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The Bochner–Schoenberg–Eberlein Property for Vector-Valued ℓ^p-Spaces

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Abstract. Let X be a non-empty set, $\mathcal{A}$ be a commutative Banach algebra, and $1 \leq p < \infty$. In this paper, we establish some basic properties of $\ell^p(X, \mathcal{A})$, inherited from $\mathcal{A}$. In particular, we characterize the Gelfand space of $\ell^p(X, \mathcal{A})$, denoted by $\Delta(\ell^p(X, \mathcal{A}))$. Mainly, we investigate the BSE property of the Banach algebra $\ell^p(X, \mathcal{A})$. In fact, we prove that $\ell^p(X, \mathcal{A})$ is a BSE algebra if and only if X is finite and $\mathcal{A}$ is a BSE algebra. Furthermore, in the case that $\mathcal{A}$ is unital, we show that for any natural number n, all continuous bounded functions on $\Delta(\ell^p(X, \mathcal{A}))$ are n-BSE functions. However, through an example, we indicate that there is some continuous bounded function on $\Delta(\ell^p(X, \mathcal{A}))$ which is not BSE. Finally, we prove that if $\ell^1(X, \mathcal{A})$ is a BSE-norm algebra, then $\mathcal{A}$ is so. We also prove the converse of this statement, whenever $\mathcal{A}$ is a supremum norm algebra.

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1. Introduction

The notion of BSE algebras and BSE functions was first introduced and studied by Takahashi and Hatori in 1990 [18] and subsequently by several authors for various kinds of Banach algebras, such as Fourier and Fourier–Stieltjes algebras, semigroup algebras, abstract Segal algebras, etc. The interested reader is referred to [5,8,11–14,19,20]. Moreover, in a recent work, Dabhi and Upadhyay proved that $\ell^1(\mathbb{Z}^2, \max)$ is a BSE algebra [4]. Furthermore, in [1], we investigated the BSE property for vector-valued Lipschitz algebra $\mathrm{Lip}_\alpha(X, \mathcal{A})$, and proved that for unital commutative semisimple Banach algebra $\mathcal{A}$, $\mathrm{Lip}_\alpha(X, \mathcal{A})$ is a BSE algebra if and only if $\mathcal{A}$ is so.

The acronym BSE stands for Bochner–Shoenberg–Eberlein famous theorem which characterizes the Fourier–Stieltjes transforms of the bounded Borel measures on locally compact abelian groups; that is, in fact, the BSE property of the group algebra $L^1(G)$ for a locally compact abelian group G; see [2,7,17]. This has led the Japanese mathematicians to introduce the BSE property for an arbitrary commutative Banach algebra as follows:

Let $\mathcal{A}$ be a commutative Banach algebra. Denote by $\Delta(\mathcal{A})$ to be the Gelfand space of $\mathcal{A}$; i.e., the space consisting of all nonzero multiplicative linear functionals on $\mathcal{A}$.

A bounded continuous function σ on $\Delta(\mathcal{A})$ is called a BSE function if there exists a constant $C > 0$, such that for every finite number of $\varphi_1, \ldots, \varphi_n$ in $\Delta(\mathcal{A})$ and complex numbers $c_1, \ldots, c_n$, the inequality:

$$\left|\sum_{j=1}^{n} c_j \sigma(\varphi_j)\right| \leq C \left\|\sum_{j=1}^{n} c_j \varphi_j\right\|_{\mathcal{A}^*}$$

holds. The BSE norm of σ ($\|\sigma\|_{\mathrm{BSE}}$) is defined to be the infimum of all such C. The set of all BSE functions is denoted by $C_{\mathrm{BSE}}(\Delta(\mathcal{A}))$. Takahasi and Hatori [18] showed that under the norm $\|.\|_{\mathrm{BSE}}$, $C_{\mathrm{BSE}}(\Delta(\mathcal{A}))$ is a commutative semisimple Banach algebra, embedded in $C_b(\Delta(\mathcal{A}))$ as a subalgebra.

Here, we provide some preliminaries, which will be required throughout the paper. See [15] for more information. A bounded linear operator on a commutative Banach algebra $\mathcal{A}$ is called a *multiplier* if it satisfies $xT(y) = T(xy)$, for all $x, y \in$. The set $M(\mathcal{A})$ of all multipliers of $\mathcal{A}$ is a unital commutative Banach algebra, called the *multiplier algebra* of $\mathcal{A}$. Set:

$$\widehat{M(\mathcal{A})} = \{\widehat{T} : T \in M(\mathcal{A})\}.$$

Remark 1.1. Let $\mathcal{A}$ be a commutative semisimple Banach algebra. Suppose that $\Phi : \Delta(\mathcal{A}) \to \mathbb{C}$ be a continuous function, such that $\Phi.\widehat{\mathcal{A}} \subseteq \widehat{\mathcal{A}}$. We call Φ a multiplier of $\mathcal{A}$. This is another definition of a multiplier of a Banach algebra. In the presence of supersimplicity, this definition is equivalent to the above definition, by considering $\Phi = \widehat{T}$; see [16] for more details. Define:

$$\mathcal{M}(\mathcal{A}) = \{\Phi : \Delta(\mathcal{A}) \to \mathcal{C} : \Phi \text{ is continuous and } \Phi.\widehat{\mathcal{A}} \subseteq \widehat{\mathcal{A}}\}.$$

When $\mathcal{A}$ is semisimple, $\widehat{M(\mathcal{A})} = \mathcal{M}(\mathcal{A})$.

A commutative Banach algebra $\mathcal{A}$ is called without order if $a\mathcal{A} = \{0\}$ implies $a = 0$ ($a \in \mathcal{A}$). A commutative and without order Banach algebra $\mathcal{A}$ is called a BSE algebra (or has the BSE property) if it satisfies the condition:

$$C_{\mathrm{BSE}}(\Delta(\mathcal{A})) = \widehat{M(\mathcal{A})}.$$

Furthermore, $\mathcal{A}$ is called a BSE algebra of type I if:

$$C_{\mathrm{BSE}}(\Delta(\mathcal{A})) = \widehat{M(\mathcal{A})} = C_b(\Delta(\mathcal{A})).$$

By Remark 1.1, in the case that $\mathcal{A}$ is a semisimple commutative Banach algebra, the BSE property of $\mathcal{A}$ is equivalent to the following equality:

$$C_{\mathrm{BSE}}(\Delta(\mathcal{A})) = \mathcal{M}(\mathcal{A}).$$

It is worthy to note that all semisimple Banach algebras are without order.

Let X be an arbitrary non-empty set and consider the Banach algebra $\ell^1(X)$ with pointwise multiplication. In [19], the authors proved that the Banach algebra $\ell^1(X)$ is BSE if and only if X is finite.

Throughout the paper, let X be a non-empty set and $\mathcal{A}$ be a commutative Banach algebra. In this paper, at first, we investigate several properties of the vector-valued Banach algebra $\ell^p(X, \mathcal{A})$ $(1 \leq p < \infty)$, inherited from $\mathcal{A}$. Moreover, we characterize the Gelfand space of the Banach algebra $\ell^p(X, \mathcal{A})$ by the set X and the Gelfand space of $\mathcal{A}$. Then, we present necessary and sufficient conditions for $\ell^p(X, \mathcal{A})$ to be a BSE algebra. In fact, we prove that $\ell^p(X, \mathcal{A})$ is a BSE algebra if and only if X is finite and $\mathcal{A}$ is a BSE algebra. Furthermore, we show that for any $n \in \mathbb{N}$ and a unital Banach algebra $\mathcal{A}$, the Banach algebra $C_{\mathrm{BSE}(n)}(\Delta(l^p(X, \mathcal{A}))) = C_{\mathrm{BSE}(n)}(X \times \Delta(\mathcal{A}))$ is equal to the Banach algebra $C_b(X \times \Delta(\mathcal{A}))$. However, with an example, we show that this result is not true for $C_{\mathrm{BSE}}(X \times \Delta(\mathcal{A}))$, even for a unital Banach $\mathcal{A}$. Moreover, we investigate BSE-norm property for $\ell^1(X, \mathcal{A})$ and prove that if $\ell^1(X, \mathcal{A})$ is a BSE-norm algebra, then $\mathcal{A}$ is so. We also prove that the converse of this results is valid, whenever $\mathcal{A}$ is a supremum norm algebra.

Finally, we present a different proof, from abstract Segal algebras point of view, to show that $\ell^p(X)$ is a BSE algebra if and only if X is finite.

2. Some Basic Properties $\ell^p(X, \mathcal{A})$ Inherited from $\mathcal{A}$

Let X be a non-empty set, $\mathcal{A}$ be a commutative Banach algebra, and $1 \leq p < \infty$. Let:

$$\ell^p(X, \mathcal{A}) = \left\{ f : X \to \mathcal{A} : \sum_{x \in X} \|f(x)\|^p < +\infty \right\}.$$

It is easily verified that $\ell^p(X, \mathcal{A})$ is a commutative Banach algebra, endowed with the norm:

$$\|f\|_p = \left(\sum_{x \in X} \|f(x)\|^p \right)^{1/p} \qquad (f \in \ell^p(X, \mathcal{A}))$$

and pointwise product. In this section, we investigate some elementary and basic properties about $\ell^p(X, \mathcal{A})$, which will be useful for further results. Let us first introduce some noteworthy vector-valued functions on X, which play an important role in our results. For any finite subset F of X and nonzero element $a \in \mathcal{A}$, we define the function δ_a^F as follows:

$$\delta_a^F(t) = \begin{cases} a & t \in F \\ 0 & t \notin F. \end{cases}$$

These functions belong clearly to $\ell^p(X, \mathcal{A})$. In the case that F is a singleton, namely $F = \{x\}$, then we simply rewrite δ_a^F as δ_a^x.

Proposition 2.1. *Let X be a set, $\mathcal{A}$ be a commutative Banach algebra, and $1 \leq p < \infty$. Then, $\ell^p(X, \mathcal{A})$ is unital if and only if X is finite and $\mathcal{A}$ is unital.*

Proof. At first, suppose that X is finite and $\mathcal{A}$ has an identity e. It is not hard to see that the constant function:

$$I : X \to A, \quad x \mapsto e$$

is the identity element of $\ell^p(X, \mathcal{A})$. Conversely, suppose that $I \in \ell^p(X, \mathcal{A})$ is the identity element of $\ell^p(X, \mathcal{A})$. Then, for each $f \in \ell^p(X, \mathcal{A})$, we have:

$$f(x)I(x) = f(x) \quad (x \in X).$$

Specially:

$$\|I(x)\| = \|I(x)I(x)\| \leq \|I(x)\| \, \|I(x)\| \quad (x \in X).$$

Thus, for each $x \in X$, $I(x) = 0$ or $\|I(x)\| \geq 1$. Note that since $I \in \ell^p(X, \mathcal{A})$, $I(x) = 0$, except for finitely many $x_1, \ldots, x_n \in X$. Now, we show that:

$$X = \{x_1, \ldots, x_n\}.$$

Suppose on the contrary that there exists $x \in X$, such that $x \notin \{x_1, \ldots, x_n\}$. Take $a \in \mathcal{A}$ to be nonzero and consider the function $\delta_a^x(t)$. Thus:

$$0 = I(x)\delta_a^x(x) = \delta_a^x(x) = a,$$

which is impossible. It follows that $X = \{x_1, \ldots, x_n\}$. In the sequel, we show that $\mathcal{A}$ is unital. For all $x \in X$ and $a \in \mathcal{A}$, we have:

$$I(t)\delta_x^a(t) = \delta_a^x(t) \quad (t \in X).$$

Consequently:

$$I(x)a = a \quad (x \in X, a \in \mathcal{A}).$$

It follows that I is a constant function. Indeed, for all $x, y \in X$ with $x \neq y$:

$$I(x)I(y) = I(x) = I(y).$$

Therefore, $I(x)$ is the identity element of $\mathcal{A}$. □

Proposition 2.2. *Let X be a set, $\mathcal{A}$ be a commutative Banach algebra, and $1 \leq p < \infty$. Then, $\ell^p(X, \mathcal{A})$ is without order if and only if $\mathcal{A}$ is without order.*

Proof. First, suppose that $\mathcal{A}$ is without order and $0 \neq f \in \ell^p(X, \mathcal{A})$. Then, there exists $x_0 \in X$, such that $f(x_0) = a \neq 0$. By the hypothesis, there exists $b \in \mathcal{A}$ such that $ab \neq 0$. It follows that:

$$f(x_0)\delta_b^{x_0}(x_0) = ab \neq 0,$$

and so, $f \ \delta_b^{x_0} \neq 0$. Consequently, $\ell^p(X, \mathcal{A})$ is without order. Conversely, suppose that $\ell^p(X, \mathcal{A})$ is without order and $0 \neq a \in \mathcal{A}$. For any $x_0 \in X$, we have:

$$\delta_{x_0}^a(x_0) = a \neq 0.$$

By the hypothesis, there exists $f \in \ell^p(X, \mathcal{A})$, such that $f \ \delta_{x_0}^a \neq 0$. Thus:

$$f(x_0)a = f(x_0)\delta_{x_0}^a(x_0) \neq 0.$$

Take $b := f(x_0)$. It follows that $ba \neq 0$. Therefore, $\mathcal{A}$ is without order. □

Theorem 2.3. *Let X be a set, $\mathcal{A}$ be a commutative Banach algebra, and $1 \leq p < \infty$. Then, the Gelfand space of $\ell^p(X, \mathcal{A})$ is homeomorphic to $X \times \Delta(\mathcal{A})$.*

Proof. Define the function Θ as:

$$\Theta : X \times \Delta(\mathcal{A}) \to \Delta(\ell^p(X, \mathcal{A}))$$
$$(x, \varphi) \mapsto \Theta_{(x,\varphi)},$$

where:

$$\Theta_{(x,\varphi)}(f) = \varphi(f(x)) \quad (f \in \ell^p(X, \mathcal{A})).$$

It is obvious that $\Theta_{(x,\varphi)} \in \Delta(\ell^p(X, \mathcal{A}))$ and so Θ is well defined. Now, we show that Θ is injective. Suppose that $\Theta_{(x,\varphi)} = \Theta_{(y,\psi)}$, for some $x, y \in X$ and $\varphi, \psi \in \Delta(\mathcal{A})$. Then, for any $f \in \ell^p(X, \mathcal{A})$, we have $\varphi(f(x)) = \psi(f(y))$. For each $a \in \mathcal{A}$, consider the function $\delta_a^{\{x,y\}}$. Thus:

$$\varphi(\delta_a^{\{x,y\}}(x)) = \psi\left(\delta_a^{\{x,y\}}(y)\right).$$

It follows that:

$$\varphi(a) = \psi(a) \quad (a \in A),$$

and we obtain $\varphi = \psi$. Moreover, the equality $\varphi(f(x)) = \varphi(f(y))$ $(f \in \ell^p(X, \mathcal{A}))$, which implies that $\varphi(f(x) - f(y)) = 0$ for each $f \in \ell^p(X, \mathcal{A})$. If $x \neq y$, then:

$$\varphi(\delta_a^x(x) - \delta_a^x(y)) = 0 \quad (a \in \mathcal{A}).$$

This implies that $\varphi(a) = 0$ for all $a \in A$ and so $\varphi = 0$. This contradiction implies that $x = y$. Consequently, Θ is injective. To prove the surjectivity, let $\Phi \in \Delta(\ell^p(X, \mathcal{A}))$. Since Φ is nonzero, there exists $f = \sum_{t \in X} \delta_{f(t)}^t$, such that $\Phi(f) \neq 0$. It follows that $\Phi(\delta_a^{x_0}) \neq 0$, for some $x_0 \in X$. Such $x_0 \in X$ is unique. Indeed, let there exists $x \neq x_0$, such that $\Phi(\delta_a^x) \neq 0$. Since $\delta_a^{x_0}.\delta_a^x = 0$, we have:

$$0 = \Phi(\delta_a^{x_0}.\delta_a^x) = \Phi(\delta_a^{x_0})\Phi(\delta_a^x) \neq 0,$$

which is a contradiction. Now, define:

$$\varphi_0 : \mathcal{A} \to \mathbb{C}$$
$$\varphi_0(a) = \Phi(\delta_a^{x_0}).$$

We show that $\Phi = \Theta_{(x_0,\varphi_0)}$. For $f \in \ell^p(X, \mathcal{A})$, we may rewrite f as:

$$f(x) = \sum_{t \in X} \delta_{f(t)}^t(x).$$

Thus, we obtain:

$$\Theta_{(x_0,\varphi_0)}(f) = \varphi_0(f(x_0)) = \Phi\left(\delta_{f(x_0)}^{x_0}\right) = \Phi(f).$$

This implies that Θ is surjective. To prove the continuity of Θ, consider the net $\{(x_\alpha, \varphi_\alpha)\}_{\alpha \in \Lambda}$ converges to (x, φ), in the topology of $X \times \Delta(A)$. So that there exists $\alpha_0 \in \Lambda$, such that for all $\alpha \geq \alpha_0$, $x_\alpha = x$. Moreover, for each $f \in \ell^p(X, A)$, we have:

$$\lim_\alpha \Theta_{(x,\varphi_\alpha)}(f) = \lim_\alpha \varphi_\alpha(f(x)) = \varphi(f(x)) = \Theta_{(x,\varphi)}(f).$$

Thus, Θ is continuous. For openness, let $\Theta_{(x_\alpha,\varphi_\alpha)}$ tends to $\Theta(x,\varphi)$, in the Gelfand topology of $\Delta(\ell^p(X,\mathcal{A}))$. It follows that for any $f\in\ell^p(X,\mathcal{A})$:

$$\lim_\alpha \Theta_{(x_\alpha,\varphi_\alpha)}(f)=\Theta_{(x,\varphi)}(f),$$

and so:

$$\lim_\alpha \varphi_\alpha(f(x_\alpha))=\varphi(f(x)).$$

In particular, for $a\in A$ with $\varphi(a)\neq 0$, we have:

$$\lim_\alpha \varphi_\alpha(\delta_a^x(x_\alpha))=\varphi(\delta_a^x(x))=\varphi(a).$$

Consequently, for $\varepsilon=\dfrac{|\varphi(a)|}{2}$, there exists $\alpha_0\in\Lambda$, such that for all $\alpha\geq\alpha_0$:

$$|\varphi_\alpha(\delta_a^x(x_\alpha))-\varphi(a)|<\frac{|\varphi(a)|}{2}.$$

We show that there exists $\alpha_1\in\Lambda$, such that for any $\alpha\geq\alpha_1$, $x_\alpha=x$. Suppose on the contrary that for any $\alpha\in\Lambda$, there exists $\beta_\alpha\geq\alpha$ such that $x_{\beta_\alpha}\neq x$. It follows that there exists $\beta_{\alpha_0}\geq\alpha_0$, such that $x_{\beta_{\alpha_0}}\neq x$. Thus:

$$\left|\varphi_\alpha(\delta_a^x(x_{\beta_{\alpha_0}}))-\varphi(a)\right|<\frac{|\varphi(a)|}{2}.$$

Since $\delta_a^x(x_{\beta_{\alpha_0}})=0$, this implies that $\varphi(a)=0$, which is a contradiction. So that there exists $\alpha_1\in\Lambda$, such that $x_\alpha=x$, for any $\alpha\geq\alpha_1$. This means that the net $\{x_\alpha\}_{\alpha\in\Lambda}$ tends to x, in the discrete topology of X. Furthermore, for $\alpha\geq\alpha_1$, $\delta_a^x(x_\alpha)=a$, which implies that $\lim_\alpha\varphi_\alpha(a)=\varphi(a)$. Consequently, $\{\varphi_\alpha\}_{\alpha\in\Lambda}$ tends to φ, in the Gelfand topology of $\Delta(\mathcal{A})$. This completes the proof. □

Proposition 2.4. *Let X be a set, $\mathcal{A}$ be a commutative Banach algebra, and $1\leq p<\infty$. Then, $\ell^p(X,\mathcal{A})$ is semisimple if and only if $\mathcal{A}$ is semisimple.*

Proof. Let $\ell^p(X,\mathcal{A})$ be semisimple and $0\neq a\in\mathcal{A}$. Then, for any $x\in X$, $\delta_x^a\neq 0$. Define:

$$\Theta: X\times\Delta(A)\to\Delta(\ell^p(X,\mathcal{A}))$$
$$(x,\varphi)\mapsto\Theta_{(x,\varphi)},$$

where

$$\Theta_{(x,\varphi)}(f)=\varphi(f(x))\quad(x\in X,\varphi\in\Delta(\mathcal{A})).$$

Then, there exists $\varphi\in\Delta(A)$, such that $\Theta_{(x,\varphi)}(\delta_x^a)=\varphi(\delta_x^a(x))\neq 0$. This follows that $\varphi(a)\neq 0$ and $\mathcal{A}$ is semisimple. Conversely, suppose that $\mathcal{A}$ is semisimple and $0\neq f\in\ell^p(X,A)$. There exists $x_0\in X$, such that $f(x_0)\neq 0$. Since $\mathcal{A}$ is semisimple, there exists $\varphi\in\mathcal{A}$, such that $\varphi(f(x_0))\neq 0$. This means that $\Theta_{(x_0,\varphi)}(f)\neq 0$. □

A bounded net $(e_\alpha)_{\alpha\in I}$ in a Banach algebra $\mathcal{A}$, satisfying the condition:

$$\lim_\alpha\varphi(xe_\alpha)=\varphi(x),$$

for every $x\in\mathcal{A}$ and $\varphi\in\Delta(\mathcal{A})$, is called Δ-weak bounded approximate identity for $\mathcal{A}$, in the sense of Jones-Lahr; see [6,9].

Remark 2.5. Let X be a set. In [19, Theorem 5], it has been proved that $\ell^1(X)$ has a Δ-weak bounded approximate identity if and only if X is finite. One can follow the exact arguments to prove this result for $\ell^p(X)$, where $1 \leq p < \infty$.

In the following result, we generalize [19, Theorem 5], for the vector-valued case.

Theorem 2.6. *Let X be a set, $\mathcal{A}$ be a commutative Banach algebra and $1 \leq p < \infty$. Then, $\ell^p(X, \mathcal{A})$ has a Δ-weak bounded approximate identity if and only if X is finite and $\mathcal{A}$ has a Δ-weak bounded approximate identity.*

Proof. First suppose that $X = \{x_1, \ldots, x_n\}$ is finite and $(e_\alpha)_{\alpha \in I}$ is a Δ-weak bounded approximate identity for $\mathcal{A}$ with $\sup_{\alpha \in I} \|e_\alpha\| \leq \beta$. For any $\alpha \in I$, define the constant function f_α on X as:

$$f_\alpha(x) = e_\alpha \quad (x \in X).$$

It is easily verified that $f_\alpha \in \ell^p(X, \mathcal{A})$, for all $\alpha \in I$. Moreover, for all $i = 1, \ldots, n$ and $\varphi \in \Delta(\mathcal{A})$:

$$\lim_\alpha (x_i, \varphi)(f_\alpha) = \lim_\alpha \varphi(f_\alpha(x_i)) = \lim_\alpha \varphi(e_\alpha) = 1.$$

It follows that $(f_\alpha)_{\alpha \in I}$ is a bounded Δ-weak approximate identity for $\ell^p(X, \mathcal{A})$. Conversely, suppose that $(f_\alpha)_{\alpha \in I}$ is a bounded Δ-weak approximate identity for $\ell^p(X, \mathcal{A})$ with $\sup_{\alpha \in I} \|f_\alpha\|_p \leq \beta$. We first show that $\ell^p(X)$ has a bounded Δ-weak bounded approximate identity. For a fixed element $\psi \in \Delta(\mathcal{A})$, define:

$$g_\alpha := \psi \circ f_\alpha : X \to \mathbb{C} \quad (\alpha \in I).$$

Then, we have:

$$\begin{aligned} \sum_{x \in X} |g_\alpha(x)|^p &= \sum_{x \in X} |\psi \circ f_\alpha(x)|^p \\ &= \sum_{x \in X} |\psi(f_\alpha(x))|^p \\ &\leq \|\psi\|^p \sum_{x \in X} \|f_\alpha(x)\|^p \\ &\leq \beta^p \|\psi\|^p. \end{aligned}$$

It means that $g_\alpha \in \ell^p(X)$, for all $\alpha \in I$ and:

$$\sup_\alpha \|g_\alpha\|_p \leq \beta \|\psi\|.$$

Furthermore, for any $x \in X$, we have:

$$\lim_\alpha \varphi_x(g_\alpha) = \lim_\alpha g_\alpha(x) = \lim_\alpha \psi(f_\alpha(x)) = \Theta_{(x,\psi)}(f_\alpha) = 1.$$

It follows that $(g_\alpha)_{\alpha \in I}$ is a Δ-weak bounded approximate identity for $\ell^p(X)$. Therefore, X is finite by Remark 2.5. Now, take $x_0 \in X$ to be fixed and for any $\alpha \in \Lambda$, and define $e_\alpha := f_\alpha(x_0) \in \mathcal{A}$. Thus, we have:

$$\|e_\alpha\| = \|f_\alpha(x_0)\| \leq \|f_\alpha\|_p \leq \beta.$$

Moreover, for any $\varphi \in \Delta(\mathcal{A})$:

$$\lim_{\alpha} \varphi(e_\alpha) = \lim_{\alpha} \varphi(f_\alpha(x_0)) = \lim_{\alpha} \Theta_{(x_0,\varphi)}(f_\alpha) = 1.$$

Therefore, $(e_\alpha)_{\alpha \in I}$ is a Δ-weak bounded approximate identity for $\mathcal{A}$. $\square$

3. The BSE Property for $\ell^p(X, \mathcal{A})$

In this section, we state the main result of the present paper. In fact, we provide a necessary and sufficient condition for $\ell^p(X, \mathcal{A})$ to be a BSE algebra.

Theorem 3.1. *Let X be a set and $\mathcal{A}$ be a commutative semisimple Banach algebra. Then, $\ell^p(X, \mathcal{A})$ is a BSE algebra if and only if X is finite and $\mathcal{A}$ is a BSE algebra.*

Proof. First, suppose that $\ell^p(X, \mathcal{A})$ is a BSE algebra. Then, by [18, Corollary 5], $\ell^p(X, \mathcal{A})$ admits a Δ-weak bounded approximate identity. Proposition 2.6 implies that X is finite and $\mathcal{A}$ has a Δ-weak bounded identity. Again, by [18, Corollary 5], we have:

$$\mathcal{M}(A) \subseteq C_{\mathrm{BSE}}(\Delta(\mathcal{A})). \tag{3.1}$$

Now, we prove the reverse of inclusion (3.1). Suppose that $\sigma \in C_{\mathrm{BSE}}(\Delta(\mathcal{A}))$. Then, there exists a bounded net $(a_\lambda)_{\lambda \in \Lambda} \subseteq \mathcal{A}$, such that for each $\varphi \in \Delta(\mathcal{A})$, $\lim_\lambda \widehat{a_\lambda}(\varphi) = \sigma(\varphi)$. For any $\lambda \in \Lambda$, define the constant function $f_\lambda : X \to \mathcal{A}$ by $f_\lambda(x) = a_\lambda$, $(x \in X)$. Since X is finite, then $f_\lambda \in \ell^p(X, \mathcal{A})$, for all $\lambda \in \Lambda$. Moreover:

$$\lim_{\lambda} \Theta_{(x,\varphi)}(f_\lambda) = \lim_{\lambda} \varphi(f_\lambda(x)) = \lim_{\lambda} \varphi(a_\lambda) = \sigma(\varphi) \quad (x \in X, \varphi \in \Delta(\mathcal{A})).$$

Define the function σ' as follows:

$$\begin{aligned} \sigma' : X \times \Delta(A) &\longrightarrow \mathbb{C} \\ \sigma'(x, \varphi) &= \sigma(\varphi), \end{aligned}$$

for all $x \in X$ and $\varphi \in \Delta(\mathcal{A})$. Thus, we have:

$$\sigma'(x, \varphi) = \sigma(\varphi) = \lim_{\lambda} \widehat{a_\lambda}(\varphi) = \lim_{\lambda} \widehat{f_\lambda} \Theta_{(x,\varphi)} \quad (x \in X, \varphi \in \Delta(\mathcal{A})).$$

This implies that $\sigma' \in C_{\mathrm{BSE}}(\Delta(\ell^p(X, \mathcal{A})))$. Since $\ell^p(X, \mathcal{A})$ is a BSE algebra, it follows that $\sigma' \in \mathcal{M}(\ell^p(X, \mathcal{A}))$. Now, take $a \in \mathcal{A}$ and consider the constant function $f : X \to \mathcal{A}$ defined by $f(x) = a$ $(x \in X)$. Then, $f \in \ell^p(X, \mathcal{A})$, and so there exists $g \in \ell^p(X, \mathcal{A})$, such that $\sigma' \widehat{f} = \widehat{g}$. Consequently for all $x \in X$ and $\varphi \in \Delta(\mathcal{A})$:

$$\sigma'(x, \varphi) \widehat{f}(x, \varphi) = \widehat{g}(x, \varphi).$$

It follows that:

$$\sigma(\varphi)\varphi(f(x)) = \sigma(\varphi)\varphi(a) = \varphi(g(x)), \tag{3.2}$$

and so:

$$\varphi(g(x)) = \varphi(g(y)) \quad (x, y \in X, \varphi \in \Delta(\mathcal{A})).$$

Semisimplicity of $\mathcal{A}$ implies that $g(x) = g(y)$, for all $x, y \in X$. So that g is a constant function and thus $g(x) = b$ $(x \in X)$, for some $b \in \mathcal{A}$. Now, the equality (3.2) implies that:

$$\sigma(\varphi)\widehat{a}(\varphi) = \sigma(\varphi)\varphi(a) = \varphi(g(x)) = \varphi(b) = \widehat{b}(\varphi).$$

Therefore, $\sigma\,\widehat{a} = \widehat{b}$, and so, $\sigma \in \mathcal{M}(\mathcal{A})$, as claimed.

Conversely, suppose that X is finite and $\mathcal{A}$ is a BSE algebra. By [18, Corollary 5], $\mathcal{A}$ has a Δ-weak bounded approximate identity. By proposition 2.6, $\ell^p(X, \mathcal{A})$ also has a Δ-weak bounded approximate identity. Again, by [18, Corollary 5], we have:

$$\mathcal{M}(\ell^p(X, \mathcal{A})) \subseteq C_{\mathrm{BSE}}(\Delta(\ell^p(X, \mathcal{A}))).$$

For the reverse of the above inclusion, suppose that $\sigma \in C_{\mathrm{BSE}}(\Delta(\ell^p(X, \mathcal{A})))$. We show that $\sigma \in \mathcal{M}(\ell^p(X, \mathcal{A}))$. To that end, take $h \in \ell^p(X, \mathcal{A})$. We find $g \in \ell^p(X, \mathcal{A})$, such that $\sigma\,\widehat{h} = \widehat{g}$. By [18, Theorem 4], there exists a bounded net $(f_\lambda)_{\lambda\in\Lambda}$ in $\ell^p(X, \mathcal{A})$, such that $\sup_\lambda \|f_\lambda\|_p \leq \beta$, for some $\beta > 0$, and:

$$\lim_\lambda \widehat{f_\lambda}(x, \varphi) = \sigma(x, \varphi) \quad (x \in X, \varphi \in \Delta(\mathcal{A})).$$

Thus:

$$\lim_\lambda \varphi(f_\lambda(x)) = \widehat{f_\lambda(x)}(\varphi) = \sigma(x, \varphi).$$

For each $x \in X$, we define the function σ_x as follows:

$$\begin{aligned} \sigma_x : \Delta(\mathcal{A}) &\longrightarrow \mathbb{C} \\ \sigma_x(\varphi) \;&=\; \sigma(x, \varphi) = \lim_\lambda \widehat{f_\lambda(x)}(\varphi). \end{aligned}$$

This follows from [18, Theorem 4] that $\sigma_x \in C_{\mathrm{BSE}}(\Delta(\mathcal{A}))$. Since $\mathcal{A}$ is a BSE algebra, $\sigma_x \in \mathcal{M}(\mathcal{A})$. Thus, for each $x \in X$, there exists $a_x \in \mathcal{A}$, such that $\sigma_x\widehat{h(x)} = \widehat{a_x}$. Now, define the function g on X as follows:

$$\begin{aligned} g : X &\longrightarrow A \\ x &\mapsto a_x. \end{aligned}$$

For all $x \in X$ and $\varphi \in \Delta(\mathcal{A})$, we have:

$$\begin{aligned} \sigma(x, \varphi)\widehat{h}(x, \varphi) &= \sigma_x(\varphi)\varphi(h(x)) \\ &= \sigma_x(\varphi)\widehat{h(x)}(\varphi) \\ &= \widehat{a_x}(\varphi) \\ &= \widehat{g(x)}(\varphi) \\ &= \widehat{g}(x, \varphi). \end{aligned}$$

Thus, $\sigma\,\widehat{h} = \widehat{g}$. So that $\ell^p(X, \mathcal{A})$ is a BSE algebra. □

Remark 3.2. In [19, Theorem 5], it is shown that for an arbitrary non-empty set X:

$$\widehat{\ell^1(X)} = C_{\mathrm{BSE}}(X) \subset \mathcal{M}(\ell^1(X)) = C_b(X).$$

Moreover, $C_{\mathrm{BSE}}(X)$ and $C_b(X)$ coincide if and only if X is finite. In fact, $\ell^1(X)$ is a BSE algebra of type I if and only if X is finite. Note that, by some similar arguments as in the proof of [19, Theorem 5], we can deduce the same results for $\ell^p(X)$ $(1 < p < \infty)$, as well. Moreover, it is easily verified that if X is finite and $\mathcal{A}$ is a unital BSE algebra, then:

$$\widehat{\ell^p(X,\mathcal{A})} = \mathcal{M}(\ell^p(X,\mathcal{A})) = C_{\mathrm{BSE}}(X \times \Delta(\mathcal{A})) \subseteq C_b(X \times \Delta(\mathcal{A})).$$

However, these equalities are not valid in general. For instance, take X to be a finite set and $\mathcal{A}$ to be a non-unital BSE algebra. Then:

$$\widehat{\ell^p(X,\mathcal{A})} \subsetneqq \mathcal{M}(\ell^p(X,\mathcal{A})) = C_{\mathrm{BSE}}(X \times \Delta(\mathcal{A})) \subseteq C_b(X \times \Delta(\mathcal{A})).$$

It is worth to note that even in the case that X is finite, $C_{\mathrm{BSE}}(X \times \Delta(\mathcal{A}))$ may not be equal to $C_b(X \times \Delta(\mathcal{A}))$, as the following example shows.

Example 3.3. Let X be a finite set with $card(X) = n > 1$ and $\mathcal{A} = \ell^\infty(X)$. Then, $\Delta(\ell^\infty(X)) = X$ and since $\ell^\infty(X)$ is a unital BSE algebra, it follows that $\ell^p(X, \ell^\infty(X))$ is also a unital BSE algebra. Consequently:

$$\widehat{\ell^p(X,\ell^\infty(X))} = C_{\mathrm{BSE}}(X \times X).$$

Suppose on the contrary that:

$$C_{\mathrm{BSE}}(X \times X) = C_b(X \times X). \tag{3.3}$$

It follows that $\ell^p(X, \ell^\infty(X))$ is a BSE algebra of type I, and so, by [18, Theorem 3], $\ell^p(X, \ell^\infty(X))$ is a C^*-algebra. Consider the function $f \in \ell^p(X, \ell^\infty(X))$, defined by $f(x) = \mathbf{1}$ $(x \in X)$, where $\mathbf{1} \in \ell^\infty(X)$ is the constant function $\mathbf{1}(x) = 1$ $(x \in X)$. Then:

$$\|f\ \bar{f}\|_p = \|f^2\|_p = n^{1/p} \neq \|f\|_p^2 = n^{2/p}.$$

This contradiction indicates that the equality (3.3) is not satisfied and:

$$C_{\mathrm{BSE}}(X \times X) \subsetneqq C_b(X \times X).$$

In other words, there are continuous bounded functions on $\Delta(\ell^p(X, \ell^\infty(X)))$ which are not BSE.

For a natural number n, a function $\sigma \in C_b(\Delta(\mathcal{A}))$ is called a n-BSE function, if there exists positive real numbers β (depending only on n), such that for any choice of $\varphi_1, \ldots, \varphi_n$ in $\Delta(\mathcal{A})$ and complex numbers $c_1, \ldots, c_n$, the inequality:

$$\left|\sum_{j=1}^{n} c_j \sigma(\varphi_j)\right| \leq C \ \left\|\sum_{j=1}^{n} c_j \varphi_j\right\|_{\mathcal{A}^*}$$

holds. The set of all n-BSE functions on $\Delta(\mathcal{A})$ will be denoted by $C_{\mathrm{BSE}(n)}(\Delta(\mathcal{A}))$. We denote by $\|\sigma\|_{\mathrm{BSE}(n)}$, the infimum of such β. By [19, Lemma 1]:

$$C_{\mathrm{BSE}(n)}(\Delta(\mathcal{A})) = C_b(\Delta(\mathcal{A}))$$

if and only if there exists a positive real numbers β_n (depending only on n), such that for any choice of $\varphi_1, \ldots, \varphi_n$ in $\Delta(\mathcal{A})$ and complex numbers $c_1, \ldots, c_n$ in the closed unit disk $\mathbb{C}_1$, there exists $x \in \mathcal{A}$, such that $\|x\| \leq \beta_n$ and $\hat{x}(\varphi_i) = c_i$.

Let:

$$C_{\mathrm{BSE}(\infty)} = \bigcap_{n\in\mathbb{N}} C_{\mathrm{BSE}(n)}(\Delta(\mathcal{A})).$$

Evidently, $\|\sigma\|_{\mathrm{BSE}} = \sup_{n\in\mathbb{N}} \|\sigma\|_{\mathrm{BSE}(n)}$ and:

$$C_{\mathrm{BSE}}(\Delta(\mathcal{A})) = \{\sigma \in C_{\mathrm{BSE}(\infty)} :\ \|\sigma\|_{\mathrm{BSE}} < \infty\}.$$

Moreover, we have the following inclusions:

$$\begin{aligned}\widehat{\mathcal{A}} &\subseteq C_{\mathrm{BSE}}(\Delta(\mathcal{A})) \subseteq C_{\mathrm{BSE}(\infty)}(\Delta(\mathcal{A})) \subseteq \cdots \\ &\subseteq C_{\mathrm{BSE}(2)}(\Delta(\mathcal{A})) \subseteq C_{\mathrm{BSE}(1)}(\Delta(\mathcal{A})) \\ &= C_b(\Delta(\mathcal{A})).\end{aligned}$$

See [19], for more information.

In Example 3.3, we observe that a continuous bounded function on the Gelfand space $\ell^p(X, \mathcal{A})$ needs not be a BSE function. However, in the sequel, we prove that for any unital commutative Banach algebra $\mathcal{A}$ and natural number n:

$$C_{\mathrm{BSE}(n)}(\Delta(\ell^p(X, \mathcal{A}))) = C_b(\Delta(\ell^p(X, \mathcal{A}))) = C_b(X \times \Delta(\mathcal{A})).$$

In fact, all continuous bounded functions on $\Delta(\ell^p(X, \mathcal{A}))$ are n-BSE functions.

Proposition 3.4. *Let X be a set and $\mathcal{A}$ be a commutative semisimple and unital Banach algebra with unit e. Then, $C_{\mathrm{BSE}(n)}(\ell^p(X, \mathcal{A})) = C_b(X \times \Delta(\mathcal{A}))$, for each $n \in \mathbb{N}$.*

Proof. To prove, we use [19, Lemma 1]. Take $c_1 \ldots, c_n \in \Delta$, $x_1, \ldots x_n \in X$, and $\varphi_1, \ldots, \varphi_n \in \Delta(\mathcal{A})$. Define the function f on X as:

$$f(x) = \begin{cases} c_i e & x \in \{x_1, \ldots, x_n\} \\ 0 & \textit{otherwise}. \end{cases}$$

Then, for each $i = 1, \ldots, n$ we have:

$$\hat{f}(x_i, \varphi_i) = \varphi_i(f(x_i)) = \varphi_i(c_i e) = c_i.$$

Moreover:

$$\|f\|_p = \left(\sum_{i=1}^{n} \|f(x_i)\|^p\right)^{1/p} = \left(\sum_{i=1}^{n} |c_i|^p\right)^{1/p} = n^{1/p}.$$

Thus, it is sufficient to take $\beta_n = n^{1/p}$, and so, the proof is completed. $\square$

It is known that in any commutative Banach algebra $\mathcal{A}$, $\|\hat{x}\|_{\mathrm{BSE}} \leq \|x\|$, for all $x \in \mathcal{A}$. In [20], the authors were interested in a class of commutative Banach algebras which satisfy the condition $\|\hat{x}\|_{\mathrm{BSE}} = \|x\|$, for each $x \in \mathcal{A}$. These algebras are called BSE norm algebras. All function algebras on a locally compact Hausdorff space, endowed with the supremum norm, and also the algebra $\ell^1(X)$ belong to such a class. In the sequel, we show that under some circumstances, $\ell^1(X, \mathcal{A})$ also belongs to this class. To that end, we require the following elementary lemma.

Lemma 3.5. *Let X be a set and $\mathcal{A}$ be a commutative semisimple Banach algebra. Suppose that $c_1, \dots c_n$ and $(x_1, \varphi_1), \dots, (x_n, \varphi_n)$ are disjoint elements of $\mathbb{C}$ and $X \times \Delta(\mathcal{A})$, respectively, such that $x_{k_1} = \cdots = x_{k_m}$, where $1 \leq k_1, \dots, k_m \leq n$. Then:*

$$\left\| \sum_{i=1}^{m} c_{k_i} \varphi_{k_i} \right\|_{\mathcal{A}^*} \leq \left\| \sum_{i=1}^{n} c_i (x_i, \varphi_i) \right\|_{\ell^1(X,\mathcal{A})^*}.$$

Proof. Let $x_{k_1} = \cdots = x_{k_m} = x$. Then, we have:

$$\begin{aligned} \left\| \sum_{i=1}^{m} c_{k_i} \varphi_{k_i} \right\|_{\mathcal{A}^*} &= \sup \left\{ \left| \sum_{i=1}^{m} c_{k_i} \varphi_{k_i}(a) \right| : \|a\| \leq 1 \right\} \\ &= \sup \left\{ \left| \sum_{i=1}^{m} c_{k_i} \varphi_{k_i}(\delta_a^x(x_{k_i})) \right| : \|a\| \leq 1 \right\} \\ &= \sup \left\{ \left| \sum_{i=1}^{n} c_i \varphi_i(\delta_a^x(x_i)) \right| : \|\delta_a^x\|_1 \leq 1 \right\} \\ &= \sup \left\{ \left| \sum_{i=1}^{n} c_i (x_i, \varphi_i)(\delta_a^x) \right| : \|\delta_a^x\|_1 \leq 1 \right\} \\ &\leq \sup \left\{ \left| \sum_{i=1}^{n} c_i (x_i, \varphi_i)(f) \right| : \|f\|_1 \leq 1 \right\} \\ &= \left\| \sum_{i=1}^{n} c_i (x_i, \varphi_i) \right\|_{\ell^1(X,\mathcal{A})^*}. \end{aligned}$$

Thus, the proof is completed. □

Recall from [16] that a Banach algebra $\mathcal{A}$ is called a supremum norm algebra if $\|\hat{a}\|_\infty = \|a\|$, for each $a \in \mathcal{A}$. For example, all C^*-algebras are supremum norm algebra.

Theorem 3.6. *Let X be a set and $\mathcal{A}$ be a commutative semisimple Banach algebra. If $\ell^1(X, \mathcal{A})$ is a BSE norm algebra, then $\mathcal{A}$ is so. The converse is true if $\mathcal{A}$ is a supremum norm algebra.*

Proof. Suppose that $\ell^1(X, \mathcal{A})$ is a BSE norm algebra. Thus, for each $f \in \ell^1(X, \mathcal{A})$, we have:

$$\|f\|_1 = \|\hat{f}\|_{\mathrm{BSE}}.$$

It follows that:

$$\|a\| = \|\delta_a^x\|_1 = \|\widehat{\delta_a^x}\|_{\mathrm{BSE}} \quad (x \in X, a \in \mathcal{A}). \tag{3.4}$$

Let $a \in \mathcal{A}$ and take $x \in X$ to be fixed. Then, for any finitely many complex numbers $c_1, \dots, c_n$ and the same number of elements $(x_1, \varphi_1), \dots, (x_n, \varphi_n)$ of

$X \times \Delta(\mathcal{A})$ with $x_{k_1} = \cdots = x_{k_m} = x$, we have:

$$\begin{aligned}\left|\sum_{i=1}^{n} c_i \widehat{\delta_a^x}(x_i,\varphi_i)\right| &= \left|\sum_{i=1}^{n} c_i \varphi_i(\delta_a^x(x_i))\right| \\ &= \left|\sum_{i=1}^{m} c_{k_i} \varphi_{k_i}(\delta_a^x(x_{k_i}))\right| \\ &= \left|\sum_{i=1}^{m} c_{k_i} \varphi_{k_i}(a)\right| \\ &= \left|\sum_{i=1}^{m} c_{k_i} \hat{a}(\varphi_{k_i})\right| \\ &\leq \|\hat{a}\|_{\mathrm{BSE}} \left\|\sum_{i=1}^{m} c_{k_i} \varphi_{k_i}\right\|_{\mathcal{A}^*} \\ &\leq \|\hat{a}\|_{\mathrm{BSE}} \left\|\sum_{i=1}^{n} c_i(x_i,\varphi_i)\right\|_{\ell^1(X,\mathcal{A})^*},\end{aligned}$$

where the last inequality is obtained from Lemma 3.5. Consequently:

$$\left|\sum_{i=1}^{n} c_i \widehat{\delta_a^x}(x_i,\varphi_i)\right| \leq \|\hat{a}\|_{\mathrm{BSE}} \left\|\sum_{i=1}^{n} c_i(x_i,\varphi_i)\right\|_{\ell^1(X,\mathcal{A})^*}. \tag{3.5}$$

Note that if all $x_1,\ldots,x_n$ are different from x, then the inequality 3.5 is obviously satisfied. Thus, we have:

$$\|\widehat{\delta_a^x}\|_{\mathrm{BSE}} \leq \|\hat{a}\|_{\mathrm{BSE}}. \tag{3.6}$$

Now, the equality (3.4) and inequality (3.6) imply that:

$$\|a\| \leq \|\hat{a}\|_{\mathrm{BSE}} \quad (a \in \mathcal{A}).$$

Therefore, $\mathcal{A}$ is a BSE norm algebra.

Conversely, suppose that $\mathcal{A}$ is a supremum norm algebra. We show that $\ell^1(X,\mathcal{A})$ is a BSE norm algebra. Take $f \in \ell^1(X,\mathcal{A})$ to be nonzero. It is enough to show that $\|f\|_1 \leq \|\hat{f}\|_{\mathrm{BSE}}$. For $\varepsilon > 0$, there exists $N \in \mathbb{N}$, such that:

$$\|f\|_1 - \varepsilon < \sum_{k=1}^{N} \|f(x_k)\|.$$

By the hypothesis:

$$\|f(x_k)\| = \|\widehat{f(x_k)}\|_\infty = \sup_{\varphi\in\Delta(\mathcal{A})} |\varphi(f(x_k))|,$$

for each $k = 1,\ldots,N$. Since $\mathcal{A}$ is unital, $\Delta(\mathcal{A})$ is compact and so all $\widehat{f(x_k)}$ $(k = 1,\ldots,N)$ take their supremum on $\Delta(\mathcal{A})$. It follows that there exists $\varphi_k \in \Delta(\mathcal{A})$, such that:

$$\|\widehat{f(x_k)}\|_\infty = |\varphi_k(f(x_k))| \quad (k = 1,\ldots,N).$$

Now let:

$$C_k = \frac{\|f(x_k)\|}{\varphi_k(f(x_k))} \quad (k = 1, \ldots, N).$$

Then, $|C_k| = 1$ and:

$$\begin{aligned} \left|\sum_{k=1}^{N} C_k \hat{f}(x_k, \varphi_k)\right| &= \left|\sum_{k=1}^{N} C_k \varphi_k(f(x_k))\right| \\ &= \|f(x_1)\| + \cdots + \|f(x_N)\| \\ &= \sum_{k=1}^{N} \|f(x_k)\| \\ &> \|f\|_1 - \varepsilon. \end{aligned}$$

Thus:

$$\|f\|_1 - \varepsilon < \left|\sum_{k=1}^{N} C_k \hat{f}(x_k, \varphi_k)\right|. \tag{3.7}$$

Moreover:

$$\begin{aligned} \left\|\sum_{k=1}^{N} C_k (x_k, \varphi_k)\right\| &= \sup_{\|h\|_1 \le 1} \left|\sum_{k=1}^{N} C_k (x_k, \varphi_k)(h)\right| \\ &= \sup_{\|h\|_1 \le 1} \left|\sum_{k=1}^{N} C_k \varphi_k(h(x_k))\right| \\ &\le \sup_{\|h\|_1 \le 1} \sum_{k=1}^{N} |C_k| \|\varphi_k\| \|h(x_k)\| \\ &= \sup_{\|h\|_1 \le 1} \sum_{k=1}^{N} \|h(x_k)\| \\ &\le 1. \end{aligned}$$

The last inequality together with (3.7) yields that:

$$\begin{aligned} \|f\|_1 - \varepsilon &< \left|\sum_{k=1}^{N} C_k \hat{f}(x_k, \varphi_k)\right| \\ &\le \|\hat{f}\|_{\mathrm{BSE}} \left\|\sum_{k=1}^{N} C_k (x_k, \varphi_k)\right\| \\ &\le \|\hat{f}\|_{\mathrm{BSE}}. \end{aligned}$$

Since ε is arbitrary, it follows that $\|f\|_1 \le \|\hat{f}\|_{\mathrm{BSE}}$, as claimed. □

4. The BSE Property of $\ell^p(X)$

Let X be a nonempty set. By [19, Theorem 5], $\ell^1(X)$ is a BSE algebra if and only if X is finite. Note that this result remains valid for $\ell^p(X)$, where $1 \le p < \infty$. In this section, we provide another proof for this result, which is interesting in its own right. We first recall the definition of abstract Segal algebras; see [3] for more information.

Let $(\mathcal{A}, \|.\|_{\mathcal{A}})$ be a commutative Banach algebra. A commutative Banach algebra $(\mathcal{B}, \|.\|_{\mathcal{B}})$ is an abstract Segal algebra with respect to $\mathcal{A}$ if:

(i) $\mathcal{B}$ is a dense ideal in $\mathcal{A}$.
(ii) There exists $M > 0$, such that $\|b\|_{\mathcal{A}} \leq M\|b\|_{\mathcal{B}}$, for all $b \in B$.
(iii) There exists $C > 0$, such that $\|ab\|_{\mathcal{B}} \leq C\|a\|_{\mathcal{A}}\|b\|_{\mathcal{B}}$, for all $a, b \in B$.

Moreover, $\mathcal{B}$ is called essential if:

$$\mathcal{B} = \{ab : \ a \in \mathcal{A}, b \in \mathcal{B}\}.$$

Our new proof for [19, Theorem 5] is based on [10, Theorem 3.1], which is described below:

"If $(\mathcal{B}, \|.\|_{\mathcal{B}})$ is an essential abstract Segal algebra with respect to the BSE algebra $\mathcal{A}$, then $\mathcal{B}$ is a BSE algebra if and only if it has a Δ-weak bounded approximate identity."

For this purpose, we remind the reader of some known spaces. Recall that $c_0(X)$ is the space, consisting of all functions vanishing at infinity. Moreover, $c_0(X)$ is a Banach algebra under pointwise product and supremum norm, defined as:

$$\|f\|_\infty = \{|f(x)| : \quad x \in X\} \quad (f \in c_0(X)).$$

The subspace $c_{00}(X)$ of $c_0(X)$, consisting of all finite support functions on X, is dense in $c_0(X)$. Moreover:

$$c_{00}(X) \subseteq \ell^p(X) \subseteq c_0(X)$$

and $\|f\|_\infty \leq \|f\|_p$, for all $f \in \ell^p(X)$.

Lemma 4.1. *Let X be a set and $1 \leq p < \infty$. Then, $\ell^p(X)$ is an essential abstract Segal algebra with respect to $c_0(X)$.*

Proof. Since $\ell^p(X)$ contains $c_{00}(X)$ and $c_{00}(X)$ is dense in $c_0(X)$, it follows that $\ell^p(X)$ is also dense in $c_0(X)$. Moreover, $\ell^p(X)$ is an ideal in $c_0(X)$ and for each $f \in \ell^p(X)$ and $g \in c_0(X)$, we have:

$$\|f\ g\|_p = \left(\sum_{x \in X} |f(x)g(x)|^p\right)^{1/p} \leq \|f\|_p\|g\|_\infty < \infty.$$

Consequently, $\ell^p(X)$ is an abstract Segal algebra in $c_0(X)$. In the sequel, we show that $\ell^p(X)$ is essential. To that end, note that the collection $\mathcal{F}$, consisting of all finite subsets of X, is a directed set by the upward inclusion; that is:

$$F_1 \leq F_2 \ \text{ if and only if } \ F_1 \subseteq F_2.$$

It is easily verified that the net $(\chi_F)_{F \in \mathcal{F}}$ is a bounded approximate identity for $c_0(X)$, where χ_F is the characteristic function on X at F. To establish the essentiality of $\ell^p(X)$, by applying Cohen factorization theorem, it is sufficient to show that $(\chi_F)_{F \in \mathcal{F}}$ is an approximate identity for $\ell^p(X)$; that is:

$$\|f\chi_F - f\|_p \to_F 0 \quad (f \in \ell^p(X)). \tag{4.1}$$

Suppose that $f \in \ell^p(X)$ and take $\varepsilon > 0$ to be arbitrary. There exists $N \in \mathbb{N}$, such that:

$$\sum_{i=N+1}^{\infty} |f(x_i)|^p < \varepsilon^p.$$

Let $F_0 = \{x_1, \ldots, x_n\}$. Since $f \in c_0(X)$, there exists finite subset F_1 of X, such that $|f(x)| < \varepsilon$, for all $x \notin F_1$. Set $F_2 := F_0 \cup F_1$. Thus, for each $F_2 \leq F$, we have:

$$\|f\chi_F - f\|_p = \left(\sum_{x \notin F} |f(x)|^p\right)^{1/p} \leq \left(\sum_{i=N+1}^{\infty} |f(x)|^p\right)^{1/p} < \varepsilon,$$

and so, (4.1) is satisfied. This completes the proof. □

Note that $c_0(X)$ is a C^*-algebra, and so, it is a BSE algebra by [18, Theorem 3]. Now, Theorem 2.6 and Lemma 4.1 together with [10, Theorem 3.1] yield the following result.

Theorem 4.2. *Let X be a set and $1 \leq p < \infty$. Then, $\ell^p(X)$ is a BSE algebra if and only if X is finite.*

Proof. By Lemma 4.1, $\ell^p(X)$ is an essential abstract Segal algebra in $c_0(X)$. Since $c_0(X)$ is a BSE algebra, [10, Theorem 3.1] implies that $\ell^p(X)$ is a BSE algebra. It follows that $\ell^p(X)$ has a Δ-weak bounded approximate identity, and so, X is finite by Theorem 2.6. The converse is obvious. □

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