac{\partial}{\partial x_1} + \cos\alpha\frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_5}), N_3 = \frac{1}{2}(-\frac{\partial}{\partial x_8} + \frac{\partial}{\partial x_{10}})\}$$

and

$$S(TM^{\perp}) = Span\{W_1 = \cos u_8 \sinh u_9 \frac{\partial}{\partial x_{12}} + \cos u_8 \cosh u_9 \frac{\partial}{\partial x_{14}}$$
$$-\sqrt{2}\sin u_8 \cosh u_9 \frac{\partial}{\partial x_{16}},$$
$$W_2 = \sin u_8 \cosh u_9 \frac{\partial}{\partial x_{12}} + \sin u_8 \sinh u_9 \frac{\partial}{\partial x_{14}} + \sqrt{2}\cos u_8 \sinh u_9 \frac{\partial}{\partial x_{16}},$$
$$W_3 = -\sqrt{2}\sinh u_9 \cosh u_9 \frac{\partial}{\partial x_{12}} + \sqrt{2}(\sin^2 u_8 + \sinh^2 u_9)\frac{\partial}{\partial x_{14}}$$
$$+\sin u_8 \cos u_8 \frac{\partial}{\partial x_{16}},$$
$$W_4 = -\sqrt{2}\sinh u_9 \cosh u_9 \frac{\partial}{\partial x_{11}} + \sqrt{2}(\sin^2 u_8 + \sinh^2 u_9)\frac{\partial}{\partial x_{13}}$$
$$+\sin u_8 \cos u_8 \frac{\partial}{\partial x_{15}}\}.$$

Furthermore, $Span\{N_1, N_2\}$ is invariant with respect to $\tilde{P}$. Since $\tilde{P}N_3 = U_5$, $\tilde{P}W_1 = U_8$ and $\tilde{P}W_2 = U_9$, then $L_1 = Span\{N_3\}$, $L_2 = Span\{W_1, W_2\}$, $M_1 = Span\{U_5\}$ and $M_2 = Span\{U_8, U_9\}$. Thus, M is a proper golden GCR-lightlike submanifol of $\tilde{M}$. On the other hand, by direct calculation we obtain

$$\tilde{\nabla}_{U_i} U_j = 0,\ 1 \le i \le 7,\ 1 \le j \le 9$$

and

$$h^s(U_8, U_8) = -\frac{\sqrt{2}\sin u_8 \cosh u_9}{(\sin^2 u_8 + 2\sinh^2 u_9)(1 + \sin^2 u_8 + 2\sinh^2 u_9)} W_4,$$
$$h^s(U_9, U_9) = \frac{\sqrt{2}\sin u_8 \cosh u_9}{(\sin^2 u_8 + 2\sinh^2 u_9)(1 + \sin^2 u_8 + 2\sinh^2 u_9)} W_4.$$

Thus, we get

$$traceh_{g|S(TM)} = h^s(U_8, U_8) + h^s(U_9, U_9) = 0.$$

Therefore, M is a minimal proper golden GCR-lightlike submanifold of $\mathbb{R}^{16}_4$.

Theorem 21. *Let M be a totally umbilical golden GCR-lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$. Then, M is minimal iff M is totally geodesic.*

Proof. Assume that M is minimal, then $h^s(X, Y) = 0$ for any $X, Y \in \Gamma(Rad(TM))$. Since M is totally umbilical, then $h^l(X, Y) = H^l g(X, Y) = 0$

for any $X, Y \in \Gamma(Rad(TM))$. Now, choose an orthonormal basis $\{e_1, e_2, ..., e_{m-r}\}$ of $S(TM)$; thus from (2.23), we get

$$\begin{aligned} traceh(e_i, e_i) &= \sum_{i=1}^{m-r} \epsilon_i h^l(e_i, e_i) + \epsilon_i h^s(e_i, e_i) \\ &= (m-r)H^l + (m-r)H^s. \end{aligned} \tag{4.1}$$

Since M is minimal and $ltr(TM) \cap S(TM^{\perp}) = \{0\}$, we obtain $H^l = 0$ and $H^s = 0$. Therefore, M is totally geodesic. The converse is clear. $\square$

Theorem 22. *Let M be a totally umbilical golden GCR-lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$. Then, M is minimal iff*

$$traceA_{W_p} = 0, \ \ and \ traceA^*_{\xi_k} = 0 \ on \ D_0 \perp M_2 \tag{4.2}$$

for $W_p \in \Gamma(S(TM^{\perp}))$, where $k \in \{1, 2, ..., r\}$ and $p \in \{1, 2, ..., n-r\}$.

Proof. Definition of a golden GCR-lightlike submanifold, M is minimal iff

$$\begin{aligned} traceh \mid_{S(TM)} &= \sum_{i=1}^{a} h(Z_i, Z_i) + \sum_{j=1}^{b} h(\tilde{P}\xi_j, \tilde{P}\xi_j) \\ &\quad + \sum_{j=1}^{b} h(\tilde{P}N_j, \tilde{P}N_j) + \sum_{l=1}^{c} h(\tilde{P}W_l, \tilde{P}W_l), \end{aligned} \tag{4.3}$$

and $h^s = 0$ on $Rad(TM)$, where $a = dim(D_0)$, $b = dim(D_2)$ and $c = dim(L_2)$. Since M is totally umbilical then from (2.22), we derive $h(\tilde{P}\xi_j, \tilde{P}\xi_j) = h(\tilde{P}N_j, \tilde{P}N_j) = 0$. Similarly, $h^s = 0$ on $Rad(TM)$. Thus from (4.3), we have

$$\begin{aligned} traceh \mid_{S(TM)} &= \sum_{i=1}^{a} h(Z_i, Z_i) + \sum_{l=1}^{c} h(\tilde{P}W_l, \tilde{P}W_l) \\ &= \sum_{i=1}^{a} \frac{1}{r} \sum_{k=1}^{r} \tilde{g}(h^l(Z_i, Z_i), \xi_k) N_k \\ &\quad + \sum_{i=1}^{a} \frac{1}{n-r} \sum_{p=1}^{n-r} \tilde{g}(h^s(Z_i, Z_i), W_p) W_p \\ &\quad + \sum_{l=1}^{c} \frac{1}{r} \sum_{k=1}^{r} \tilde{g}(h^l(\tilde{P}W_l, \tilde{P}W_l), \xi_k) N_k \\ &\quad + \sum_{l=1}^{c} \frac{1}{n-r} \sum_{p=1}^{n-r} \tilde{g}(h^s(\tilde{P}W_l, \tilde{P}W_l), W_p) W_p, \end{aligned} \tag{4.4}$$

where $\{W_1, W_2, ..., W_{n-r}\}$ is an orthonormal basis of $S(TM^{\perp})$. Using (2.14) and (2.18) in (4.4), we obtain

$$\begin{aligned} traceh \mid_{S(TM)} = & \sum_{i=1}^{a} \frac{1}{r} \sum_{k=1}^{r} \tilde{g}(A^*_{\xi_k} Z_i, Z_i) N_k \\ & + \sum_{i=1}^{a} \frac{1}{n-r} \sum_{p=1}^{n-r} \tilde{g}(A_{W_p} Z_i, Z_i) W_p \\ & + \sum_{l=1}^{c} \frac{1}{r} \sum_{k=1}^{r} \tilde{g}(A^*_{\xi_k} \tilde{P} W_l, \tilde{P} W_l) N_k \\ & + \sum_{l=1}^{c} \frac{1}{n-r} \sum_{p=1}^{n-r} \tilde{g}(A_{W_p} \tilde{P} W_l, \tilde{P} W_l) W_p. \end{aligned} \tag{4.5}$$

Thus, $traceh \mid_{S(TM)} = 0$ iff $traceA_{W_p} = 0$ and $traceA^*_{\xi_k} = 0$ on $D_0 \perp M_2$, which completes the proof. □

Theorem 23. *Let M be an irrotational golden GCR-lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$. Then M is minimal iff $traceA^*_{\xi_k} \mid_{S(TM)} = 0$ and $traceA_{W_p} \mid_{S(TM)} = 0$, where $W_p \in \Gamma(S(TM^{\perp}))$, $k \in \{1, 2, ..., r\}$ and $p \in \{1, 2, ..., n-r\}$.*

Proof. Since M is irrotational, then $h^s(X, \xi) = 0$ for any $X \in \Gamma(TM)$ and $\xi \in \Gamma(Rad(TM))$. Thus, $h^s = 0$ on $Rad(TM)$. Moreover, we have

$$\begin{aligned} traceh \mid_{S(TM)} &= \sum_{i=1}^{m-r} \epsilon_i (h^l(e_i, e_i) + h^s(e_i, e_i)) \\ &= \sum_{i=1}^{m-r} \epsilon_i \{ \frac{1}{r} \sum_{k=1}^{r} \tilde{g}(h^l(e_i, e_i), \xi_k) N_k \\ & \quad + \frac{1}{n-r} \sum_{p=1}^{n-r} \tilde{g}(h^s(e_i, e_i), W_p) W_p \}, \end{aligned} \tag{4.6}$$

where $\{W_1, W_2, ..., W_{n-r}\}$ is an orthonormal basis of $S(TM^{\perp})$. Using (2.14) and (2.18) in (4.6), we obtain

$$\begin{aligned} traceh \mid_{S(TM)} = & \sum_{i=1}^{m-r} \epsilon_i \{ \frac{1}{r} \sum_{k=1}^{r} \tilde{g}(A^*_{\xi_k} e_i, e_i) N_k \\ & + \frac{1}{n-r} \sum_{p=1}^{n-r} \tilde{g}(A_{W_p} e_i, e_i) W_p \}. \end{aligned} \tag{4.7}$$

Thus, the proof is completed. □

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