Mediterr. J. Math. (2020) 17:115
https://doi.org/10.1007/s00009-020-01558-8
1660-5446/20/040001-17
published online June 24, 2020
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Mediterranean Journal of Mathematics

m-Quasi-Einstein Metrics Satisfying Certain Conditions on the Potential Vector Field

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Abstract. In this paper we study Riemannian manifolds (M^n, g) admitting an m-quasi-Einstein metric with V as its potential vector field. We derive an integral formula for compact m-quasi-Einstein manifolds and prove that the vector field V vanishes under certain integral inequality. Next, we prove that if the metrically equivalent 1-form $V^\flat$ associated with the potential vector field is a harmonic 1-form, then V is an infinitesimal harmonic transformation. Moreover, if M is compact then it is Einstein. Some more results were obtained when (i) V generates an infinitesimal harmonic transformation, (ii) V is a conformal vector field.

Mathematics Subject Classification. 53C25, 53C20, 53D10.

Keywords. Ricci soliton, m-quasi-Einstein metric , infinitesimal harmonic transformation , conformal vector field.

1. Introduction

In the recent years, Einstein metrics and several of their generalizations [1] have received a lot of importance in geometry and physics. These are Ricci solitons, Ricci almost solitons, m-quasi-Einstein metrics and generalized quasi-Einstein metrics. Ricci solitons have been extensively studied, also because of their connection with the study of the Ricci flow. A Ricci soliton is a Riemannian manifold (M^n, g) together with a vector field V that satisfies

$$\pounds_V g + 2S = 2\lambda g,$$

where $\pounds_V$ denotes the Lie-derivative operator along a vector field V, and S the Ricci tensor of g and λ a constant. It is said to be trivial (Einstein) if either $V = 0$, or V is Killing. This is said to be a *gradient Ricci soliton* if $V = Df$, for some smooth function f on M, where D is the gradient operator. For details about Ricci soliton, we refer to [2].

Generalizing the notion of *gradient Ricci soliton*, Case et al. [3] introduced the notion *quasi-Einstein* metric. This is closely related to the warped product spaces (see [1]) and appears from the *m-Bakry-Emery* Ricci tensor S_f^m, defined by (see [4])

$$S_f^m = S + \nabla^2 f - \frac{1}{m}\mathrm{d}f \otimes \mathrm{d}f.$$

A Riemannian manifold M together with a Riemannian metric g is said to quasi-Einstein respect to the function f and the constant m if $S_f^m = \lambda g$, i.e., if its Ricci tensor S satisfies

$$S + \nabla^2 f - \frac{1}{m}\mathrm{d}f \otimes \mathrm{d}f = \lambda g, \tag{1.1}$$

where λ is a constant, $0 < m \leq \infty$ and $\nabla^2 f$ denotes the Hessian tensor of the smooth function f on M. This also appears from the warped product of the base of an $(n+m)$-dimensional Einstein manifold (see [5]). If λ is a smooth function in the defining condition (1.1), then it is known as *generalized m-quasi-Einstein*. For more details we refer to [6–8]). Equation (1.1) reduces to the usual Einstein condition when f is constant. Moreover, when $m = \infty$ it reduces to exactly the gradient Ricci soliton [2]. Thus, Eq. (1.1) can be regarded as a generalization of gradient Ricci soliton. In [3], several results were proved extending rigidity results for gradient Ricci solitons presented by Petersen–Wylie [9].

Recently, Nurowski and Randall [10] extended the notion of Ricci soliton by introducing a class of overdetermined system of equations

$$\pounds_V g = 2\alpha S - 2\beta V^\flat \otimes V^\flat + 2\lambda g.$$

on pseudo-Riemannian manifolds (M^n, g) for some vector field V and some real constants α, β and λ, where $V^\flat$ is a 1-form associated with V. In this paper, we consider a particular type of generalized Ricci soliton, called m-quasi-Einstein metric, studied first by Limoncu [11] (see also [12]). This also arises as a generalization of the quasi-Einstein metric [3], by taking the 1-form $V^\flat$ instead of $\mathrm{d}f$ in the defining Eq. (1.1). Explicitly, this can be presented as

$$S + \frac{1}{2}\pounds_V g - \frac{1}{m}V^\flat \otimes V^\flat = \lambda g, \tag{1.2}$$

where $\pounds_V$ denotes the Lie-derivative operator along a vector field V, known as potential vector field. A Riemannian manifold M satisfying Eq. (1.2) is said to be a m-quasi-Einstein manifold and along the manuscript is denoted by (M^n, V, g). Using the terminology of Ricci soliton, an m-quasi-Einstein metric is said to be *expanding, steady or shrinking,* respectively, if $\lambda < 0$, $\lambda = 0$, or $\lambda > 0$. If $V = 0$, Eq. (1.2) simply reduces to the Einstein condition and in this case, we say that the m-quasi-Einstein metric is trivial. It is also interesting to remark that if the potential vector field V is the gradient of a smooth function f, then Eq. (1.2) reduces to the quasi-Einstein condition, as defined by Eq. (1.1). It may be also mentioned that the study of Eq. (1.1) depends mostly on the behavior of the potential function f, whereas the study of (1.2) only depends on the potential vector field V. Moreover, we remark that if V is Killing (or conformal Killing [12]), the m-quasi-Einstein metric is not trivial like Ricci soliton. Further, it is interesting to note that

Eq. (1.2) reduces to the so called *Ricci soliton* when $m = \infty$. For this reason, one may consider that a Riemannian manifold M with an m-quasi-Einstein metric g is a direct generalization of the Ricci soliton and as well as gradient Ricci soliton. In [11], Limoncu first studied Eq. (1.2) to generalize Qian's [13] results, which were the natural generalization of Myers' compactness theorem on Riemannian manifolds [14]. Recently, the author studies m-quasi-Einstein metrics within the framework of contact metric manifolds [15]. In the present paper, we study Eq. (1.2) under some conditions on the potential vector field V and the scalar curvature.

The organization of this paper is as follows. In Sect. 2, we recall basic definitions of rough Laplacian, infinitesimal harmonic transformations and harmonic vector fields. Section 3 has been devoted to derive several non trivial examples of m-quasi-Einstein metric. In Sect. 4, we have proved an integral formula for compact m-quasi-Einstein Riemannian manifolds and we extend a result of Barros–Gomes [16]. In Sect. 5, we have studied m-quasi-Einstein metric when the 1-form $V^\flat$ is harmonic. In this case, V generates an infinitesimal harmonic transformation. Next, we prove that if the potential vector field V generates an infinitesimal harmonic transformation on a compact m-quasi-Einstein manifold M, then V is Killing and the Ricci tensor S is a Killing tensor, i.e., cyclic parallel. Finally, we consider m-quasi-Einstein metric when the potential vector field is conformal Killing and the scalar curvature is constant.

2. Preliminaries

Let (M^n, g) be a Riemannian manifold and consider a diffeomorphism $f : M \to M$. Let ∇' be the pull-back connection of the Levi–Civita connection ∇ of (M^n, g) by f. If we have $\mathrm{trace}_g(\nabla' - \nabla) = 0$, then f is said to be harmonic (see for instance [17].

Next, consider a vector field V and let $\{f_t\}$ be any local 1-parameter group of transformations of V. Using each f_t as previously we can define a connection ∇^t from ∇. Thus, we arrive to the Lie-derivative ∇ with respect to V, $\pounds_V \nabla$. We say that V is an infinitesimal harmonic transformation if $\mathrm{trace}_g(\pounds_V \nabla) = 0$ (see [17,18]). An interesting characterization of such vector field was given by Stepanov–Shandra in [17]. They proved that

"A vector field V generates an infinitesimal harmonic transformation on a Riemannian manifold (M^n, g) if and only if $\Delta V = 2QV$".

The operator Δ is known as the Laplacian and it is determined by the Weitzenböck formula

$$\Delta V = \nabla^* \nabla V + QV,$$

where ∇^* is the formal adjoint of ∇, given by $\bar{\Delta} V = \nabla^* \nabla V$ and Q is the Ricci operator associated with the Ricci tensor S. The operator $\bar{\Delta}$ is known as the rough Laplacian of vector field V, and is defined by $\bar{\Delta} V = -\mathrm{trace}_g \nabla^2 V$. Explicitly, if $\{e_i\}$ be any local orthonormal frame field, then the rough Laplacian of the vector field is defined by

$$\bar{\Delta}V = \sum_i \{\nabla_{\nabla_{e_i}e_i} - \nabla_{e_i}\nabla_{e_i}\}V. \tag{2.1}$$

The following examples are well-known for infinitesimal harmonic transformations:

- Any Killing vector field on a Riemannian manifold generates an infinitesimal harmonic transformation (see [17]).
- The potential vector field V of the Ricci soliton is necessarily an infinitesimal harmonic transformation (see [19]).
- Since the Reeb vector field ξ of a K-contact manifold is Killing, it generates an infinitesimal harmonic transformation (see [20]).
- Let (M, g, J) be a nearly Kaehlerian manifold, where $J^2 = -I$, $g(J, J) = g$ and $(\nabla_X J)Y + \nabla_Y J)X = 0$ for any vector field X, Y on M and let V be a vector field on M such that $\pounds_V J = 0$ (i.e., holomorphic). Then V is necessarily an infinitesimal harmonic transformation (see [17]).
- Any vector field V on a contact metric manifold $M^{2n+1}(\varphi, \xi, \eta, g)$ that leaves the tensor φ invariant (i.e., $\pounds_V\varphi = 0$) is necessarily an infinitesimal harmonic transformation (see [21]).

To end this section, recall that a vector field V, on a Riemannian manifold (M^n, g) it is said to be harmonic if the associated 1-form $V^\flat$ is closed and co-closed, i.e., $\mathrm{d}V^\flat = 0$ and $\delta V^\flat = 0$, respectively (see for instance [22]). Thus, a harmonic vector field V satisfies $\Delta V = 0$, and the converse is true for compact case. We denote the divergence of a along the manuscript as div and define the divergence of a vector field V as $\operatorname{div} V = \sum_{i=1}^n g(\nabla_{e_i}V, e_i)$.

3. Some Examples

Example 3.1. We consider the hyperbolic space $\mathbb{H}^n$ given by $\mathbb{R} \times \mathbb{R}^{n-1}$ with the warped product metric $g = \mathrm{d}t^2 + e^{2t}g_0$, where g_0 is the standard metric of $\mathbb{R}^{n-1}$. Let us define a function $f : \mathbb{H}^n \to \mathbb{R}$ given by $f(t) = mt$, where m is a positive integer. Then it is easy to verify that $(\mathbb{H}^n, g, Df, \lambda = -m - n + 1)$ is an expanding gradient m-quasi-Einstein manifold [23].

Example 3.2. Let us consider the Berger sphere $(\mathbb{S}^3, g_{k,\sigma})$, where $\mathbb{S}^3 = \{(z, w) \in \mathbb{C}^2 \ : |z|^2 + |w|^2 = 1\}$ and $g_{k,\sigma} = \frac{4}{k}\{g_0 + (\frac{4\sigma^2}{k} - 1)U^\flat \otimes U^\flat\}$, g_0 stands for the usual Riemannian metric on $\mathbb{S}^3$, $k > 0$, $\sigma \neq 0$ are constants and U is the Killing vector on $\mathbb{S}^3$. For each $(z, w) \in \mathbb{S}^3$ given by $U(z, w) = (iz, iw)$. Set $E_3 = \frac{k}{4\sigma}U$, which is also Killing. Then we can deduce $(\mathbb{S}^3, V = \sqrt{m(4\sigma^2 - k)}E_3, \lambda = k - 2\sigma^2)$ is a non gradient m-quasi-Einstein manifold. For details we refer to [16, 24].

Some more non gradient examples can be found in [15]. In our next examples, we prove that there exists an m-quasi-Einstein metric on certain product manifolds whose potential vector field is parallel but not vanishes anywhere.

Example 3.3. Let $M^{2n+1}(\varphi, \xi, \eta, g)$ be an almost contact metric manifold and consider the product manifold $M^{2n+1} \times \mathbb{R}$. Let J be the almost complex structure on $M^{2n+1} \times \mathbb{R}$ defined by

$$J\left(X, f\frac{\mathrm{d}}{\mathrm{d}t}\right) = \left(\varphi X - f\xi, \eta(X)\frac{\mathrm{d}}{\mathrm{d}t}\right),$$

where X is tangent to M^{2n+1}, f a smooth function on $M^{2n+1} \times \mathbb{R}$ and t the coordinate on $\mathbb{R}$. Then (J, G) induces an Hermitian structure on the product manifold $M^{2n+1} \times \mathbb{R}$ with the product metric $G = g + \mathrm{d}t^2$. If one assumes that this structure is Kaehler, then $M^{2n+1}(\varphi, \xi, \eta, g)$ becomes a cosymplectic manifold [25].

One may also construct such structure through the product of a Kaehler manifold $N^{2n}(J, G)$ and the real line $\mathbb{R}$. If t denotes the coordinate of $\mathbb{R}$ and X be any vector field of N, then any vector field of $M = N \times \mathbb{R}$ can be written as $(X, f\frac{\mathrm{d}}{\mathrm{d}t})$. Define a tensor field φ of type $(1, 1)$ by $\varphi(X, f\frac{\mathrm{d}}{\mathrm{d}t}) =: (JX, 0)$, a 1-form $\eta =: \mathrm{d}t$, a vector field $\xi =: \frac{\mathrm{d}}{\mathrm{d}t}$ and a Riemannian metric $g = G + \mathrm{d}t^2$. Then it follows that (φ, ξ, η, g) defines a cosymplectic structure on M^{2n+1} [26,27].

A cosymplectic manifold M whose φ-sectional curvature does not depend on the point is called cosymplectic space form and we denote it by $M(c)$. A straight forward computation shows that a cosymplectic manifold has constant φ-sectional curvature c at a point if and only if the curvature tensor R is given by [25].

$$\begin{aligned} R(X,Y)Z = \frac{c}{4}\{&g(Y,Z)X - g(X,Z)Y + \eta(X)\eta(Z)Y \\ &-\eta(Y)\eta(Z)X + \eta(Y)g(X,Z)\xi - \eta(X)g(Y,Z)\xi \\ &-g(\varphi X,Z)\varphi Y + g(\varphi Y,Z)\varphi X + 2g(X,\varphi Y)\varphi Z\} \end{aligned} \tag{3.1}$$

From this we can compute the Ricci tensor as

$$S(X,Y) = \frac{(n+1)c}{2}[g(X,Y) - \eta(X)\eta(Y)]. \tag{3.2}$$

Comparing previous formula with (1.2) it follows that M^{2n+1} admits an m-quasi-Einstein metric with $\lambda = \frac{(n+1)c}{2}$ and the potential vector field $V = \sqrt{\frac{-m(n+1)c}{2}}\xi$, where $c < 0$. Since the Reeb vector field ξ is parallel, V is also parallel. This shows that there exists non trivial m-quasi-Einstein metric with non vanishing parallel potential vector field.

Example 3.4. Let $N(c)$ be a Kaehlerian manifold with constant holomorphic sectional curvature c. Then the Riemannian product $N(c) \times \mathbb{R}$ (where $\mathbb{R}$ is the real line) becomes a cosymplectic space form. Particularly, if we take $N(c)$ as a complex hyperbolic space $\mathbb{CH}^n$ of constant holomorphic sectional curvature -4, then the product $\mathbb{CH}^n(-4) \times \mathbb{R}$ is a cosymplectic space form $M(c)$. Consequently, $M(c)(\varphi, \xi, \eta, g)$ admits m-quasi-Einstein metric with $V = \pm\xi$, $m = \frac{1}{2(n+1)}$ and $\lambda = 2(n+1)$.

Example 3.5. Consider the warped product $M = \mathbb{R} \times_f \mathbb{CH}^n(-4)$, where $(\mathbb{CH}^n(-4), \bar{g}, J)$ is a Kaehler manifold, $f(t) = e^t$ is the warping function on the line $\mathbb{R}$ with the metric

$$g = \mathrm{d}t^2 + f^2(t)\bar{g}$$

We define a $(1,1)$ tensor field φ on M by $\varphi X = JX$, for any vector field X on N and $\varphi X = 0$ for any vector field X tangent to $\mathbb{R}$. We also define a unit vector field $\xi = \frac{\partial}{\partial t}$, a 1-form $\eta = dt$. Then in [28] it was proved that (φ, ξ, η, g) defines an almost contact metric structure on M, which satisfies

$$\nabla_X \xi = X - \eta(X)\xi. \tag{3.3}$$

Clearly the Ricci tensor of M satisfies $S = -2ng$ (see [28]). Thus, if we take $V = -k\xi$, where $k > 0$, then we see that $M(\varphi, \xi, \eta, g)$ admits an m-quasi-Einstein structure with $m = \frac{k}{2}$ and $\lambda = -2(n+k)$.

4. Triviality of m-Quasi-Einstein Metrics

In [29], Perelman proved that any compact Ricci soliton is necessarily a gradient soliton; by Hamilton [30] and Ivey [31] that any compact gradient non shrinking Ricci soliton is Einstein. Hence, any compact non-shrinking Ricci soliton must be Einstein. On the other hand, Barros–Gomes [16] proved that a compact m-quasi-Einstein manifold (M^n, g, V) has vanishing potential vector field (i.e., $V = 0$) if M is Einstein. Here we extend all these results for compact m-quasi-Einstein manifold (M^n, g, V) satisfying certain integral inequalities. First, we establish the following

Theorem 4.1. *Let (M^n, V, g) be a compact m-quasi-Einstein manifold. Then the following integral formula is valid*

$$\int_M \left[\frac{1}{2}Vr - |Q|^2 + \frac{1}{m}S(V,V) + \lambda r\right] \mathrm{d}v_g = 0, \tag{4.1}$$

where $\mathrm{d}v_g$ denotes the volume form of M.

Proof. Using Koszul formula [1], one can write the following

$$2(\nabla_Y V, Z) = (\pounds_V g)(Y, Z) + \mathrm{d}V^\flat(Y, Z), \tag{4.2}$$

for any vector field V on M (where $V^\flat$ is the 1-form dual to V, that is $V^\flat(Y) = g(Y, V)$. We now define a skew symmetric tensor field ψ of type $(1,1)$ on M by

$$\mathrm{d}V^\flat(Y, Z) = 2g(\psi Y, Z), \tag{4.3}$$

for all $Y, Z \in \chi(M)$. Therefore, using Eqs. (4.2) and (4.3) in (1.2), we immediately obtain

$$\nabla_Y V = -QY + \frac{1}{m}g(Y,V)V + \lambda Y + \psi Y, \tag{4.4}$$

where Q as previously denotes the Ricci operator. Taking covariant derivative of (4.4) along an arbitrary vector field X yields

$$\begin{aligned}\nabla_X \nabla_Y V = &-(\nabla_X Q)Y - Q(\nabla_X Y) + \frac{1}{m}\{g(Y,V)\nabla_X V + g(\nabla_X V, Y)V \\ &+ g(\nabla_X Y, V)V\} + \lambda \nabla_X Y + (\nabla_X \psi)Y + \psi(\nabla_X Y).\end{aligned} \tag{4.5}$$

Making use of this and (4.4) and (4.5) we obtain that the curvature tensor of (M, g) satisfies

$$R(X,Y)Z = (\nabla_Y Q)X - (\nabla_X Q)Y + \frac{1}{m}\{g(Y,V)\nabla_X V + g(\nabla_X V, Y)V \\ -g(X,V)\nabla_Y V - g(\nabla_Y V, X)V\} + (\nabla_X \psi)Y - (\nabla_Y \psi)X. \tag{4.6}$$

Contracting Eq. (4.6) over X with respect to an orthonormal frame shows that

$$S(Y,V) = \frac{1}{2}Yr + \frac{1}{m}\{(\mathrm{div}V)g(Y,V) + g(\nabla_V V, Y) - 2g(\nabla_Y V, V)\} \\ +\sum_{i=1}^{n} g(\nabla_{e_i}\psi)Y, e_i),$$

where $r = \mathrm{trace}_g S$ is the scalarcurvature of g. Since $\nabla_Y |V|^2 = 2g(\nabla_Y V, V)$, the foregoing equation can be exhibited as

$$QV = \frac{1}{2}Dr + \frac{1}{m}\{(\mathrm{div}V)V + \nabla_V V - D|V|^2\} + \delta\psi, \tag{4.7}$$

for all vector field Y on M and $g(\delta\psi, Y) = -\sum_{i=1}^{n} g(\nabla_{e_i}\psi)e_i, Y)$. Differentiating (4.7) along an arbitrary vector field X and making use of (4.4) implies

$$(\nabla_X Q)V - Q^2X + \frac{1}{m}g(X,V)QV + \lambda QX + \psi QX = \frac{1}{2}\nabla_X Dr \\ +\frac{1}{m}\{(\mathrm{div}V)\nabla_X V + (X\mathrm{div}V)V + \nabla_X\nabla_V V - \nabla_X D|V|^2\} + \nabla_X \delta\psi.$$

Tracing this over X with respect to an orthonormal frame $\{e_i : i = 1, 2, \ldots, n\}$ we find

$$\sum_{i=1}^{n} g((\nabla_{e_i} Q)V, e_i) - |Q|^2 + \frac{1}{m}S(V,V) + \lambda r + \sum_{i=1}^{n} g(\psi Q e_i, e_i) = \frac{1}{2}\Delta r \\ +\frac{1}{m}\{(\mathrm{div}V)^2 + V(\mathrm{div}V) + \mathrm{div}\nabla_V V - \Delta|V|^2\} + \mathrm{div}\delta\psi, \tag{4.8}$$

where D is the gradient operator and $\Delta = \mathrm{div}D$. From the contraction of second Bianchi identity it follows that $\frac{1}{2}Vr = \sum_{i=1}^{n} g((\nabla_{e_i} Q)V, e_i)$. Next, we note that $\nabla_X((\mathrm{div}V)V) = X(\mathrm{div}V)V + (\mathrm{div}V)\nabla_X V$, for any vector field X on M. Contracting the last equation over X yields

$$\mathrm{div}((\mathrm{div}V)V) = V(\mathrm{div}V) + (\mathrm{div}V)^2. \tag{4.9}$$

On the other hand, choosing an orthonormal frame $(e_1, e_2, \ldots, e_n)$ such that $Qe_i = \lambda_i e_i$ at an arbitrary fixed point of M we get

$$\sum_{i=1}^{n} g(\psi Q e_i, e_i) = \sum_{i=1}^{n} \lambda_i g(\psi e_i, e_i) = 0. \tag{4.10}$$

In view of Eqs. (4.9) and (4.10), Eq. (4.8) reduces to

$$\frac{1}{2}Vr-|Q|^2+\frac{1}{m}S(V,V)+\lambda r=\frac{1}{2}\Delta r$$
$$+\frac{1}{m}\{\mathrm{div}((\mathrm{div}V)V)+\mathrm{div}\nabla_V V-\Delta|V|^2\}+\mathrm{div}\delta\psi. \tag{4.11}$$

By hypothesis M is compact, so integrating (4.11) and using divergence Theorem, we see that the integral on the left hand side vanishes on M. This concludes the proof. □

For an Einstein manifold of negative scalar curvature, we have $\frac{1}{m}S(V,V)=\frac{r}{n}|V|^2\le 0$. On the other hand, Petersen–Wylie [9] proved that a shrinking compact gradient soliton is rigid with trivial potential function f if $\int_M S(Df,Df)dM\le 0$. So, extending this here we prove

Corollary 4.1. *Let* (M^n,V,g) *be a compact steady or shrinking* m*-quasi-Einstein manifold such that* $\int_M[\frac{1}{m}S(V,V)+\frac{1}{2}Vr]\mathrm{d}M\le 0$. *Then* V *is identically zero and* M *is Einstein.*

Proof. Taking trace of (1.2) yields

$$\mathrm{div}V=n\lambda-r+\frac{1}{m}|V|^2. \tag{4.12}$$

Note that $|Q-\lambda I|^2=|Q|^2-2\lambda r+n\lambda^2$. So, we may rewrite (4.1) as

$$\begin{aligned}\int_M\left[\frac{1}{m}S(V,V)+\frac{1}{2}g(Dr,V)\right]\mathrm{d}M&=\int_M|Q-\lambda I|^2+\lambda(r-n\lambda)\\&=\int_M[|Q-\lambda I|^2-\lambda\mathrm{div}V+\frac{\lambda}{m}|V|^2]\mathrm{d}M\\&=\int_M[|Q-\lambda I|^2+\frac{\lambda}{m}|V|^2]\mathrm{d}M.\end{aligned}$$

By our assumption on the integral inequality we have

$$\int_M[|Q-\lambda I|^2+\frac{\lambda}{m}|V|^2]\mathrm{d}M\le 0. \tag{4.13}$$

Now, if M is steady, then $\lambda=0$. Hence from Eq. (4.13) it follows that M is Einstein. Thus using the result of Barros–Gomes [16] we can conclude that $V=0$. On the other hand, if $\lambda>0$, then the conclusion follows from (4.13). This completes the proof. □

Applying the Einstein condition on the potential vector field V, the equality $S(V,V)-\frac{r}{n}g(V,V)=0$ holds trivially on M. So, replacing the equality with an inequality we prove

Corollary 4.2. *Let* (M^n,V,g) *be a compact* m*-quasi-Einstein manifold with constant scalar curvature. If it satisfies* $\frac{1}{m}\int_M[S(V,V)-\frac{r}{n}g(V,V)]\mathrm{d}M\le 0$, *then* M *is Einstein.*

Proof. Since $|Q-\frac{r}{n}I|^2=|Q|^2-\frac{r^2}{n}$ and the scalar curvature r is constant, it follows from the integral formula (4.1)

$$\int_M\left[\frac{1}{m}S(V,V)+\frac{r}{n}(n\lambda-r)\right]\mathrm{d}M=\int_M|Q-\frac{r}{n}I|^2\mathrm{d}M.$$

Thus using (4.12) and since the scalar curvature r is constant, we obtain.

$$\frac{1}{m}\int_M[S(V,V)-\frac{r}{n}g(V,V)]\mathrm{d}M=\int_M|Q-\frac{r}{n}I|^2\mathrm{d}M.$$

By our assumption on the integral inequality we complete the proof. □

Next, we present another extension of the result of Barros–Gomes [16]. For this we recall the following two well known results on a Riemannian manifold

Lemma 4.1. [32] *For any vector field V on a Riemannian manifold (M^n,g), we have*
$\mathrm{div}(\nabla_V V)-\mathrm{div}((\mathrm{div}V)V)=S(V,V)+\frac{1}{2}|\pounds_V g|^2-|\nabla V|^2-(\mathrm{div}V)^2.$

Lemma 4.2. [12] *For any m-quasi-Einstein manifold (M,V,g) we have*
$\frac{1}{2}\Delta|V|^2=|\nabla V|^2-S(V,V)+\frac{2}{m}|V|^2\mathrm{div}V.$

For quasi-Einstein metrics (i.e. satisfying (1.1)) it is known that a compact quasi-Einstein metric with constant scalar curvature is trivial (see [3]). Here we extend this result and prove

Theorem 4.2. *Let (M,V,g) be a compact m-quasi-Einstein metric. If its scalar curvature is constant, then V is Killing.*

Proof. From the above two lemmas, we deduce

$$\mathrm{div}(\nabla_V V)-\mathrm{div}((\mathrm{div}V)V)+\frac{1}{2}\Delta|V|^2=\frac{1}{2}|\pounds_V g|^2$$
$$-(\mathrm{div}V)^2+\frac{2}{m}|V|^2\mathrm{div}V.$$

Since M is compact we may integrate the foregoing equation over M to achieve

$$\int_M[\frac{1}{2}|\pounds_V g|^2-(\mathrm{div}V)^2+\frac{2}{m}|V|^2\mathrm{div}V]\mathrm{d}M=0. \tag{4.14}$$

By virtue of (4.12), Eq. (4.14) can be written as

$$\int_M[\frac{1}{2}|\pounds_V g|^2-(\mathrm{div}V)^2+2\mathrm{div}V(\mathrm{div}V+r-n\lambda]\mathrm{d}M$$
$$=\int_M[\frac{1}{2}|\pounds_V g|^2+(\mathrm{div}V)^2+2(r-n\lambda)\mathrm{div}V]\mathrm{d}M=0. \tag{4.15}$$

Since the scalar curvature is constant, the foregoing equation yields

$$\int_M[\frac{1}{2}|\pounds_V g|^2+(\mathrm{div}V)^2]\mathrm{d}M=0.$$

From which it follows that $\pounds_V g=0$ (and $\mathrm{div}V=0$). □

Remark 4.1. By virtue of this result we can easily prove the result of Barros–Gomes [16] "Any compact m-quasi-Einstein manifold M of dimension ≥ 3 has vanishing potential vector field provided M is Einstein." Hence from the Theorem 4.2 the potential vector field V is Killing. Therefore from (1.2) the Ricci tensor satisfies

$$S(X,Y) = \frac{1}{m}g(X,V)g(Y,V) + \lambda g(X,Y). \tag{4.16}$$

Tracing this the scalar curvature r fulfills $r - n\lambda = \frac{1}{m}|V|^2$. On the other hand, using $S = \frac{r}{n}g$ in (4.16) yields $(\frac{r}{n} - \lambda)g(X,Y) = \frac{1}{m}g(X,V)g(Y,V)$. Setting $X = Y = V$ and making use of $r - n\lambda = \frac{1}{m}|V|^2$ shows that $V = 0$.

Theorem 4.3. *Let (M^n, V, g) be a compact steady or shrinking m-quasi-Einstein manifold. If the Ricci tensor S is Codazzi, then V is identically zero.*

Proof. Since S is Codazzi, the scalar curvature is constant. Hence by the Theorem 4.2 V is Killing and therefore $\mathrm{div}V = 0$. Consequently, from (4.12) $|V|$ is constant, i.e., $g(\nabla_X V, V) = 0$, for any vector field X. On the other hand, using the hypothesis $(\nabla_X S)(Y,Z) = (\nabla_Y S)(X,Z)$ in (4.16) gives

$$\begin{aligned} &g(\nabla_X V, Y)g(V,Z) - g(\nabla_Y V, X)g(V,Z) \\ &\quad + g(Y,V)g(\nabla_X V, Z) - g(X,V)g(\nabla_Y V, Z) = 0. \end{aligned} \tag{4.17}$$

Contracting (4.17) over Y and Z and noting that V is constant we see that $\nabla_V V = 0$. Next, setting $Y = V$ in (4.17) shows that $|V|\nabla_X V = 0$. If $V = 0$, then the conclusion follows. So, we suppose that $|V| \neq 0$. Then V is parallel and hence $QV = 0$. Using this in (4.16) we have $(\lambda + \frac{1}{m}|V|^2)|V|^2 = 0$. Since $\lambda \geq 0$, we must have $V = 0$. This concludes the proof. □

5. Infinitesimal Harmonic Transformations

In [19], the authors established that the condition that a vector field be infinitesimal harmonic transformation is related to the existence of Ricci soliton by proving that *"the potential vector field V of a Ricci soliton on a Riemannian manifold (M, g) is an infinitesimal harmonic transformation on M"*. This result suggests to find conditions on the potential vector field of the m-quasi-Einstein metric that also generates an infinitesimal harmonic transformation. First, we prove

Theorem 5.1. *Let (M^n, g, V) be an m-quasi-Einstein manifold. If $V^\flat$ is a harmonic 1-form, then V is an infinitesimal harmonic transformation. Moreover, if M is compact then V is parallel.*

Before entering into the proof we prove our key result

Lemma 5.1. *For any m-quasi-Einstein manifold (M^n, g, V) we have*

$$\begin{aligned} g(QV, Z) - g(\bar{\Delta}V, Z) = \frac{2}{m}\{&(\mathrm{div}V)g(Z,V) \\ &+ g(\nabla_V V, Z) - g(\nabla_Z V, V)\}. \end{aligned} \tag{5.1}$$

Proof. Differentiating (1.2) along an arbitrary vector field X gives

$$(\nabla_X \pounds_V g)(Y,Z) + 2(\nabla_X S)(Y,Z) = \frac{2}{m}\{g(Y,\nabla_X V)g(Z,V) + g(Y,V)g(Z,\nabla_X V)\}. \tag{5.2}$$

Making use of this in the commutation formula (see Yano [22])

$$(\pounds_V \nabla_X g - \nabla_X \pounds_V g - \nabla_{[V,X]} g)(Y,Z) = -g((\pounds_V \nabla)(X,Y),Z) - g((\pounds_V \nabla)(X,Z),Y)$$

we deduce

$$g((\pounds_V \nabla)(X,Y),Z) + g((\pounds_V \nabla)(X,Z),Y) = -2(\nabla_X S)(Y,Z) + \frac{2}{m}\{g(\nabla_X V,Y)g(Z,V) + g(Y,V)g(\nabla_X V,Z)\}.$$

By a straightforward combinatorial computation and using the symmetric property of $\pounds_V \nabla$ the foregoing equation yields

$$\begin{aligned} g((\pounds_V \nabla)(X,Y),Z) = {} & (\nabla_Z S)(X,Y) - (\nabla_Y S)(Z,X) - (\nabla_X S)(Y,Z) \\ & + \frac{1}{m}[g(Z,V)\{g(\nabla_X V,Y) + g(\nabla_Y V,X)\} \\ & + g(Y,V)\{g(\nabla_X V,Z) - g(\nabla_Z V,X)\} \\ & + g(X,V)\{g(\nabla_Y V,Z) - g(\nabla_Z V,Y)\}]. \end{aligned} \tag{5.3}$$

Now applying the well-known formula (see Yano [22])

$$g((\pounds_V \nabla)(X,Y),Z) = g(\nabla_X \nabla_Y V - \nabla_{\nabla_X Y} V + R(V,X)Y,Z)$$

in (5.3) and then setting $X = Y = e_i$, where $e_1, e_2, \ldots, e_n$ is a local orthonormal frame field, implies

$$g\left(\sum_{i=1}^{n}(\nabla_{e_i}\nabla_{e_i} V - \nabla_{\nabla_{e_i} e_i} V + R(V,e_i)e_i), Z\right) = \frac{2}{m}\{(\mathrm{div}V)g(Z,V) + g(\nabla_V V,Z) - g(\nabla_Z V,V)\}.$$

By virtue of (2.1) the first two terms of the left hand side of the foregoing equation give $-\bar{\Delta}V$, while the last term gives the Ricci operator in the direction of V. This completes the proof. □

Proof of Theorem 5.1. As $V^\flat$ is a harmonic 1-form, we have $\mathrm{div}V = 0$ and $V^\flat$ is closed, i.e., $\mathrm{d}V^\flat = 0$. Therefore, Eq. (5.1) transforms to

$$-\bar{\Delta}V + QV = 0. \tag{5.4}$$

Making use of (5.4) in the Weitzenböck formula:

$$\Delta V = \bar{\Delta}V + QV \tag{5.5}$$

provides $\Delta V = 2QV$. This shows that V is an infinitesimal harmonic transformation and we complete the proof of the first part. We now prove the second part. Since $V^\flat$ is closed, Eq. (1.2) reduces to

$$\nabla_Y V + QY = \lambda Y + \frac{1}{m}V^\flat(Y)V. \tag{5.6}$$

By virtue of (5.6) one can easily deduce

$$\nabla_X\nabla_Y V-\nabla_{\nabla_X Y}V=\frac{1}{m}\{g(Y,V)\nabla_X V+g(\nabla_X V,Y)V\}-(\nabla_X Q)Y. \quad (5.7)$$

Making use of (5.6) and (5.7) we can easily deduce

$$\begin{aligned}R(X,Y)V=&\frac{1}{m}\{g(Y,V)\nabla_X V-g(X,V)\nabla_Y V\}\\&-(\nabla_X Q)Y+(\nabla_Y Q)X.\end{aligned} \quad (5.8)$$

Contracting (5.8) over X we get

$$S(Y,V)=\frac{1}{m}\{g(Y,V)\mathrm{div}V-g(\nabla_Y V,V)\}+\frac{1}{2}(Yr). \quad (5.9)$$

On the other hand, tracing (5.6) and taking into account $\mathrm{div}V=0$, we obtain $r=n\lambda+\frac{1}{m}|V|^2$. From which we find that $Yr=\frac{2}{m}g(\nabla_Y V,V)$. From the last formula and (5.9), it follows $S(Y,V)=0$, for all Y. Since the 1-form $V^\flat$ associated with the vector field V is harmonic, we integrate the Bochner formula:

$$\int_M\{|\nabla V|^2+S(V,V)\}\mathrm{d}M=\int_M\frac{1}{2}\Delta|V|^2\mathrm{d}M=0,$$

to deduce that $\nabla V=0$. This finishes the proof. □

Any Killing vector field on a Riemannian manifold is clearly an infinitesimal harmonic transformation (i.e., $\Delta V+2QV=0$) and satisfies $\mathrm{div}V=0$. The converse is true in the compact case [33]. Here we prove the later for the case in which the potential vector field of an m-quasi-Einstein manifold is an infinitesimal harmonic transformation. Precisely we prove

Theorem 5.2. *Let (M^n,g,V) be an m-quasi-Einstein manifold whose potential vector field is an infinitesimal harmonic transformation. Then V is Killing and the Ricci tensor S is a Killing tensor (i.e. cyclic parallel) if it satisfies any one of the following conditions*

(i) *the scalar curvature r is constant,*
(ii) *the norm of the potential vector field V is constant.*

Proof. (i) By hypothesis we have $\Delta V=2QV$. Using this in the well known formula (Weitzenböck) $\Delta V=\bar{\Delta}V+QV$ implies that $\bar{\Delta}V=QV$. By virtue of this Eq. (5.1) provides

$$(\mathrm{div}V)g(Z,V)+g(\nabla_V V,Z)-g(\nabla_Z V,V)=0. \quad (5.10)$$

Taking $Z=V$ the foregoing equation yields $|V|^2(\mathrm{div}V)=0$. Since the m-quasi-Einstein metric is non-trivial (i.e., not Einstein), we assume that $|V|^2\neq 0$ on an open set O of M. Thus, on O we have $\mathrm{div}V=0$. As a consequence, (5.10) can be written as

$$\nabla_V V-\frac{1}{2}D|V|^2=0, \quad (5.11)$$

where we have used $X|V|^2 = 2g(\nabla_X V, V)$ and D is the gradient operator. Taking covariant derivative of the foregoing equation along an arbitrary vector field X, and then contracting the resulting equation over X yields

$$\mathrm{div}\nabla_V V - \frac{1}{2}\Delta|V|^2 = 0. \tag{5.12}$$

On the other hand, tracing (1.2) we have

$$\mathrm{div}V + r - \frac{1}{m}|V|^2 = n\lambda. \tag{5.13}$$

Using $\mathrm{div}V = 0$ (5.13) gives $r + \frac{1}{m}|V|^2 = n\lambda$. Since the scalar curvature is constant, $|V|^2$ is also constant on O and hence $|V|^2 \neq 0$ on M. Consequently, (5.12) provides $\mathrm{div}\nabla_V V = 0$. By virtue of this and $\mathrm{div}V = 0$, Lemma 4.1 shows that

$$S(V, V) + \frac{1}{2}|\pounds_V g|^2 - |\nabla V|^2 = 0. \tag{5.14}$$

Moreover, from Lemma 4.2, we have $S(V, V) - |\nabla V|^2 = 0$. Using this in (5.14), we can conclude that V is Killing. Hence (1.2) reduces to

$$S(Y, Z) = \lambda g(Y, Z) + \frac{1}{m}g(Y, V)g(X, V). \tag{5.15}$$

Next, we take covariant differentiation of (5.15) in the direction of X to get

$$(\nabla_X S)(Y, Z) = \frac{1}{m}[g(Y, V)g(\nabla_X V, Z) + g(Z, V)g(\nabla_X V, Y)]. \tag{5.16}$$

Taking cyclic permutation of (5.16) over $\{X, Y, Z\}$ and making use of the fact that V is Killing, we deduce S is cyclically parallel, i.e.

$$\bigoplus_{X,Y,Z} (\nabla_X S)(Y, Z) = 0.$$

This finishes the proof of item (i).

By assumption $|V|$ is constant. Since g represents a non trivial m-quasi-Einstein metric we have $|V| \neq 0$ on M. Hence from $|V|^2\mathrm{div}V = 0$ it follows that $\mathrm{div}V = 0$. Thus, by virtue of (5.13) we can conclude that the scalar curvature r is constant. Hence the rest of the proof follows from item (i). $\square$

It is known that if the Ricci tensorof a Riemannian metric is of Codazzi type then the scalar curvature is constant. Thus, as a consequence of Theorem 5.2, we have

Corollary 5.1. *Let (M, V, g) be an m-quasi-Einstein manifold of dimension n such that the Ricci tensor S is Codazzi. If V is an infinitesimal harmonic transformation, then either M is Einstein, or V is parallel.*

Proof. Since the scalar curvature r is constant, from Theorem 5.2, the vector field V is Killing, and hence $\mathrm{div}V = 0$. Consequently, Eq. (5.10) implies that $g(\nabla_V V, Z) = g(\nabla_Z V, V) = 0$. This shows that $|V|$ is constant. On the other hand, since the Ricci tensor is cyclically parallel and Codazzi, we must have S is parallel. Thus, it follows from (5.16) that $g(Y, V)g(\nabla_X V, Z) + g(Z, V)g(\nabla_X V, Y) = 0$. Taking V instead of Z in the foregoing equation and since $|V|$ is constant, we see that $|V|^2 g(\nabla_X V, Y) = 0$. Thus, we have either

$|V| = 0$ or, $|V| \neq 0$. The former shows that M is Einstein and the latter implies that V is parallel. □

It is known [16] that if the potential vector field of a compact m-quasi-Einstein manifold is conformal Killing, then it must be Killing. Waiving the compactness assumption here we prove

Theorem 5.3. *Let (M^n, g, V) be an m-quasi-Einstein manifold with constant scalar curvature. If V is conformal, then either V is Killing or (M^n, g) is Einstein.*

Proof. By hypothesis V is conformal. Thus, there exists a smooth function ρ such that $(\pounds_V g)(X, Y) = g(\nabla_X V, Y) + g(\nabla_Y V, X) = 2\rho g(X, Y)$. Therefore, Eq. (1.2) reduces to

$$S(X, Y) = (\lambda - \rho)g(X, Y) + \frac{1}{m} V^\flat(X)V^\flat(Y). \tag{5.17}$$

Differentiating this along an arbitrary vector field Z we get

$$(\nabla_Z S)(X, Y) = -(Z\rho)g(X, Y) + \frac{1}{m}\{V^\flat(X)(\nabla_Z V^\flat)Y + V^\flat(Y)(\nabla_Z V^\flat)X\}.$$

Taking the cyclic sum over $\{X, Y, Z\}$ and remembering that V is conformal the preceding equation yields

$$\bigoplus_{X,Y,Z} [(\nabla_Z S)(X, Y) + (Z\rho)g(X, Y) - \frac{2\rho}{m} V^\flat(X)g(Y, Z)] = 0, \tag{5.18}$$

where $\bigoplus_{X,Y,Z}$ denotes one more time the cyclic permutation sum over $\{X, Y, Z\}$. Contracting (5.18) over Y, Z provides

$$\frac{2}{n+2}(Xr) + (X\rho) - \frac{2\rho}{m} V^\flat(X) = 0. \tag{5.19}$$

By virtue of this, Eq. (5.18) transforms into

$$\bigoplus_{X,Y,Z} [(\nabla_Z S)(X, Y) - \frac{2}{n+2}(Xr)] = 0.$$

Moreover, the scalar curvature being constant, the foregoing equation entails that the Ricci tensor is cyclically parallel. At this point, we rewrite Eq. (5.17) as

$$S(X, Y) = \alpha g(X, Y) + \beta\omega(X)\omega(Y), \tag{5.20}$$

where $\alpha = \lambda - \rho$, $\beta = \frac{|V|^2}{m}$ and ω is a 1-form associated with the unit vector field $\omega^\sharp$. Now, the covariant differentiation of (5.20) along an arbitrary vector field gives

$$(\nabla_Z S)(X, Y) = (Z\alpha)g(X, Y) + (Z\beta)\omega(X)\omega(Y) \\ +\beta\{\omega(Y)(\nabla_Z\omega)X + \omega(X)(\nabla_Z\omega)Y\}. \tag{5.21}$$

Since the Ricci tensor is cyclically parallel Eq. (5.21) entails

$$\bigoplus_{X,Y,Z} [(X\alpha)g(Y, Z) + (X\beta)\omega(Y)\omega(Z) \\ +\beta\{\omega(Y)(\nabla_X\omega)Z + \omega(Z)(\nabla_X\omega)Y\}] = 0. \tag{5.22}$$

Setting $Y = Z = \omega^\sharp$, where $\omega^\sharp$ is the vector field defined by $g(\omega^\sharp, X) = \omega(X)$, for all X, in (5.22) we find

$$(X\alpha) + (X\beta) + 2\{(\omega^\sharp\alpha) + (\omega^\sharp\beta)\}\omega(X) + 2\beta(\nabla_{\omega^\sharp}\omega)X = 0. \tag{5.23}$$

On the other hand, contracting (5.21) and remembering that the scalar curvature is constant we obtain

$$X\alpha + (\omega^\sharp\beta)\omega(X) + \beta[(\nabla_{\omega^\sharp}\omega)X - \delta\omega\omega(X)] = 0. \tag{5.24}$$

Tracing (5.20) and then differentiating yields

$$n(X\alpha) + (X\beta) = 0. \tag{5.25}$$

Replacing X by $\omega^\sharp$ in (5.23) shows $(\omega^\sharp\alpha) + (\omega^\sharp\beta) = 0$. Using this in (5.25) provides $\omega^\sharp\alpha = \omega^\sharp\beta = 0$. By virtue of this and taking $X = \omega^\sharp$ in (5.24) one easily verifies $\beta\delta\omega = 0$. Making use of this and $(\omega^\sharp\beta) = 0$ in (5.24) provides $X\alpha + \beta(\nabla_{\omega^\sharp}\omega)X = 0$. Using all these consequences it follows from (5.23) that $X\alpha - X\beta = 0$. Combining this with (5.25) implies $X\alpha = X\beta = 0$. This shows that the functions α and β are constant. In other words, ρ and $|V|^2$ are constant on M. As V is conformal $\rho|V|^2 = 0$. Thus, either $\rho = 0$ or $V = 0$. The former shows that V is Killing and the latter implies that M is Einstein. This establishes the proof. □

Waving the condition on the scalar curvature and assuming that the potential vector field V is homothetic, i.e., $\pounds_V g = \rho g$ with ρ a constant, we prove

Corollary 5.2. *Let (M^n, g, V) be an m-quasi-Einstein manifold. If V is a homothetic vector field, then either V is Killing or, (M^n, g) is Einstein.*

Proof. As V is homothetic, we have

$$g(\nabla_X V, Y) + g(\nabla_Y V, X) = 2\rho g(X, Y), \tag{5.26}$$

where ρ is a constant. Therefore, from (5.19) it follows that

$$\frac{2}{n+2}(Xr) - \frac{2\rho}{m}V^\flat(X) = 0. \tag{5.27}$$

Writing this as: $\frac{1}{n+2}\mathrm{d}r = \frac{\rho}{m}V^\flat$, and applying d (operator of exterior differentiation) to this, we achieve $\rho \mathrm{d}V^\flat = 0$. If $\rho = 0$, then V is Killing and if $\rho \neq 0$, then $V^\flat$ is closed. The last condition together with (5.26)implies $\nabla_X V = \rho X$. Making use of this we deduce $R(X, Y)V = 0$. From which it follows that $S(Y, V) = 0$. Use of this in (5.17) gives $\{(\lambda - \rho) - \frac{1}{m}|V|^2\}|V|^2 = 0$. If $V = 0$, then M is Einstein. So, we assume that $|V|^2 \neq 0$ in some open set N of M. Then on N, we have $(\lambda - \rho) - \frac{1}{m}|V|^2 = 0$. Now, the trace of (5.17) gives $r = n(\lambda - \rho) - \frac{1}{m}|V|^2$. The last two equations imply that $r = (n-1)(\lambda - \rho)$ and clearly this is constant. Hence, from (5.27) we obtain $V = 0$. Thus we arrive at a contradiction. This completes the proof. □

Acknowledgements

The author is grateful to the reviewers for their valuable comments and suggestions.

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Received: May 6, 2019.

Revised: January 31, 2020.

Accepted: June 6, 2020.