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Multiplicity of Solution for a Quasilinear Equation with Singular Nonlinearity

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Abstract. For an open, bounded domain Ω in $\mathbb{R}^N$ which is strictly convex with smooth boundary, we show that there exists a $\Lambda > 0$ such that for $0 < \lambda < \Lambda$, the quasilinear singular problem

$$\begin{aligned} -\Delta_p u &= \lambda u^{-\delta} + u^q \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega;\; u > 0 \text{ in } \Omega \end{aligned}$$

admits at least two distinct solutions u and v in $W^{1,p}_{loc}(\Omega) \cap L^\infty(\Omega)$ provided $\delta \geq 1$, $\frac{2N+2}{N+2} < p < N$ and $p-1 < q < \frac{Np}{N-p} - 1$.

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Introduction

In this paper, we study the multiplicity of weak solution to the quasilinear singular problem given by

$$\begin{aligned} -\Delta_p u &= \lambda u^{-\delta} + u^q \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega;\; u > 0 \text{ in } \Omega \end{aligned} \tag{1}$$

where $\Omega(\subset \mathbb{R}^N)$ is a strictly convex bounded domain with smooth boundary. Here

$$\Delta_p u := \text{div}(|\nabla u|^{p-2}\nabla u)$$

is the p-Laplacian operator for $1 < p < \infty$. We also assume that $\lambda > 0$, $\delta \geq 1$, $\frac{2N+2}{N+2} < p < N$ and $p-1 < q < p^*-1$, where $p^* = \frac{Np}{N-p}$ is the critical Sobolev exponent.

We start with a brief background of the problem (1) which were available in the literature and is critical for a clear understanding of the issues and the framework of our study. About 3 decades of work on the study of singular

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elliptic equation can be traced back to the pioneering work of of Crandall et al. [1], where the problem

$$-\Delta u = u^{-\delta} \text{ in } \Omega;\ u = 0 \quad \text{on} \quad \partial\Omega$$

was shown to admit a unique classical solution for any $\delta > 0$ provided Ω bounded. Following this Lazer–Mckenna [2] elaborating that the unique classical solution u is also in $H_0^1(\Omega)$ iff $0 < \delta < 3$. They also showed that the solution belongs to $C^1(\bar{\Omega})$ provided $0 < \delta < 1$. This was followed by the work of Haitao [3] who studied the perturbed singular problem

$$\begin{aligned} -\Delta u &= \lambda u^{-\delta} + u^q \quad \text{in} \quad \Omega \\ u &= 0 \text{ on } \partial\Omega;\ \ u > 0 \quad \text{in} \quad \Omega \end{aligned} \tag{2}$$

and showed the existence of $\Lambda > 0$ such that there exists at least two solutions $u, v \in H_0^1(\Omega)$ to problem (2) for $\lambda < \Lambda$, no solution for $\lambda > \Lambda$ and at least one solution for $\lambda = \Lambda$ provided $0 < \delta < 1 < q \leq \frac{N+2}{N-2}$ using fibering method on Nehari manifold. These results were generalised for p-Laplacian by Giacomoni et al. [4] who showed among other results the existence of at least two solutions for $0 < \delta < 1$ and $p-1 < q \leq p^*-1$. It should be noted that in the above-mentioned works on perturbed problem, the solution so obtained satisfied the equation in the trace sense and the restriction $0 < \delta < 1$ is due to the use of variational methods which requires the associated functional to be well defined on $W_0^{1,p}(\Omega)$.

Boccardo and Orsina [5] took a different approach and showed that the problem

$$-\operatorname{div}(M(x)\nabla u) = \frac{f(x)}{u^\delta} \quad \text{in} \quad \Omega;\ \ u = 0 \ \text{ in } \ \partial\Omega \tag{3}$$

admits a solution $u \in H_{loc}^1(\Omega)$ for any non-negative $f \in L^1(\Omega)$ in the sense that

$$\int_\Omega \nabla u \nabla \phi = \int_\Omega \frac{f\phi}{u^\delta},\ \phi \in C_0^1(\Omega)$$

for $u \geq c_\omega$ in ω where $\omega \subset\subset \Omega$ and $\delta > 0$ among other results. The boundary condition is understood as such that $u^{\frac{1+\delta}{2}}$ belongs to $H_0^1(\Omega)$. Recently, the problem (3) has been generalised by Canino et al. [6] for the p-Laplacian, where existence of a solution $u \in W_{loc}^{1,p}(\Omega)$ was shown for $\delta > 0$ and $f \in L^1(\Omega)$ such that $u^{\frac{p-1+\delta}{p}} \in W_0^{1,p}(\Omega)$. Moreover, the solution was proved to be unique provided Ω is star-shaped w.r.t the origin. The perturbed problem (2) was studied by Arcoya-Mérida [7] and the existence of at least two solutions was proved in $H_{loc}^1(\Omega) \cap L^\infty(\Omega)$ for any $\delta > 0$. Moreover, any solution u so obtained satisfies $u^{\frac{1+\delta}{2}} \in H_0^1(\Omega)$. This was done by regularising the singular problem and showing the multiplicity result using a combination of some a priori estimates and bifurcation theory and then passing to the limit. In this work, we aim to provide a generalization of the results of Arcoya-Mérida [7] for any $\frac{2N+2}{N+2} < p < N$. We finish our survey of the literature by providing some references for nonexistence results concerning singular nonlinearity which can be found in [8]. Interested readers may also find the corresponding parabolic problem which was studied in [9,10] and the references therein.

Now that the history of the problem is clear, let us move to discuss the difficulties one encounters while studying the problem (1) for any $1 < p < \infty$ and the strategy we employ to circumvent those difficulties. To obtain the multiplicity result we start by the standard approach of studying the multiplicity of solution to the regularized problem which is given by

$$\begin{aligned} &-\Delta_p u = \lambda f_n(u) + u^q \text{ in } \quad \Omega; \\ &u = 0 \text{ on } \partial\Omega; \quad u > 0 \text{ in } \Omega, \end{aligned} \tag{4}$$

where $f_n(x) := (x + \frac{1}{n})^{-\delta}$ with $\delta \geq 1$, $n \in \mathbb{N}$ and $\lambda > 0$. Note that this problem is non-singular for any $n \in \mathbf{N}$. We start by showing multiplicity of solution to Eq. (4) for every fixed n by proving an uniform a priori estimate and then using Leray–Schauder degree. We conclude by passing to the limit to obtain two distinct solutions to our main problem (1). One of the main challenges in this study is to find the uniform a priori estimates independent of n. For $p = 2$, Kelvin transform has been employed to obtain the boundary estimate on the solutions of the regularized problem in [7] which fails for p-Laplacian, see Lindqvist [11]. Moreover, application of the moving plane method is also tricky owing to the degeneracy of the p-Laplacian at the critical points. We overcome this difficulty by proving an uniform Höpf Lemma by modifying the arguments of Vázquez [12] (also see Peral [13]) in combination with a delicate application of moving plane technique by combining some of our ideas with that of Castorina–Sanchón [14] to arrive at the required estimate. This process requires the strict convexity of the domain. Once we have an uniform neighbourhood of the boundary, the blow-up analysis of Gidas–Spruck [15] goes through, which required segregating the maxima's of u_n in some interior of the boundary independent of n.

Following Arcoya–Mérida [7], the solutions of (1) has been understood here in the following sense:

Definition 0.1. We say $u \in W^{1,p}_{loc}(\Omega) \cap L^\infty(\Omega)$ is a weak solution to the problem (1) if for every open subset $\omega \subset\subset \Omega$ and $\phi \in W^{1,p}_0(\omega)$ one has $u^{-\delta}\phi \in L^1(\Omega)$ and also satisfying

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla\phi \,\mathrm{d}x = \lambda \int_\Omega u^{-\delta}\phi \,\mathrm{d}x + \int_\Omega u^q \phi \,\mathrm{d}x. \tag{5}$$

The boundary condition $u = 0$ on $\partial\Omega$ is understood as in Arcoya–Mérida [7], i.e. we require that a suitable power of u is in $W^{1,p}_0(\Omega)$.

Main Result

We denote the set

$$\mathbb{E} = \left\{ (p, q) : \frac{2N+2}{N+2} < p < N \quad \text{and} \quad p - 1 < q < p^* - 1 \right\}.$$

For the rest of the paper, we will assume $(p, q) \in \mathbb{E}$, $\delta \geq 1$ and Ω is a strictly convex bounded domain with smooth boundary unless otherwise mentioned.

Theorem 0.1. *Given $\delta \geq 1$, there exists $\Lambda > 0$ such that for any $0 < \lambda < \Lambda$, problem (1) admits at least two solution $u, v \in W^{1,p}_{loc}(\Omega) \cap L^\infty(\Omega)$ provided $(p,q) \in \mathbb{E}$. The solutions so obtained fulfils the boundary data in the sense that*

$$u^\alpha, v^\alpha \in W^{1,p}_0(\Omega) \quad \textit{for all} \quad \alpha > \frac{(p-1)(\delta+p-1)}{p^2}.$$

Remark 0.1. Note that for $p = 2$, Theorem 0.1 boils down to the main result of Arcoya–Mérida [7] provided Ω is strictly convex. It is worth noting that this assumption of strict convexity on Ω was not required in Arcoya–Mérida as opposed to our case which is due to the non-degeneracy of the Laplace operator.

Remark 0.2. For $1 \leq \delta < 2 + \frac{1}{p-1}$, one has $\frac{(p-1)(\delta+p-1)}{p^2} < 1$. Therefore, choosing $\alpha = 1$ in Theorem 0.1, we obtain $u, v \in W^{1,p}_0(\Omega)$. This implies that for $\delta \in [1, 2 + \frac{1}{p-1})$ one has the existence of at least two solutions $u, v \in W^{1,p}_0(\Omega)$, hence improving the result of Giacomoni et al. [4] when $p - 1 < q < p^* - 1$.

1. Preliminary Lemmas

We begin this section by extending a few results for the p-Laplacian case:

Lemma 1.1. *Given $\lambda > 0$, the regularized singular problem*

$$-\Delta_p u = \lambda \left(u + \frac{1}{n}\right)^{-\delta} \textit{ in } \Omega; \quad u = 0 \textit{ on } \partial\Omega \tag{6}$$

admits an unique positive solution u_n in $W^{1,p}_0(\Omega) \cap L^\infty(\Omega)$ for each $n \in \mathbb{N}$. Moreover, one has the following:

(i) *u_n is increasing w.r.t n.*
(ii) *$u_n > c_\omega > 0$ for all $\omega \subset\subset \Omega$, where c_ω depends only on ω and not on n.*
(iii) *$||u_n||_\infty \leq M\lambda^{\frac{1}{\delta+p-1}}$ for all $n \in \mathbb{N}$ with $M > 0$ is a constant independent of n.*

Lemma 1.2. *There exists $\delta_0 > 0$ such that every bounded non-trivial positive solution u of the problem $-\Delta_p u = u^q$ in Ω satisfies $||u||_\infty > \delta_0$.*

Lemma 1.3. *There exists $\bar{\Lambda} > 0$ (independent of n) such that for all $\lambda \geq \bar{\Lambda}$ the problem (4) does not admit any weak solution $u_n \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega)$.*

Lemma 1.4. *Then there exists $K > 0$ (independent of n) such that $||u_n||_\infty \leq K$, where $u_n \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega)$ solves (4) for $\lambda > 0$.*

Lemma 1.5. *Let $\delta \geq 1$. Then there exists $N \in \mathbb{N}$ and $\Lambda > 0$ such that for any $n \geq N$, the problem (4) admits at least two distinct solution $u_n, v_n \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega)$ provided $0 < \lambda < \Lambda$.*

Before we start with the proof of the lemmas, we state some useful results.

Lemma 1.6 (Lemma B.1, Stampacchia [16]). *Let $\phi(t)$, $k_0 \leq t < \infty$, be non-negative and non-increasing such that*

$$\phi(h) \leq \left[\frac{c}{(h-k)^l}\right] |\phi_k|^m, \quad h > k > k_0,$$

where c, l, m are positive constants with $\beta > 1$. Then

$$\phi(k_0 + d) = 0,$$

where

$$d^l = C[\phi(k_0)]^{m-1} 2^{\frac{lm}{m-1}}.$$

Theorem 1.1 (Liouville theorem for p-Laplacian, Corollary 3 of Serrin-Zou [17]). *Let $\Omega = \mathbb{R}^N$ and assume $1 < p < N$. Then the problem $-\Delta_p u = u^q$ has a bounded positive C^1 solution on Ω iff $q \geq p^* - 1$.*

Before proceeding further, we state the following strong comparison principle which follows arguing similarly as in the proof of Theorem 2.3 of Giacomoni et al. [4].

Theorem 1.2. *Let $n \in \mathbb{N}$ and $u, v \in C^{1,\alpha}(\overline{\Omega})$ for some $0 < \alpha < 1$ be positive in Ω such that*

$$-\Delta_p u - \lambda \left(u + \frac{1}{n}\right)^{-\delta} = f, \tag{7}$$

$$-\Delta_p v - \lambda \left(v + \frac{1}{n}\right)^{-\delta} = g, \tag{8}$$

with $u = v = 0$ on $\partial\Omega$, where $f, g \in C(\Omega)$ are such that $0 \leq f < g$ pointwise everywhere in Ω. Then the following strong comparison principle holds:

$$0 < u < v \ \text{in } \Omega \quad \text{and} \quad \frac{\partial v}{\partial \eta} < \frac{\partial u}{\partial \eta} < 0 \ \text{on } \partial\Omega,$$

where η is the outward unit normal to the boundary of Ω.

Theorem 1.3 (Strong comparison principle, Theorem 3.7 of Damascelli–Sciunzi [18].). *Let Ω be a bounded smooth domain in $\mathbb{R}^N$, $N \geq 2$. Assume $\frac{2N+2}{N+2} < p < \infty$ and $u, v \in C^1(\overline{\Omega})$ be positive in Ω such that*

$$-\Delta_p u - f(u) \leq -\Delta_p v - f(v) \ \text{weakly in } \Omega,$$

and f satisfy the following conditions:

(a) f is a positive continuous on $[0, \infty)$,
(b) f is locally Lipschitz on $(0, \infty)$.

Let u solves the following equation:

$$\begin{cases} -\Delta_p w = f(w) \ \text{in } \Omega, \\ w > 0 \ \text{in } \Omega, \quad w = 0 \ \text{on } \partial\Omega. \end{cases} \tag{9}$$

If $u \leq v$ and $u \neq v$ in Ω, then $u < v$ in Ω.

One can also have a strong comparison principle with sign changing nonlinearity f which generalizes Lemma 9 due to Roselli–Sciunzi [19]. Before we move to the proof of the lemmas, let us note that the first part of the proof of Lemma 1.1 can be done using similar techniques from Canino et al. [6] but we still provide it here for completeness.

Proof of Lemma 1.1. Fix $v \in L^p(\Omega)$ and $n \in \mathbb{N}$. Consider $J_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined as

$$J_\lambda(u) = \frac{1}{p}\int_\Omega |\nabla u|^p \mathrm{d}x - \lambda \int_\Omega \frac{u}{(|v| + \frac{1}{n})^\delta} \mathrm{d}x$$

Clearly, J_λ is continuous, coercive and strictly convex in $W_0^{1,p}(\Omega)$. Hence, there exists a unique minimizer $u \in W_0^{1,p}(\Omega)$ solving

$$-\Delta_p u = \frac{\lambda}{(|v| + \frac{1}{n})^\delta}. \tag{10}$$

Define $S : L^p(\Omega) \to L^p(\Omega)$ by

$$S(v) = (-\Delta_p)^{-1}\left(\frac{\lambda}{(|v| + \frac{1}{n})^\delta}\right) := u.$$

Arguing exactly as in A.0.3 of Peral [13] in conjugation with Poincaré inequality and Sobolev inequality yields the continuity and compactness of S. Note that multiplying u with equation (10) and integrating we have,

$$\int_\Omega |\nabla u|^p \, \mathrm{d}x \leq \lambda n^\delta \int_\Omega |u| \, \mathrm{d}x \leq C(\lambda, n, \delta, \Omega)\left(\int_\Omega |u|^p \, \mathrm{d}x\right)^{\frac{1}{p}},$$

where $C(\lambda, n, \delta, \Omega)$ is a positive constant which depend only on λ, n, δ and Ω. Hence, using Poincaré inequality on the left side of the above relation, we have

$$||u||_p \leq (C(\lambda, n, \delta, \Omega))^{\frac{1}{p-1}}.$$

This essentially shows that there exists a ball in $L^p(\Omega)$ which remains invariant under the action of S. Hence, Schauder fixed point theorem gives the existence of a fixed point $u_n \in W_0^{1,p}(\Omega)$ thus solving (6). Again by Vazquez strong maximum principle [12], we have $u_n > 0$ in Ω satisfying

$$-\Delta_p u_n = \lambda\left(u_n + \frac{1}{n}\right)^{-\delta}; \quad u_n \in W_0^{1,p}(\Omega).$$

Again using Lemma A.1 in Perera-Silva [20], we have $u_n \in L^\infty(\Omega)$ for any fixed n. For monotonicity, we denote u_i to be the solution of the equation:

$$-\Delta_p u = \lambda\big(u + \frac{1}{i}\big)^{-\delta} \text{ in } \Omega; \;\; u = 0 \text{ on } \partial\Omega \tag{11}$$

for $i = 1, 2, \ldots$

Subtracting Eq. (11) for $i = n$ from $i = n + 1$ and multiplying with $(u_n - u_{n+1})^+$, we have

$$\int_\Omega (|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_{n+1}|^{p-2}\nabla u_{n+1}) \cdot \nabla (u_n - u_{n+1})^+ \mathrm{d}x$$
$$\leq \lambda \int_\Omega \big[\big(u_n + \frac{1}{n+1}\big)^{-\delta} - \big(u_{n+1} + \frac{1}{n+1}\big)^{-\delta}\big](u_n - u_{n+1})^+ \mathrm{d}x \quad (12)$$

From the algebraic inequality (Lemma 4.1 of Ghoussoub and Yuan [21]), we get for $p \geq 2$

$$\int_\Omega (|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_{n+1}|^{p-2}\nabla u_{n+1}) \cdot \nabla (u_n - u_{n+1})^+ \mathrm{d}x$$
$$\geq C_p ||\nabla (u_n - u_{n+1})^+||^p \geq 0.$$

Also when $1 < p < 2$, we have

$$\int_\Omega (|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_{n+1}|^{p-2}\nabla u_{n+1}) \cdot \nabla (u_n - u_{n+1})^+ \mathrm{d}x$$
$$\geq C_p \frac{||\nabla u_n - \nabla u_{n+1}||^2}{(||\nabla u_n|| + ||\nabla u_{n+1}||)^{2-p}} \geq 0.$$

Again from the monotonicity of $f_n(x) = (x + \frac{1}{n})^{-\delta}$ w.r.t x, we have

$$\int_\Omega \left[\left(u_n + \frac{1}{n+1}\right)^{-\delta} - \left(u_{n+1} + \frac{1}{n+1}\right)^{-\delta}\right](u_n - u_{n+1})^+ \mathrm{d}x \leq 0.$$

Combining this with (12), we have

$$||(u_n - u_{n+1})^+||_{1,p} = 0.$$

Employing with the boundary condition gives

$$(u_n - u_{n+1})^+ = 0;$$

therefore, u_n is monotonically increasing w.r.t n. Uniqueness follows arguing as above.

The positivity of u_n on compact subsets follows by noting that $u_1 > 0$ in Ω, where u_1 solves the equation

$$-\Delta_p u = \frac{\lambda}{(u+1)^\delta} \text{ in } \Omega; \quad u = 0 \text{ in } \partial\Omega.$$

Hence, using regularity theorem of Lieberman and DiBenedetto [22,23] one can conclude that $u_n \in C^{1,\alpha(n)}(\bar{\Omega})$ for each $n \in \mathbb{N}$ for some $0 < \alpha(n) < 1$. Therefore, from monotonicity of solutions, we can conclude that $u_n > u_1$ in Ω and hence

$$u_n > c_\omega > 0 \quad \text{for} \quad \omega \subset\subset \Omega$$

with c_ω is independent of n.

Now to show the uniform boundedness of the solutions we assume, $v = u_n$ be a solution to Eq. (6) and let $\lambda = 1$.

For $k \geq 1$, choose

$$\phi := G_k(v) = \begin{cases} v-k \text{ if } v > k \\ 0 \quad \text{if} \quad v \leq k \end{cases}$$

and define, $A(k) = \{x \in \Omega : v > k\}$. So for $0 < k < h$ we have $A(h) \subset A(k)$.

Since $-\Delta_p v = \frac{1}{(v+\frac{1}{n})^\delta} < \frac{1}{v^\delta}$; hence,

$$\int_{A(k)} |\nabla v|^p dx < C \int_{A(k)} \frac{v-k}{v^\delta} dx \leq |A(k)|^{\frac{1}{p'}} ||(v-k)||_{L^p(A(k))} < C|A(k)|^{\frac{1}{p'}} ||\nabla v||_{L^p(A(k))}.$$

By Poincaré and Sobolev inequalities, we have

$$||v||^{p-1}_{L^{p^*}(A(k))} < \frac{C}{S^{p-1}} |A(k)|^{\frac{1}{p'}}.$$

where $C > 0$ and $S > 0$ are the Poincaré and Sobolev constant respectively with $p' = \frac{p}{p-1}$. Using the above inequalities we get,

$$|A(h)| \leq \big(\frac{c}{S^{p-1}}\big)^{\frac{p^*}{p-1}} \frac{1}{(h-k)^{p^*}} |A(k)|^{\frac{p^*}{p}}.$$

Using Lemma 1.6 we have for $h > k > 0$,

$$|A(T)| = 0,$$

which implies $v \in L^\infty(\Omega)$ and $||v||_\infty \leq T$ for some T independent of n.

Now, for any $\lambda > 0$, suppose v satisfies

$$\int_\Omega |\nabla v|^{p-2} \nabla v \nabla \phi \, dx < \lambda \int_\Omega \frac{\phi}{v^\delta} \, \mathrm{d}x$$

for all positive $\phi \in [W_0^{1,p}(\Omega)]$.

Choosing $w = (\frac{1}{\lambda})^{\frac{1}{\delta+p-1}} u_n$, we see that w satisfies:

$$\int_\Omega |\nabla w|^{p-2} \nabla w \nabla \phi \, \mathrm{d}x < \int_\Omega \frac{\phi}{w^\delta} \, \mathrm{d}x$$

for all positive $\phi \in W_0^{1,p}(\Omega)$. Hence, from the case $\lambda = 1$, we have

$$||w||_\infty \leq T \text{ which implies } ||u_n||_\infty \leq T\lambda^{\frac{1}{\delta+p-1}}.$$

□

Proof of Lemma 1.2. Assume there exists a sequence $u_n \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ of non-trivial solutions of $-\Delta_p u = u^q$ such that $||u_n||_\infty \to 0$ as $n \to \infty$. Define $v_n(x) := u_n(x)||u_n||_\infty^{-1}$ then $||v_n||_\infty = 1$.

Since u_n satisfies $-\Delta_p u = u^q$, we have

$$-\Delta_p v_n = ||u_n||_\infty^{q-p+1} v_n^q := f_n.$$

Since f_n are uniformly bounded for sufficiently large n, we have by Tolksdorf, Dibendetto and Lieberman regularity results [22–24] that $||v_n||_{C^{1,\beta}(\overline{\Omega})} \leq M$ for some $\beta \in (0,1)$ and M independent of n. By Ascoli-Arzelá theorem up to a subsequence, $v_n \to v$ in $C_0^1(\overline{\Omega})$, but that would imply $v = 0$, thanks to Lemma 1.1 of Azizieh-Clément [25] contradicting that $||v_n||_\infty = 1$. □

Proof of Lemma 1.3. Let us assume $\phi_1 \in [W_0^{1,p}(\Omega)]^+$ to be the first eigenfunction corresponding to the first eigenvalue λ_1 of the operator $-\Delta_p$, i.e.,

$$-\Delta_p \phi_1 = \lambda_1 {\phi_1}^{p-1} \text{ in } \Omega;\ \phi_1 = 0 \text{ on } \partial\Omega.$$

Let $u = u_n$ be a weak solution of Eq. (4) for any fixed n, then by strong maximum principle [12], we have $\frac{\phi_1^p}{u^{p-1}} \in W_0^{1,p}(\Omega)$ and hence using Picone identity (Theorem 1.1 of Allegretto-Huang [26] or Theorem 2.1 of Bal [27]), we have

$$\int_\Omega |\nabla \phi_1|^p \mathrm{d}x - \int_\Omega |\nabla u|^{p-2} \nabla u \nabla \left(\frac{\phi_1^p}{u^{p-1}} \right) \mathrm{d}x \geq 0.$$

This implies $\int_\Omega (\lambda_1 u^{p-1} - \lambda f_n(u) - u^q) \phi_1^p \mathrm{d}x \geq 0$.

Define $\overline{\Lambda} := \max\limits_{x \in \Omega} \Big[\dfrac{\lambda_1 u^{p-1} - u^q}{f_1(u)} \Big]$.

Using the boundedness of u, we have for every $\epsilon > 0$ there exists a $\delta_0 > 0$ such that $s^q < \epsilon s^{p-1}$ for all $s = ||u||_\infty \in [0, \delta_0]$. So for a suitable choice of ϵ, we have $\overline{\Lambda} > 0$.

Therefore,

$$\overline{\Lambda} > \frac{\lambda_1 u^{p-1} - u^q}{f_n(u)} \geq \lambda.$$

Hence, the result follows. □

We divide the proof of Lemma 1.4 into several steps. The idea of the proof comes from combining and modifying some ideas from work of Castorina-Sanchón [14] and that of Bal-Giacomoni [28]. Note that similar ideas as in step 1 can also be found in papers of Canino et al. [?] and that of Esposito-Sciunzi [29] for semilinear and quasilinear problems, respectively, and step 4 in Canino et al. [30,31].

Proof of Lemma 1.4. We will prove the lemma in several steps:

Step 1 (Uniform Höpf Lemma) We start by showing that for any $n \in \mathbb{N}$ we have $\frac{\partial u_n}{\partial \eta}(x) < c < 0$ for some c which is independent of n but depends on x and η is the outward unit normal to $\partial\Omega$ at the point x.

Since Ω has a C^2 boundary it also satisfies the interior ball condition. Hence, for $x_0 \in \partial\Omega$, there exists $B_r(y) \subset \Omega$ such that $\partial B_r(y) \cap \partial\Omega = \{x_0\}$.

Define the function $w : B_r(y) \to \mathbb{R}$ such that

$$w(x) = [2^{\frac{N-p}{p-1}} - 1]^{-1} r^{\frac{N-p}{p-1}} |x - y|^{\frac{p-N}{p-1}} - [2^{\frac{N-p}{p-1}} - 1]^{-1}.$$

Hence, w satisfies the following:

1. $w(x) \equiv 1$ on $\partial B_{\frac{r}{2}}(y)$ and $w(x) = 0$ on $\partial B_r(y)$.
2. $0 < w(x) < 1$ if $x \in B_r(y) \cap B_{\frac{r}{2}}(y)$ with $|\nabla w(x)| > c > 0$ for some positive constant c depending on x .

Define $\tau = \inf\{u_n(x) | x \in \partial B_{\frac{r}{2}}(y)\}$, where u_n satisfies Eq. (4). We aim to show that $\tau > c_{B_{\frac{2r}{3}}(y)}$ independent of n. Using Theorem 1.2, we also have $u_n(x) > v_n(x)$ for all $n \in \mathbb{N}$ and a.e. $x \in \Omega$, where v_n solves (6). Hence, from Lemma 1.1

$$u_n > v_n \geq v_1 > c_{B_{\frac{2r}{3}}(y)} \quad \text{for all} \quad n \in \mathbb{N}.$$

Set $v=\tau w$ and note that v satisfies the following equation:

$$\begin{aligned} -\Delta_p v =&0 \quad \text{in} \quad B_r(y)-\overline{B_{\frac{r}{2}}(y)} \\ v =&\tau \quad \text{if} \quad x\in\partial B_{\frac{r}{2}}(y);\;\; v=0 \quad \text{if} \quad x\in\partial B_r(y). \end{aligned}$$

We also have that $u_n\geq v$ on the boundary of $B_r(y)-\overline{B_{\frac{r}{2}}(y)}$ and $-\Delta_p v\leq -\Delta_p u_n$ in $B_r(y)$.

So using Theorem 1.2 of Lucia-Prashanth [32], we have $u_n\geq v$ in $B_r(y)-\overline{B_{\frac{r}{2}}(y)}$.

Now since $u_n(x_0)=v(x_0)=0$, one has from properties of w:

$$\begin{aligned} \frac{\partial u_n}{\partial\eta}(x_0) &= \lim_{t\to 0^-}\frac{u_n(x_0+t\eta)}{t}\leq \lim_{t\to 0^-}\frac{v(x_0+t\eta)}{t} \\ &=\frac{\partial v(x_0)}{\partial\eta}=\tau\frac{\partial w}{\partial\eta}(x_0)<c<0, \end{aligned}$$

where $c<0$ is independent of n and η is the outward normal at x_0.

Step 2 (Existence of a neighbourhood of the boundary which is independent of critical points of u_n) Define $Z(u_n)=\{x\in\Omega:\nabla u_n(x)=0\}$ to be the the critical set of u_n, where u_n satisfies equation (4). Since $u_n\in C^1(\bar{\Omega})$ from Step 1 we have that $\frac{\partial u_n}{\partial\eta}<0$ on the boundary. So using the compactness of $\partial\Omega$ and $Z(u_n)$, we deduce that $\text{dist}(\partial\Omega,Z(u_n))=d_n>0$ for all $n\in\mathbb{N}$. We assert that there exists $\epsilon_0>0$ independent of n such that $d_n>\epsilon_0>0$, i.e, there exists a neighbourhood of boundary given by $\Omega_{\epsilon_0}=\{x\in\Omega:\text{dist}(x,\partial\Omega)<\epsilon_0\}$ such that $Z(u_n)\cap\Omega_{\epsilon_0}=\phi$ for any $n\in\mathbb{N}$. If not, then $\exists\, x_m\in Z(u_n)$ s.t $\text{dist}(x_m,\partial\Omega)\to 0$ as $n\to\infty$ and $\nabla u_n(x_m)=0$. Up to a subsequence, $x_{m_k}\to y_0$. Clearly, $y_0\in\partial\Omega$ and let $\eta(y_0)$ is the unit outward normal to y_0 be such that $\frac{\partial u_n}{\partial\eta}(y_0)<c<0$, thanks to the Uniform Hópf Lemma. Hence, there exists $\iota>0$ such that for all $y\in B_\iota(y_0)\cap\Omega$ one has $|\nabla u_n(y)|>\frac{c}{2}$, where c is independent of n. This is a contradiction since we can always choose $x_{m_0}\in B_\iota(y_0)\cap\Omega$ such that $\nabla u_{n_0}(x_{m_0})=0$.

Step 3 (Monotonicity of u_n) For $e\in\mathbb{S}^n$, $\gamma\in\mathbb{R}$ and a fixed $n\in\mathbb{N}$ define

(i) The hyperplane $\mathbb{T}:=\mathbb{T}_{\gamma,e}=\{x\in\mathbb{R}^N:x.e=\gamma\}$ and the corresponding cap $\Sigma=\Sigma_{\gamma,e}=\{x\in\mathbb{R}^N:x.e<\gamma\}$.

(ii) $a(e)=\inf\limits_{x\in\Omega}x.e$

(iii) $x'=x_{\gamma,e}$ be the reflection of x w.r.t $\mathbb{T}$ i.e, $x'=x+2(\gamma-x.e)e$.

(iv) Σ' be the non-empty reflected cap of Σ w.r.t $\mathbb{T}$ for any $\gamma>a(e)$.

(v) $\Lambda_1(e):=\{\mu>a(e):\forall\gamma\in(a(e),\mu),$ condition $(\mathcal{A})$ holds$\}$ and $\Lambda'(e):=\sup\Lambda_1(e)$,

where condition $(\mathcal{A})$ is given by the following two conditions:

- Σ' is not internally tangent to $\partial\Omega$ at some point $p\notin T_{\gamma,e}$.
- For all $x\in\partial\Omega\cap T_{\gamma,e}$, $e(x).e\neq 0$, where $e(x)$ is the unit inward normal to $\partial\Omega$ at x.

From Proposition 2 of Azizieh-Lumaire [33], we have that the map $e\to\Lambda'(e)$ is continuous, provided Ω is strictly convex.

Further, define $v_n(x)=u_n(x_{\gamma,e})$. Using the boundedness and the strict convexity of the Ω we have Σ' is contained in Ω for any $\gamma\leq\gamma_1$, where γ_1

depends only on Ω, independent of e. Define $\gamma_0 = \min(\gamma_1, \epsilon_0)$. For $\gamma - a(e)$ small consider any such Σ. Now since v_n and u_n both satisfies Eq. (4) and Δ_p is invariant under reflection hence on the hyperplane $\mathbb{T}$ both functions coincides. Moreover, for $x \in \partial\Sigma \cap \partial\Omega$, we have $u_n(x) = 0$ and $v_n(x) = u_n(x') > 0$ since $x' \in \Omega$. Hence, we have

$$\begin{aligned} &\Delta_p u_n + u_n^q + f_n(u_n) = \Delta_p v_n + v_n^q + f_n(v_n) \quad \text{in} \quad \Sigma \\ &u_n \leq v_n \quad \text{on} \quad \partial\Sigma. \end{aligned} \tag{13}$$

Using the comparison principle of Damascelli-Sciunzi [34] for narrow domain we have $u_n \leq v_n$ in Σ. Again using the comparison principle, we have $u_n \leq v_n$ in $\Sigma_{\gamma,e}$ for any $\gamma \in (a(e), \gamma_0]$. So u_n is non-decreasing in the e-direction for all $x \in \Sigma_{\gamma_0,e}$.

Step 4 (Existence of a non-zero measurable set away from boundary where u is non-decreasing) Fix $x_0 \in \partial\Omega$ and let $e = \eta(x_0)$ be the unit outward normal to $\partial\Omega$ at x_0. From step 3, we have that u_n is non-decreasing in e direction for all $x \in \Sigma_{\gamma,e}$ and $a(e) < \gamma < \gamma_0$.

If $\theta \in \mathbb{S}^{N-1}$ be any other direction close to e, then the reflection of $\Sigma_{\gamma,\theta}$ w.r.t $\mathbb{T}_{\gamma,\theta}$ will still be in Ω due to the strict convexity of the domain and so u_n will be non-decreasing in the θ direction. Choose $\gamma = \frac{\gamma_0}{2}$ and consider the region $\Sigma_{\frac{\gamma_0}{2},e}$, since Ω is strictly convex there exists a small neighbourhood $\Theta \in \mathbb{S}^{N-1}$ such that $\Sigma_{\frac{\gamma_0}{2},e} \subset \Sigma_{\gamma_0,\theta}$ for all $\theta \in \Theta$. Hence, u_n is non-decreasing in every direction $\theta \in \Theta$ and for any x with $x.e < \frac{\gamma_0}{2}$.

Set

$$\Sigma_0 = \left\{x \in \Omega : \frac{\gamma_0}{8} < x.e < \frac{3\gamma_0}{8}\right\}.$$

Clearly $\Sigma_0 \subset \Sigma_{\frac{\gamma_0}{2},e}$ and u_n is non-decreasing in any direction $\theta \in \Theta$ and $x \in \Sigma_0$. Finally, choose $\epsilon = \frac{\gamma_0}{8}$ and fix any point $x \in \Omega_{\epsilon'}$. If x_0 is the projection of this point on $\partial\Omega$, then

$$u_n(x) \leq u_n(x_0 - \epsilon e) \leq u_n(y)$$

for all $y \in I_x$, where $I_x \subset \Sigma_0$ is the truncated cone with vertex at $x_0 - \epsilon' e$ and opening angle $\frac{\Theta}{2}$. Moreover, I_x has the following properties:

- $|I_x| > \kappa$ for some κ depending only on Ω and ϵ.
- $u_n(x) \leq u_n(y)$ for all $y \in I_x$ and $n \in \mathbb{N}$.

Step 5 (Deriving the boundary estimates) Using Picone's identity (Allegretto-Huang [26]) on e_1 the first eigenfunction of the p-Laplacian on Ω and u_n one has using the strong maximum principle of Vázquez [12] that $\frac{e_1^p}{u_n^{p-1}} \in W_0^{1,p}(\Omega)$.

Therefore,

$$\begin{aligned} \int_\Omega \frac{[u_n^q + f_n(u_n)]e_1^p}{u_n^{p-1}} \,\mathrm{d}x &= \int_\Omega |\nabla u_n|^{p-2} \nabla u_n \nabla \left(\frac{e_1^p}{u_n^{p-1}}\right) \,\mathrm{d}x \\ &\leq \int_\Omega |\nabla e_1|^p \mathrm{d}x \leq C(\Omega). \end{aligned} \tag{14}$$

Let $e_1(z) \geq \zeta > 0$ for all $z \in \Omega - \Omega_{\frac{\epsilon'}{2}}$. Hence from (14), we deduce

$$\zeta^p \int\limits_{\Omega-\Omega_{\frac{\epsilon'}{2}}} \frac{[u_n^q + f_n(u_n)]}{u_n^{p-1}} \,\mathrm{d}x \leq C(\Omega)$$

which would then imply that

$$\int\limits_{I_x} \frac{[u_n^q + f_n(u_n)]}{u_n^{p-1}} \,\mathrm{d}x \leq \frac{C(\Omega)}{\zeta^p}.$$

Now, since

$$\int\limits_{I_x} \frac{[u_n^q + f_n(u_n)]}{u_n^{p-1}} \,\mathrm{d}x \geq \int\limits_{I_x} u_n^{q-p+1}(y)\mathrm{d}y \geq u_n^{q-p+1}(x)|I_x|, \tag{15}$$

we have

$$u_n^{q-p+1}(x) \leq \frac{C'(\Omega)}{\zeta^p}$$

for some constant $C' > 0$, i.e, $u_n(x) \leq \bar{C}$ for all $x \in \Omega_\epsilon$ and for all $n \in \mathbb{N}$.

Step 6 (Initiating the blow-up analysis) For any open set $\Omega' \subset\subset \Omega$, there exists $C(\Omega')$ such that $||u||_\infty < C(\Omega')$ for every solution u_n of $(P_{n,\lambda})$.

Assume by contradiction that there is a sequence (u_n) of positive solutions of $(P_{n,\lambda})$ and a sequence of points $x_n \in \Omega$ such that $M_n = u_n(P_n) = \max\{u_n(x) : x \in \bar{\Omega}'\} \to \infty$ as $n \to \infty$. Using the boundary estimates, we can safely assume that $x_n \to x_0 \in \bar{\Omega}'$ as $n \to \infty$. Let $2d$ be the distance of $\bar{\Omega}'$ to $\partial\Omega$ and assume $\Omega_d = \{x \in \Omega : \mathrm{dist}(x, \Omega') < d\}$.

Let R_n be the sequence of positive numbers such that $R_n^{\frac{p}{q-p+1}} M_n = 1$. Clearly, $R_n \to 0$ as $M_n \to \infty$.

Define the scaled function $v_n : B(0, \frac{d}{R_n}) \to \mathbb{R}$ such that

$$v_n(y) = R_n^{\frac{p}{q-p+1}} u_n(P_n + R_n y).$$

Since u_n attains its maxima at P_n, we have $||v_n||_\infty = v_n(0) = 1$.

Again as $R_n \to 0$, we can choose a n_0 such that $B(0, R) \subset B(0, \frac{d}{R_n})$ for a fixed $R > 0$ and $n \geq n_0$.

Also we have that v_n satisfies the following:

$$\nabla v_n(y) = R_n^{\frac{p}{q-p+1}+1} \nabla u_n(P_n + R_n y)$$

$$\text{hence,} \quad -\Delta_p v_n(y) = R_n^{\frac{pq}{q-p+1}} [\lambda f_n(u_n(P_n + R_n y)) + R_n^{\frac{-pq}{q-p+1}} v_n^q(P_n + R_n y)]$$

Since $P_n + R_n y \in \bar{\Omega}_d \subset \Omega$ for any $y \in B(0, R)$, we have from Lemma 1.1 and Theorem 1.3,

$$R_n^{\frac{pq}{q-p+1}} [\lambda f_n(u_n(P_n + R_n y)) + R_n^{\frac{-pq}{q-p+1}} v_n^q(P_n + R_n y)] \leq C(\bar{\Omega}_d)$$

for all $n \geq n_0$. Fixing a ball $\bar{B} \in B(0, \frac{d}{R_n})$ for all $n \geq n_0$, from the interior estimates of Tolksdorf [24] and Lieberman [22], we get the existence of some constant $K > 0$ and $\beta \in (0, 1)$ depending only on N, p, B such that

$$v_n \in C^{1,\beta}(\bar{B}) \text{ and } ||v_n||_{1,\beta} \leq K.$$

This allows us to deduce the existence of a function $v \in C^1(\bar{B})$ and a convergence subsequence $v_n \to v$ in $C^1(\bar{B})$ from Ascoli–Arzelá theorem. Passing to the limit, we have

$$\begin{aligned} &\int_B |\nabla v|^{p-2}\nabla v \nabla \phi \,\mathrm{d}x = \int_B v^q \phi \,\mathrm{d}x \ \text{ in } \ B, \ \phi \in C_c^\infty(B) \\ &v \in C^1(\bar{B}), \ \ v \geq 0 \ \text{ on } \ \bar{B}. \end{aligned}$$

Moreover, we also have $||v||_\infty = 1$. Using strong maximum principle of Vázquez [12], we also have $v(x) > 0$ for all $x \in B$. Taking larger and larger balls, we obtain a Cantor diagonal subsequence which converges to $v \in C^1(\mathbb{R}^N)$ on all compact subsets of $\mathbb{R}^N$ and satisfy

$$\begin{aligned} &\int_{\mathbb{R}^N} |\nabla v|^{p-2}\nabla v \nabla \phi \,\mathrm{d}x = \int_{\mathbb{R}^N} v^q \phi \,\mathrm{d}x \ \text{ in } \ \mathbb{R}^N, \ \phi \in C_c^\infty(\mathbb{R}^N) \\ &v \in C^1(\mathbb{R}^N), \ \ v > 0 \ \text{ in } \ \mathbb{R}^N \end{aligned}$$

which is a contradiction to Theorem 1.1. □

Before we begin with the proof of Lemma 1.5, we state some lemmas. We will provide proof in cases where they are generalised for p-Laplacian.

Lemma 1.7. (DeFigueiredo et al. [35]). *Let C be a cone in a Banach space X and $\phi : C \to C$ be a compact map such that $\phi(0) = 0$. Assume that there exists $0 < r < R$ such that*

1. *$x \neq t\phi(x)$ for $0 \leq t \leq 1$ and $||x|| = r$,*
2. *a compact homotopy, $F : \overline{B}_R \times [0, \infty) \to C$ such that $F(x, 0) = \phi(x)$ for $||x|| = R$, $F(x, t) \neq x$ for $||x|| = R$ and $0 \leq t < \infty$ and $F(x, t) = x$ has no solution $x \in \overline{B}_R$ for $t \geq t_0$.*

Then if $U = \{x \in C : r < ||x|| < R\}$ and $B_\rho = \{x \in C : ||x|| < \rho\}$, we have $deg(I - \phi, B_R, 0) = 0, deg(I - \phi, B_r, 0) = 1$ and $deg(I - \phi, U, 0) = -1$.

Let us define the set

$$\mathbb{P} = \{u \in C_0^{1,\alpha}(\bar{\Omega}) : u(x) \geq 0 \ \text{ in } \ \overline{\Omega}\}.$$

Clearly

$$\mathbb{P}^\sim = \left\{ u \in C^{1,\alpha}(\bar{\Omega}) : u(x) > 0 \text{ in } \Omega \text{ and } \frac{\partial u}{\partial \eta}(x) < 0 \text{ for all } x \in \partial\Omega \right\}$$

is the interior of $\mathbb{P}$, where η is the unit outward normal to $\partial\Omega$.

Lemma 1.8. *Suppose u and $\overline{u}$ are the solution and super-solution to Eq. (4) in $C_0^{1,\alpha}(\bar{\Omega})$. If $u \neq \overline{u}$, then $\overline{u} - u$ is not on $\partial\mathbb{P}$, where $\partial\mathbb{P}$ is the boundary of $\mathbb{P}$.*

Proof. Assume $\overline{u} - u \in \partial\mathbb{P}$. Hence, we have $\overline{u}(x) \geq u(x)$ in Ω. Using Theorem 1.3, we have $\bar{u} - u > 0$ in Ω. Now Theorem 1.2 gives $\overline{u} - u \in \mathbb{P}^\sim$. Since $\mathbb{P}^\sim \cap \partial\mathbb{P} = \emptyset$, we arrive at a contradiction to our assumption. □

Lemma 1.9. *Suppose $I \subset \mathbb{R}$ is an interval and let $\Sigma \subset I \times C_0^{1,\alpha}(\overline{\Omega})$ be a connected set of solutions of Eq. (4). Consider a continuous map $U : I \to C_0^{1,\alpha}(\overline{\Omega})$ such that $U(\lambda)$ is a super-solution of (4) for every $\lambda \in I$, but not a solution. If $u_0 \leq U(\lambda_0)$ in Ω but $u_0 \neq U(\lambda_0)$ for some $(\lambda_0, u_0) \in \Sigma$ then $u < U(\lambda)$ in Ω for all $(\lambda, u) \in \Sigma$.*

Proof. Consider a continuous map,

$$T : I \times C_0^{1,\alpha}(\overline{\Omega}) \to C_0^{1,\alpha}(\overline{\Omega}) \text{ given by } T(\lambda, u) = U(\lambda) - u.$$

Since T is a continuous operator, $T(\Sigma)$ is connected in $C_0^{1,\alpha}(\overline{\Omega})$. By Lemma 1.8, $T(\sum)$ completely lies in $P^{\sim}$ or completely outside P. Since $T(\lambda_0, u_0) \in \mathbb{P}$, we have $T(\Sigma) \subset \mathbb{P}^{\sim}$ and, therefore, $u < U(\lambda)$ for all $(\lambda, u) \in \Sigma$. □

Lemma 1.10 (Ambrosetti-Arcoya [36]). *Given X be a real Banach space with $U \subset X$ be open, bounded set. Let $a, b \in \mathbb{R}$ such that the equation $u - T(\lambda, u) = 0$ has no solution on ∂U for all $\lambda \in [a, b]$ and that $u - T(\lambda, u) = 0$ has no solution in $\overline{U}$ for $\lambda = b$.*

Also let $U_1 \subset U$ be open such that $u - T(\lambda, u) = 0$ has no solution in ∂U_1 for $\lambda = a$ and $deg(I - K_a, U, 0) \neq 0$. Then there exists a continuum C in $\Gamma = \{(\lambda, u) \in [a, b] \times X : u - T(\lambda, u) = 0\}$ such that

$$C \cap (\{a\} \times U_1) \neq \emptyset \ \text{ and } \ C \cap (\{a\} \times (U - U_1)) \neq \emptyset$$

Proof of Lemma 1.5. We proceed by splitting the proof into several steps:

Step 1: (Existence of a super-solution which is not a solution) Define, $A(s) = \frac{1}{2}\Big((\frac{s}{T})^{\delta+p-1} - s^{\delta+q}\Big)$ for $s \in [0, \infty)$ and T is as in Lemma 1.4 and define

$$\beta = \max_{0 \leq s \leq \min\{\delta_0, \delta_1\}} A(s),$$

where $\delta_1 = (2q - 2p + 3)^{\frac{1}{p-q-1}} T^{\frac{\delta+p-1}{p-q-1}}$. Clearly for $\lambda_0 \in (0, \beta)$ and $\delta_2 = \min\{\delta_0, \delta_1\}$, we have A is strictly positive on $(0, \delta_2)$ and so $\beta > 0$. Hence, by I.V.P of continuous functions, there exists a $\mu \in (0, \delta_2)$ such that $A(\mu) = \lambda_0$.

If we set $\lambda_* = (\frac{\mu}{T})^{\delta+p-1}$, then

$$\lambda_* > \lambda_0 + \mu^{\delta+q} = \lambda_0 + [T(\lambda_*)^{\frac{1}{\delta+p-1}}]^{\delta+q}.$$

Hence, for w_{n,λ_*} satisfying Eq. (6) and $n \geq n_0$ one has

$$\lambda_* > \lambda_0 + ||w_{n,\lambda_*}||_\infty^q \big(||w_{n,\lambda_*}||_\infty + \frac{1}{n}\big)^\delta$$

which can be rewritten as

$$\lambda_* > \lambda + w_{n,\lambda_*}^q \big(w_{n,\lambda_*} + \frac{1}{n}\big)^\delta \ \text{ for } \ \lambda \leq \lambda_0.$$

Therefore,

$$-\Delta_p w_{n,\lambda_*} = \frac{\lambda_*}{\big(w_{n,\lambda_*} + \frac{1}{n}\big)^\delta} > \frac{\lambda}{(w_{n,\lambda_*} + \frac{1}{n})^\delta} + w_{n,\lambda_*}^q \ \text{ for } \ \lambda \leq \lambda_0 \ \text{ and } \ n \geq n_0.$$

Hence, we have the existence of a super-solution $w_{n,\lambda_*} \in C^{1,\alpha}(\bar{\Omega})$ for some $\alpha > 0$ with $||w_{n,\lambda_*}||_\infty \leq \mu$ which is not a solution to (4).

Step 2: (Existence of an unique solution with a particular norm) Define

$$F_n(s) = \frac{\lambda(s+\frac{1}{n})^{-\delta} + s^q}{s^{p-1}} \text{ for } s \in (0,\infty).$$

Using the convexity of the function $s^q(s+\frac{1}{n})^{1+\delta}$, we can derive the existence of a unique $M_n > 0$ which is increasing w.r.t λ such that

$$\lambda(p+\delta-1)M_n + \frac{p-1}{n} = (q-p+1)M_n^q(M_n+\frac{1}{n})^{1+\delta}.$$

Moreover, one also has

$$(q-p+1)s^q\left(s+\frac{1}{n}\right)^{1+\delta} \leq \lambda(p+\delta+1)s + \frac{p-1}{n}$$

for $s \leq M_n$. From the above, we conclude that

$$F_n'(s) = \frac{1}{s^p}\Big[\frac{\lambda(1-p-\delta)s + \frac{1-p}{n}}{(s+\frac{1}{n})^{1+\delta}}\Big] + (q-p+1)s^{q-p} < 0.$$

Hence, F_n is decreasing and by Díaz-Saá [37], we have the existence of an unique solution to equation (4) s.t $||u_n||_\infty \leq M_n$.

Moreover, from step 1, we have for $\mu < \delta_1$,

$$\frac{q-p+1}{\delta+p-1}\mu^{\delta+q} < \lambda_0$$

provided $\delta > 1$. So

$$M_n(\lambda_0) \geq M_n(\lambda_n) = \mu + \epsilon$$

for all $n \geq m_1$, where λ_m is defined as

$$\lambda_m := \frac{(q-p+1)(\mu+\epsilon)^q(\mu+\epsilon+\frac{1}{m})^{1+\delta}}{(\mu+\epsilon)(\delta+p-1)+\frac{p-1}{m}} < \lambda_0.$$

Step 3: (Existence of two distinct solution) Let $n \geq N$, where $N = \max\{n_0, m_1\}$ and define $K_\lambda : C(\bar{\Omega}) \to C(\bar{\Omega})$ by

$$K_\lambda(u_n) = (-\Delta_p)^{-1}(\lambda f_n(u_n) + u_n^q);\ \ \lambda \geq 0.$$

Using the compactness of $(-\Delta_p)^{-1}$ on $C(\bar{\Omega})$, we can assume that K_λ is also compact map. Note that one can view Eq. (4) as the fixed point equation given by $u_n = K_\lambda(u_n)$.

Recall from Lemma 1.3, we have Eq. (4) does not admit any solution for $\lambda \geq \bar{\Lambda}$. So for $\lambda \in [0, \bar{\Lambda})$, choose R_n (depending on n) such that $||u_n||_\infty \leq R_n$.

Consider the positive cone of $C(\bar{\Omega})$ given by

$$C = \{u_n \in C(\overline{\Omega}) : u_n \geq 0 \text{ in } \Omega\},\ R := R_n.$$

Define

$$K_0 : C \to C \text{ by } \phi(u_n) = (-\Delta_p)^{-1}u_n^q$$

and

$$F : \bar{B}_R \times [0,\infty) \to C \text{ by } F(u_n,\lambda) = (-\Delta_p)^{-1}(\lambda f_n(u_n) + u_n^q).$$

Using Lemmas 1.2, 1.3 and 1.4, we conclude that K_0 and F satisfies all the conditions in Lemma 1.7 for some $0 < r < R$.

Hence, we have $\deg(I - K_0, B_R, 0) = 0$ and $\deg(I - K_0, B_r, 0) = 1$.

Setting $X = C^{1,\alpha}(\bar{\Omega}), a = 0, b = \bar{\Lambda}, T(\lambda, u_n) = K_\lambda(u_n), U = B_{R_n}$ and $U_1 = B_r$ in Lemma 1.10, we get a continuum $C_n \subset \Gamma = \{(\lambda, u_n) \in [0, \bar{\Lambda}] \times X : u_n - K_\lambda(u_n) = 0\}$ such that

$$C_n \cap (\{0\} \times B_r) \neq \emptyset,\ C_n \cap (\{0\} \times (B_{R_n} - B_r)) \neq \emptyset. \tag{16}$$

Define the continuous map $U : [0, \lambda_0] \to C_0^{1,\alpha}(\overline{\Omega})$ by $U(\lambda) = w_{n,\lambda^*}\ \forall\, \lambda \in [0, \lambda_0]$.

Since Ω satisfies the interior sphere condition, we can apply Lemma 1.8 to deduce that every pair (λ, u_n) belonging to the connected component of $C_n \cap ([0, \lambda_0] \times C^{1,\alpha}(\bar{\Omega}))$ which emanates from $(0, 0)$ lies pointwise below the branch $\{(\lambda, U(\lambda)) : 0 \leq \lambda \leq \lambda_0\}$ at least until it crosses $\lambda = \lambda_0$.

In particular, there exists u_n in the slice $C_n^{\lambda_0} = \{v \in C^{1,\alpha}(\overline{\Omega}) : (\lambda_0, v) \in C_n\}$ which satisfies that $0 < u_n < w_{n,\lambda^*}$. Recalling that $||w_{n,\lambda^*}|| \leq \mu$, we have $||u_n||_\infty \leq ||w_{n,\lambda^*}||_\infty \leq \mu$.

Clearly, from step 2, we have u_n is the unique solution of equation (4) with small norm, e.g., $||u_n||_\infty \leq \mu + \epsilon$.

Again by (16) one has $C_n \cap (\{0\} \times (B_{R_n} - B_{\mu+\epsilon})) \neq \emptyset$ and so we conclude also the existence of v_n such that $||v_n||_\infty \geq \mu + \epsilon$.

Hence, we have the existence of two distinct solution for $\lambda = \lambda_0$; since $\lambda_0 < \bar{\Lambda}$ is arbitrary, we have our required result. □

2. Proof of Main Result

Proof of Theorem 0.1. From Lemma 1.5, we have the existence of at least two solutions u_n and v_n solving Eq. (4).

Note that we can choose $c > 0$ such that $\underline{u} = (c\phi_1 + n^{\frac{1+p-\delta}{p}})^{\frac{p}{\delta+p-1}} - \frac{1}{n}$ will be a weak sub-solution to problem (6) for $\lambda = \lambda_0$.

Since $\frac{\lambda_0}{(s+\frac{1}{n})^\delta} \leq \frac{\lambda_0}{(s+\frac{1}{n})^\delta} + s^q$ for $s \geq 0$, one concludes that each solution of (4) with $\lambda = \lambda_0$ is a super-solution of (6) with $\lambda = \lambda_0$.

Using Theorem 1.2, we have

$$\underline{u} \leq w_{n,\lambda_0} \leq u_n \leq \mu,\ \underline{u} \leq w_{n,\lambda_0} \leq v_n \text{ and } ||v_n||_\infty \geq \mu + \epsilon > \mu. \tag{17}$$

Let $z_n = u_n$ or v_n so from (17) and Lemma 1.4, we have

$$\underline{u} \leq z_n \leq M,$$

where M is independent of n. By Theorem 1.2 and Lemma 1.1, we have

$$\forall\, \omega \subset\subset \Omega,\ \exists\, c_\omega : z_n \geq c_\omega > 0 \text{ in } \omega \text{ and for all } n \in \mathbb{N}. \tag{18}$$

We now claim that z_n is bounded in $W^{1,p}_{loc}(\Omega)$.

Let $\phi \in C_0^1(\Omega)$ and taking $z_n\phi^p$ as test function in Eq. (4), we get

$$\int_\Omega |\nabla z_n|^p \phi^p \,\mathrm{d}x = -p\int_\Omega \phi^{p-1} z_n |\nabla z_n|^{p-2}\nabla\phi\nabla z_n \,\mathrm{d}x + \int_\Omega \frac{\lambda_0 z_n \phi^p}{(z_n + \frac{1}{n})^\delta} + \int_\Omega z_n^{q+1}\phi^p \mathrm{d}.x$$

Again using Young's inequality with ϵ, we have $\int_\Omega |\nabla z_n|^p \phi^p \leq c_\phi$ for all $n \in \mathbb{N}$ for some c_ϕ depending only on ϕ. So $z_n \in W^{1,p}_{loc}(\Omega)$.

Hence, there exists $z \in W^{1,p}_{loc}(\Omega) \cap L^\infty(\Omega)$ such that up to a subsequence $z_n \to z$ a.e. to z weakly in $W^{1,p}(\omega)$ for all $\omega \subset\subset \Omega$.

Therefore, applying dominated convergence theorem, we deduce that

$$\lim_{n\to\infty} \int_\Omega \Big(\frac{\lambda_0}{\left(z_n + \frac{1}{n}\right)^\delta} + \phi z_n^q\Big)\, \mathrm{d}x = \lambda_0 \int_\Omega \frac{\phi}{z^\delta}\, \mathrm{d}x + \int_\Omega \phi z^p\, \mathrm{d}x.$$

Again since $||u_n||_\infty \le \mu$, $||v_n||_\infty \ge \mu + \epsilon > \mu$ and $u_n \to u, v_n \to v$ a.e. we have the existence of two distinct solution u and v in $W^{1,p}_{loc}(\Omega)$. Now we will prove that for some $\alpha > 0$, we have $u^\alpha, v^\alpha \in W^{1,p}_0(\Omega)$. Fix $\alpha > \frac{(p-1)(\delta+p-1)}{p^2}$ and $\theta = p(\alpha - 1) + 1$; hence, $\theta > \frac{(\delta-1)(p-1)}{p}$.

Take $\phi = (z_n + \frac{1}{n})^\theta - (\frac{1}{n})^\theta$ as a test function in Eq. (4) to obtain

$$\begin{aligned}\int_\Omega |\nabla\big((z_n + \frac{1}{n})^\alpha - \frac{1}{n^\alpha}\big)|^p dx &= \alpha^p \int_\Omega \big(z_n + \frac{1}{n}\big)^{(\alpha-1)p} |\nabla z_n|^p \mathrm{d}x \\ &\le \lambda_0 \int_\Omega \big(z_n + \frac{1}{n}\big)^{\theta-\delta} + \int_\Omega \big(z_n + \frac{1}{n}\big)^\theta z_n^q \mathrm{d}x \\ &\le \lambda_0 \int_\Omega (z_n + 1)^{\theta-\delta} + \int_\Omega \big(z_n + \frac{1}{n}\big)^\theta z_n^q \mathrm{d}x\end{aligned}$$

provided $\theta \ge \delta$ and then the above integration is bounded thanks to Lemma 1.4.

If $\theta < \delta$, then we have

$$\left(z_n + \frac{1}{n}\right)^{\theta-\delta} \le \left(c\phi_1 + n^{\frac{\delta+p-1}{p}}\right)^{\frac{p(\theta-\delta)}{\delta+p-1}} \le (c\phi_1)^{\frac{p(\theta-\delta)}{\delta+p-1}}.$$

Since $\theta > \frac{(\delta-1)(p-1)}{p}$, $\int_\Omega \phi_1^{\frac{p(\theta-\delta)}{\delta+p-1}} \mathrm{d}x < \infty$ (See Mohammed [38]).

Therefore, $(z_n + \frac{1}{n})^\alpha - (\frac{1}{n})^\alpha$ is bounded in $W^{1,p}_0(\Omega)$ and since z_n converges a.e. to z in Ω, we have $(z_n + \frac{1}{n})^\alpha - (\frac{1}{n})^\alpha \to^w z^\alpha$ a.e. in $W^{1,p}_0(\Omega)$.

□

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