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Multiple Solutions for a Kirchhoff-Type Equation

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Abstract. In this paper, we study a class of Kirchhoff-type equation with asymptotically linear right-hand side and compute the critical groups at a point of mountain pass type under suitable Hilbert space. The existence results of three nontrivial solutions under the resonance and non-resonance conditions are established by using the minimax method and Morse theory.

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1. Introduction

In this article, we consider the following Kirchhoff-type problems with Dirichlet boundary conditions:

$$\begin{cases} -(a+b\int_\Omega |\nabla u|^2dx)\Delta u(x) = f(x,u), & \text{in } \Omega, \\ u=0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ $(N=1,2,3), a, b > 0$, and $f : \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies:

(f_1) $f \in C^1(\bar{\Omega}\times\mathbb{R},\mathbb{R}),\ f(x,0)=0,\ f(x,t)t \geq 0$ for all $x \in \Omega$, $t \in \mathbb{R}$,

(f_2) f' is subcritical in t, i.e., there is a constant $p \in (2, 2^*), 2^* = +\infty$ for $N = 1, 2$ and $2^* = 6$ for $N = 3$ such that

$$\lim_{t\to\infty} \frac{f_t(x,t)}{|t|^{p-1}} = 0 \quad \text{uniformly for} \quad x \in \bar{\Omega},$$

(f_3) $\lim_{|t|\to 0} \frac{f(x,t)}{t} = f_0, \lim_{|t|\to\infty} \frac{f(x,t)}{t^3} = l$ uniformly for $x \in \Omega$, where f_0 and l are constants;

(f_4) $\lim_{|t|\to\infty}[f(x,t)t - 4F(x,t)] = +\infty$, where $F(x,t) = \int_0^t f(x,s)ds$.

Birkhäuser

It is pointed out in [1] that the problem (1.1) models several physical and biological systems where u describes a process which depends on its average (for example, population density). Moreover, this problem is related to the stationary analogue of the Kirchhoff equation

$$u_{tt} - \left(1 + \int_\Omega |\nabla u|^2 dx\right) \Delta u = g(x,t),$$

which was proposed by Kirchhoff [16] as an extension of the classical D'Alembert's wave equation for free vibration of elastic strings. Kirchhoff's model takes into account the changes in length of the string produced by transverse vibrations. Some early studies of Kirchhoff equations may be seen [2,4,13]. Recently, by variational methods, Alves [1], Ma-Rivera [20] studied the existence of one positive solution, and He-Zou [15] studied the existence of infinitely many positive solutions for the problem (1.1), respectively; Perera-Zhang [22] studied the existence of nontrivial solutions for the problem (1.1) via the Yang index theory; Zhang-Perera [24] and Mao-Zhang [21] studied the existence of sign-changing solutions for problem (1.1) via invariant sets of descent flow. In [24], the authors considered the 4-superlinear case:

$$\text{there exists } \nu > 4 : \nu F(x,t) \le t f(x,t),\ |t| \text{ large}, \tag{1.2}$$

which implies that there exists a constant $c > 0$ such that

$$F(x,t) \ge c(|t|^\nu - 1).$$

Note that condition (1.2) plays an important role for showing the boundedness of Palais–Smale sequences. Furthermore, by a simple calculation, it is easy to see that condition (1.2) implies that

$$\lim_{t\to+\infty} \frac{F(x,t)}{t^4} = +\infty.$$

Hence $F(x,u)$ grows in a 4-superlinear rate as $|t| \to +\infty$. In the case of $N > 3$, some related work for problem (1.1), see [17,18] and their references. In particular, in [11], Cheng-Wu studied the existence and non-existence of positive solutions for problem (1.1) with the asymptotic behavior assumption of f at zero and the more general asymptotically 4-linear than our condition (f_3) of f at infinity. In the present paper, following the idea of [7,9,10,12, 23] on the study of p-Laplacian problems, we can compute mountain pass-type critical groups under suitable Hilbert space and obtain the existence of multiple solutions of asymptotically 4-linear problem (1.1) by using Morse theory.

We need the following preliminaries. Let $E := H_0^1(\Omega)$ be the Sobolev space equipped with the inner product and the norm

$$\langle u, \varphi \rangle = \int_\Omega \nabla u \nabla \varphi \mathrm{d}x,\ ||u|| = \langle u, u \rangle^{\frac{1}{2}}$$

and λ_1 be the first eigenvalue of $(-\Delta, H_0^1(\Omega))$. We denote by $|\cdot|_p$ the usual L^p-norm. Since Ω ($\Omega \subset \mathbb{R}^3$) is a bounded domain, $E \hookrightarrow L^p(\Omega)$ continuously for $p \in [1,6]$, compactly for $p \in [1,6)$, and there exists $\gamma_p > 0$ such that

$$|u|_p \le \gamma_p ||u||, \quad \forall u \in E.$$

Seeking a weak solution of problem (1.1) is equivalent to finding a critical point u^* of C^1 functional

$$I(u) := \frac{a}{2}||u||^2 + \frac{b}{4}||u||^4 - \int_\Omega F(x,u)\mathrm{d}x, \quad \forall u \in E, \tag{1.3}$$

where $F(x,u) = \int_0^u f(x,s)ds$. Then

$$\left\langle I'(u^*), \varphi \right\rangle = (a + b||u^*||^2)\int_\Omega \nabla u^* \nabla \varphi dx - \int_\Omega f(x,u^*)\varphi \mathrm{d}x = 0, \quad \forall \varphi \in E.$$

Definition 1.1. Let $(E, ||\cdot||_E)$ be a Hilbert space with its dual space $(E^*, ||\cdot||_{E^*})$ and $I \in C^1(E,\mathbb{R})$. For $c \in \mathbb{R}$, we say that I satisfies the $(PS)_c$ condition if for any sequence $\{u_n\} \subset E$ with

$$I(u_n) \to c, I'(u_n) \to 0 \quad \text{in } E^*,$$

there is a subsequence $\{u_{n_k}\}$ such that $\{u_{n_k}\}$ converges strongly in E. Also, we say that I satisfies $(C)_c$ condition (i.e., Cerami condition) if for any sequence $\{u_n\} \subset E$ with

$$I(u_n) \to c,\ ||I'(u_n)||_{E^*}(1 + ||u_n||_E) \to 0,$$

there is subsequence $\{u_{n_k}\}$ such that $\{u_{n_k}\}$ converges strongly in E.

Lastly, to state our results, we recall some basic facts on the eigenvalue problem:

$$\begin{cases} -||u||^2 \triangle u = \mu u^3, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

μ is an eigenvalue of problem (1.4) means that there is a non-zero $u \in E$ such that

$$||u||^2 \int_\Omega \nabla u \nabla \varphi \mathrm{d}x = \mu \int_\Omega u^3 \varphi \mathrm{d}x,\ \forall \varphi \in E.$$

This u is called an eigenvector corresponding to eigenvalue μ. Set

$$J(u) = ||u||^4,\ u \in S := \left\{ u \in E : \int_\Omega u^4 \mathrm{d}x = 1 \right\}.$$

Denote by $\mathcal{A}$ the class of closed symmetric subsets of S and denote by $i(A)$ the yang index of A, let

$$\mathcal{F}_m = \{A \in \mathcal{A} :\ i(A) \geq m - 1\},$$

and set

$$\mu_m := \inf_{A \in \mathcal{F}_m} \max_{u \in A} J(u).$$

By Proposition 3.2 of Perera-Zhang [22], we know that $\{\mu_m\}$ is an unbounded eigenvalues sequence of the nonlinear problem (1.4) and

$$0 < \mu_1 \leq \mu_2 \leq \cdots.$$

Let φ_i be the normalized eigenfunction corresponding to the eigenvalue μ_i. Then the first eigenvalue μ_1 of problem (1.4) can be characterized as

$$\mu_1 := \inf_{u \in S} J(u),$$

and μ_1 can be achieved at some $\varphi_1 \in S$ and $\varphi_1 > 0$ in Ω (see [24]).

Now, we give our main results.

Theorem 1.1. *Assume conditions* (f_1)*–*(f_3) *hold,* $f_0 < a\lambda_1$ *and* $l \in (b\mu_k, b\mu_{k+1})$ *for some* $k \geq 2$*, then problem* (1.1) *has at least three nontrivial solutions.*

Theorem 1.2. *Assume conditions* (f_1)*–*(f_4) *hold,* $f_0 < a\lambda_1$ *and* $l = b\mu_k$ *for some* $k \geq 3$*, then problem* (1.1) *has at least three nontrivial solutions.*

Here, we also give an example for $f(x,t)$. It satisfies all assumptions of our Theorem 1.2.

Example A. Set

$$f(x,t) = \begin{cases} -\left(3l - \frac{a\lambda_1\epsilon}{2}\right)t + lt^3 + 2l, & t > 1; \\ \frac{a\lambda_1\epsilon}{2}t, & |t| \leq 1; \\ -\left(3l - \frac{a\lambda_1\epsilon}{2}\right)t + lt^3 - 2l, & t < -1, \end{cases}$$

where $0 < \epsilon < \min\{\frac{6l}{a\lambda_1}, 1\}$ and $l = b\mu_k$.

As previous introduction, assume condition (f_3) holds, then problem (1.1) is called asymptotically 4 linear at infinity, which means that usual condition (1.2) is not satisfied. This will bring some difficulty if the mountain pass theorem is used to seek nontrivial solutions of problem (1.1). For standard Laplacian Dirichlet problem, Zhou [25] have overcome it by using some monotonicity condition. Novelties of our this paper are as following.

We consider multiple solutions of problem (1.1) in the cases of resonance and non-resonance by using the mountain pass theorem and Morse theory. At first, we use the truncated technique and mountain pass theorem to obtain a positive solution and a negative solution of problem (1.1) under our more general condition (f_1), (f_2) and (f_3) with respect to the conditions (H_1) and (H_3) in [25]. In the course of proving the existence of positive solution and negative solution, the monotonicity condition (H_2) of [25] on the nonlinear term f is not necessary, this point is very important because we can directly prove existence of positive solution and negative solution by using Rabinowitz's mountain pass theorem. That is, the proof of our compact condition is more simple than that in [25]. Furthermore, we can obtain a nontrivial solution when the nonlinear term f is resonance or non-resonance at the infinity by computing mountain pass-type critical groups under suitable Hilbert space.

The paper is organized as follows. In Sect. 2, we prove some lemmas in order to prove our main results. In Sect. 3, we give the proofs for our main results.

2. Some Lemmas

Consider the following problem

$$\begin{cases} -(a+b\int_\Omega |\nabla u|^2\mathrm{d}x)\Delta u(x) = f_+(x,u), & \text{in } \Omega,\\ u=0, & \text{on } \partial\Omega, \end{cases}$$

where

$$f_+(x,t) = \begin{cases} f(x,t), & t>0,\\ 0, & t\le 0. \end{cases}$$

Also we set $F_+(x,t) = \int_0^t f_+(x,s)\mathrm{d}s$ and introduce the functional $I_+ : E \to \mathbb{R}$ defined by

$$I_+(u) := \frac{a}{2}||u||^2 + \frac{b}{4}||u||^4 - \int_\Omega F_+(x,u)\mathrm{d}x, \quad \forall u\in E.$$

Clearly $I_+ \in C^{2-0}(E,\mathbb{R})$.

Lemma 2.1. *I_+ satisfies the (PS) condition.*

Proof. Let $\{u_n\}\subset E$ be a sequence such that $|I_+(u_n)|\le c$, $\langle I'_+(u_n),\varphi\rangle \to 0$ as $n\to\infty$. Note that

$$\begin{aligned}\langle I'_+(u_n),\varphi\rangle &= (a+b||u_n||^2)\int_\Omega \nabla u_n\nabla\varphi\mathrm{d}x - \int_\Omega f_+(x,u_n)\varphi\mathrm{d}x\\ &= o(\|\varphi\|)\end{aligned} \tag{2.1}$$

for all $\varphi\in E$. Assume that $|u_n|_4$ is bounded, taking $\varphi = u_n$ in (2.1). By (f_3), there exists $c_1, c_2>0$ such that $|f_+(x,u_n(x))|\le c_1|u_n(x)| + c_2|u_n(x)|^3$, a.e. $x\in\Omega$. So u_n is bounded in E. If $|u_n|_4\to+\infty$, as $n\to\infty$, set $v_n = \frac{u_n}{|u_n|_4}$, then $|v_n|_4 = 1$. Taking $\varphi = v_n$ in (2.1), it follows that $\|v_n\|$ is bounded. Without loss of generality, we assume that $v_n \rightharpoonup v$ in E, then $v_n\to v$ in $L^4(\Omega)$. Hence, $v_n\to v$ a.e. in Ω. Dividing both sides of (2.1) by $|u_n|_4^3$, we get

$$\begin{aligned}&(a+b||u_n||^2)(|u_n|_4^{-2})\int_\Omega \nabla v_n\nabla\varphi\mathrm{d}x - \int_\Omega \frac{f_+(x,u_n)}{|u_n|_4^3}\varphi\mathrm{d}x\\ &\quad = o\left(\frac{\|\varphi\|}{|u_n|_4^3}\right),\ \forall\varphi\in E.\end{aligned} \tag{2.2}$$

Then for a.e. $x\in\Omega$, we deduce that $\frac{f_+(x,u_n)}{|u_n|_4^3} \to lv_+^3$ as $n\to\infty$, where $v_+ = \max\{v,0\}$. In fact, when $v(x)>0$, by (f_3) we have

$$u_n(x) = v_n(x)|u_n|_4 \to +\infty$$

and

$$\frac{f_+(x,u_n)}{|u_n|_4^3} = \frac{f_+(x,u_n)}{u_n^3}v_n^3 \to lv^3.$$

When $v(x)=0$, we have

$$\frac{f_+(x,u_n)}{|u_n|_4^3} \le c_1|v_n||u_n|_4^{-2} + c_2|v_n^3| \longrightarrow 0.$$

When $v(x) < 0$, we have

$$u_n(x) = v_n(x)|u_n|_4 \longrightarrow -\infty$$

and

$$\frac{f_+(x, u_n)}{|u_n|_4^3} = 0.$$

Since $\frac{f_+(x,u_n)}{|u_n|_4^3} \leq c_1|v_n||u_n|_4^{-2} + c_2|v_n^3|$, by (2.2) and the Lebesgue dominated convergence theorem, we arrive at

$$(b\|v\|^2)\int_\Omega \nabla v \nabla \varphi dx - \int_\Omega l v_+^3 \varphi \mathrm{d}x = 0, \text{ forany } \varphi \in E. \tag{2.3}$$

From the strong maximum principle, we deduce that $v > 0$. Choosing $\varphi = \varphi_1$ in (2.3), we obtain

$$l\int_\Omega v^3\varphi_1 \mathrm{d}x = b\mu_1 \int_\Omega v^3 \varphi_1 \mathrm{d}x.$$

This is a contradiction. □

Lemma 2.2. *Let φ_1 be the eigenfunction corresponding to μ_1 with $\|\varphi_1\| = 1$. If $f_0 < a\lambda_1$ and $l > b\mu_1$, then*

(a) There exist $\rho, \beta > 0$ such that $I_+(u) \geq \beta$ for all $u \in E$ with $\|u\| = \rho$;
(b) $I_+(t\varphi_1) = -\infty$ as $t \to +\infty$.

Proof. By (f_1) and (f_3), if $l \in (b\mu_1, +\infty)$, for any $\varepsilon > 0$, there exist $A = A(\varepsilon) \geq 0$ and $B = B(\varepsilon)$ such that for all $(x, s) \in \Omega \times \mathbb{R}$,

$$F_+(x, s) \leq \frac{1}{2}(f_0 + \varepsilon)s^2 + As^{p+1}, \tag{2.4}$$

$$F_+(x, s) \geq \frac{1}{4}(l - \varepsilon)s^4 - B, \tag{2.5}$$

where $p \in (1, 5)$.

Choose $\varepsilon > 0$ such that $f_0+\varepsilon < a\lambda_1$. By (2.4) and the Sobolev inequality, we get

$$\begin{aligned} I_+(u) &= \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \int_\Omega F(x, u)\mathrm{d}x \\ &\geq \frac{a}{2}\|u\|^2 - \frac{1}{2}\int_\Omega [(f_0 + \varepsilon)u^2 + A|u|^{p+1}]\mathrm{d}x \\ &\geq \frac{1}{2}\left(a - \frac{f_0 + \varepsilon}{\lambda_1}\right)\|u\|^2 - c\|u\|^{p+1}. \end{aligned}$$

So, part (a) holds if we choose $\|u\| = \rho > 0$ small enough.

On the other hand, if $l \in (b\mu_1, +\infty)$, take $\varepsilon > 0$ such that $l - \varepsilon > b\mu_1$. By (2.5), we have

$$I_+(t\varphi_1) \leq \frac{a}{2}t^2||\varphi_1||^2 + \frac{1}{4}\left(b - \frac{l}{\mu_1}\right)t^4||\varphi_1||^4 + B_1|\Omega| \to -\infty \text{ as } t \to \infty.$$

Thus part (b) is proved. □

Lemma 2.3. *Let $E = V \oplus W$, where $V = span\{\varphi_1, \varphi_2, \cdots, \varphi_k\}$ when $l > b\mu_k$ or $V = span\{\varphi_1, \varphi_2, \cdots, \varphi_{k-1}\}$ when $l = b\mu_k$ and $W = V^{\perp}$. If f satisfies $(f_1), (f_3)$ and (f_4) then*

(i) the functional I is coercive on W, that is

$$I(u) \to +\infty \quad \text{as } \|u\| \to +\infty, \ u \in W$$

and bounded from below on W,

(ii) the functional I is anti-coercive on V.

Proof. We firstly prove this conclusion for $l > b\mu_k$.

For $u \in W$, by (f_1) and (f_3), for any $\varepsilon > 0$, there exists $B_1 = B_1(\varepsilon)$ such that for all $(x, s) \in \Omega \times \mathbb{R}$,

$$F(x, s) \le \frac{1}{4}(l + \varepsilon)s^4 + B_1. \tag{2.6}$$

So we have

$$\begin{aligned} I(u) &= \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \int_\Omega F(x, u)\mathrm{d}x \\ &\ge \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{1}{4}(l + \varepsilon)|u|_4^4 - B_1|\Omega| \\ &\ge \frac{1}{4}\left(b - \frac{l + \varepsilon}{\mu_{k+1}}\right)\|u\|^4 - B_1|\Omega|. \end{aligned}$$

Choose $\varepsilon > 0$ such that $l + \varepsilon < b\mu_{k+1}$. This proves (i).

For $u \in V$, again using (f_1) and (f_3), for any $\varepsilon > 0$, there exists $B_2 = B_2(\varepsilon)$ such that for all $(x, s) \in \Omega \times \mathbb{R}$,

$$F(x, s) > \frac{1}{4}(l - \varepsilon)s^4 + B_2. \tag{2.7}$$

From (2.7), we have

$$\begin{aligned} I(u) &= \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \int_\Omega F(x, u)\mathrm{d}x \\ &\le \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{1}{4}(l - \varepsilon)|u|_4^4 - B_2|\Omega| \\ &\le \frac{a}{2}\|u\|^2 + \frac{1}{4}\left(b - \frac{l - \varepsilon}{\mu_k}\right)\|u\|^4 - B_1|\Omega|. \end{aligned}$$

Choose $\varepsilon > 0$ such that $l - \varepsilon > b\mu_k$. This proves (ii).

Now we consider the case $l = b\mu_k$.

Write $G(x, t) = F(x, t) - \frac{b}{4}\mu_k t^4$, $g(x, t) = f(x, t) - b\mu_k t^3$. Then (f_3) and (f_4) imply that

$$\lim_{|t| \to \infty} [g(x, t)t - 4G(x, t)] = +\infty \tag{2.8}$$

and

$$\lim_{|t| \to \infty} \frac{4G(x, t)}{t^4} = 0. \tag{2.9}$$

It follows from (2.8) that for every $M > 0$, there exists a constant $T > 0$ such that

$$g(x,t)t - 4G(x,t) \geq M, \ \forall t \in \mathbb{R}, \ |t| \geq T, \ \text{a.e. } x \in \Omega. \tag{2.10}$$

For $\tau > 0$, we have

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{G(x,\tau)}{\tau^4} = \frac{g(x,\tau)\tau - 4G(x,\tau)}{\tau^5}. \tag{2.11}$$

Integrating (2.11) over $[t,s] \subset [T,+\infty)$, we deduce that

$$\frac{G(x,s)}{s^4} - \frac{G(x,t)}{t^4} \geq -\frac{M}{4}\left(\frac{1}{s^4} - \frac{1}{t^4}\right). \tag{2.12}$$

Letting $s \to +\infty$ and using (2.9), we see that $G(x,t) \leq -\frac{M}{4}$, for $t \in \mathbb{R}, \ t \geq T$, a.e. $x \in \Omega$. A similar argument shows that $G(x,t) \leq -\frac{M}{4}$, for $t \in \mathbb{R}, \ t \leq -T$, a.e. $x \in \Omega$. Hence

$$\lim_{|t|\to\infty} G(x,t) \to -\infty, \ \text{a.e. } x \in \Omega. \tag{2.13}$$

For $u \in W$, by (2.13), we get

$$\begin{aligned} I(u) &= \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \int_\Omega F(x,u)\mathrm{d}x \\ &\geq \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{b}{4}\mu_k|u|_4^4 - \int_\Omega G(x,u)\mathrm{d}x \\ &\geq \frac{a}{2}\|u\|^2 - \int_\Omega G(x,u)\mathrm{d}x \to +\infty \end{aligned}$$

for $u \in W$ with $\|u\| \to \infty$.

The proof of conclusion (ii) is completely identical to the case $l > b\mu_k$. Hence we omit it here. □

Lemma 2.4. *If $b\mu_k < l < b\mu_{k+1}$, then I satisfies the (PS) condition.*

Proof. Let $\{u_n\} \subset E$ be a sequence such that $|I(u_n)| \leq c$, $\langle I'(u_n), \varphi\rangle \to 0$. Since

$$\begin{aligned} \langle I'(u_n), \varphi\rangle &= (a + b||u_n||^2)\int_\Omega \nabla u_n \nabla\varphi \mathrm{d}x - \int_\Omega f(x,u_n)\varphi\mathrm{d}x \\ &= o(\|\varphi\|) \end{aligned} \tag{2.14}$$

for all $\varphi \in E$. If $|u_n|_4$ is bounded, we can take $\varphi = u_n$. By (f_3), there exists a constant $c_1, c_2 > 0$ such that $|f(x,u_n(x))| \leq c_1|u_n(x)| + c_2|u_n(x)|^3$, a.e. $x \in \Omega$. So u_n is bounded in E. If $|u_n|_4 \to +\infty$, as $n \to \infty$, set $v_n = \frac{u_n}{|u_n|_4}$, then $|v_n|_4 = 1$. Taking $\varphi = v_n$ in (2.14), it follows that $\|v_n\|$ is bounded. Without loss of generality, we assume $v_n \rightharpoonup v$ in E, then $v_n \to v$ in $L^4(\Omega)$. Hence, $v_n \to v$ a.e. in Ω. Dividing both sides of (2.14) by $|u_n|_4$, we get

$$\begin{aligned} &(a + b||u_n||^2)(|u_n|_4^{-2})\int_\Omega \nabla v_n \nabla\varphi dx - \int_\Omega \frac{f(x,u_n)}{|u_n|_4^3}\varphi dx \\ &= o\left(\frac{||\varphi||}{|u_n|_4^3}\right), \ \forall\varphi \in E. \end{aligned} \tag{2.15}$$

Then for a.e. $x \in \Omega$, we have $\frac{f(x,u_n)}{|u_n|_4^3} \to lv^3$ as $n \to \infty$. In fact, if $v(x) \neq 0$, by (f_3), we have

$$|u_n(x)| = |v_n(x)||u_n|_4 \to +\infty$$

and

$$\frac{f(x,u_n)}{|u_n|_4^3} = \frac{f(x,u_n)}{u_n^3}v_n^3 \to lv^3.$$

If $v(x) = 0$, we have

$$\frac{|f(x,u_n)|}{|u_n|_4^3} \leq c_1|v_n||u_n|_4^{-2} + c_2|v_n|^3 \longrightarrow 0.$$

Since $\frac{|f(x,u_n)|}{|u_n|_4^3} \leq c_1|v_n||u_n|_4^{-2} + c_2|v_n|^3$, by (2.15) and the Lebesgue dominated convergence theorem, we arrive at

$$(b\|v\|^2)\int_\Omega \nabla v \nabla \varphi dx - \int_\Omega lv^3\varphi dx = 0, \text{ for any } \varphi \in E.$$

Obviously $v \neq 0$, hence, this contradicts our assumption. □

Lemma 2.5. *Suppose $l = b\mu_k$ and I satisfies (f_4). Then the functional I satisfies the (C) condition.*

Proof. Suppose $u_n \in E$ satisfies

$$I(u_n) \to c \in \mathbb{R}, \quad (1 + \|u_n\|)\|I'(u_n)\| \to 0 \text{ as } n \to \infty. \tag{2.16}$$

In view of (f_3), it suffices to prove that u_n is bounded in E. Similar to the proof of Lemma 2.4, we have

$$(b\|v\|^2)\int_\Omega \nabla v \nabla \varphi \mathrm{d}x - \int_\Omega lv^3\varphi \mathrm{d}x = 0, \text{ for any } \varphi \in E. \tag{2.17}$$

Therefore, $v \neq 0$ is an eigenfunction of μ_k, then $|u_n(x)| \to \infty$ for a.e. $x \in \Omega_0$ $(\Omega_0 \subset \Omega)$ with positive measure. It follows from (f_4) that

$$\lim_{n\to+\infty}[f(x,u_n(x))u_n(x) - 4F(x,u_n(x))] = +\infty$$

holds uniformly in $x \in \Omega_0$, which implies that

$$\int_\Omega (f(x,u_n)u_n - 4F(x,u_n))\mathrm{d}x \to +\infty \text{ as } n \to \infty. \tag{2.18}$$

On the other hand, (2.16) implies that

$$4I(u_n) - < I'(u_n), u_n > \to 4c \text{ as } n \to \infty.$$

Thus

$$\int_\Omega (f(x,u_n)u_n - 4F(x,u_n))\mathrm{d}x \to -\infty \text{ as } n \to \infty,$$

which contradicts (2.18). Hence u_n is bounded. □

It is well known that critical groups and Morse theory are the important tools in solving elliptic partial differential equation. Let us recall some results which will be used later. We refer the readers to the books [6] for more information on Morse theory.

Let E be a Hilbert space and $I \in C^1(E, \mathbb{R})$ be a functional satisfying the (PS) condition or (C) condition, and $H_q(X, Y)$ be the qth singular relative homology group with integer coefficients. Let u_0 be an isolated critical point of I with $I(u_0) = c,\ c \in \mathbb{R}$, and U be a neighborhood of u_0. The group

$$C_q(I, u_0) := H_q(I^c \cap U, I^c \cap U \backslash \{u_0\}),\ q \in Z$$

is said to be the qth critical group of I at u_0, where $I^c = \{u \in E : I(u) \leq c\}$.

Let $K := \{u \in E : I'(u) = 0\}$ be the set of critical points of I and $a < \inf I(K)$, the critical groups of I at infinity are formally defined by (see [3])

$$C_q(I, \infty) := H_q(E, I^a),\ q \in Z.$$

The following result comes from [3,6] and will be used to prove the results in this paper.

Proposition 2.6. [3] *Assume that $E = V \oplus W$, I is bounded from below on W and $I(u) \to -\infty$ as $\|u\| \to \infty$ with $u \in V$. Then*

$$C_k(I, \infty) \ncong 0,\ if\ k = \dim V < \infty. \tag{2.19}$$

Next, we recall some similar results in [7,8]. We assume that (f_2) holds and u_0 is an isolated critical point of the functional I. The second-order differential of I in u_0 is given by

$$\begin{aligned}\langle I''(u_0)\varphi, w\rangle = a\int_\Omega \nabla\varphi\nabla w \mathrm{d}x - \int_\Omega f'(x, u_0)\varphi w \mathrm{d}x \\ +b\left(2\int_\Omega \nabla u_0\nabla w \mathrm{d}x \int_\Omega \nabla u_0 \nabla\varphi \mathrm{d}x + \|u_0\|^2 \int_\Omega \nabla\varphi\nabla w \mathrm{d}x\right),\end{aligned} \tag{2.20}$$

for any $\varphi, w \in E$. Let H_{u_0} be the closure of $C_0^\infty(\Omega)$ under the scalar product

$$\langle\varphi, w\rangle_{u_0} = a\int_\Omega \nabla\varphi\nabla w \mathrm{d}x + b\left(2\int_\Omega \nabla u_0\nabla w \mathrm{d}x \int_\Omega \nabla u_0\nabla\varphi \mathrm{d}x + \|u_0\|^2\int_\Omega \nabla\varphi\nabla w \mathrm{d}x\right),$$

then H_{u_0} is topological isomorphic to E. By (f_2), we know that $I''(u_0)$ is a Fredholm operator defined by setting

$$\langle I''(u_0)\varphi, w\rangle = \langle\varphi, w\rangle_{u_0} - \int_\Omega f'(x, u_0)\varphi w \mathrm{d}x \tag{2.21}$$

for any $\varphi, w \in H_{u_0}$. So we can consider splitting $H_{u_0} = H^- \oplus H^0 \oplus H^+$, where H^-, H^0, H^+ are, respectively, the negative, null, and positive space, according to the spectral decomposition of $I''(u_0)$ in $L^2(\Omega)$, and H^-, H^0 have finite dimensions. If we set $W = H^+$ and $V = H^- \oplus H^0$, then we get splitting

$$H_{u_0} = V \oplus W. \tag{2.22}$$

Now, by our assumptions (f_1) and (f_2), slightly modifying the proof of Lemmas 4.2–4.5 in [7], we will obtain four parallel results for Kirchhoff problem (1.1) as follows.

Lemma 2.7. *If $N = 1$, then there exist $r_0 > 0$ and $C > 0$ such that for any $\eta \in E$, $\|\eta - u_0\| < r_0$, we have*

$$\langle I^{''}(\eta)\varphi, \varphi\rangle \geq C\|\varphi\|^2_{u_0}$$

for any $\varphi \in W$.

Lemma 2.8. *Let $\tau > 0$. If $\eta \in B_\tau(u_0) \subset E$ is a solution of*

$$(a + b\|\eta\|^2)\int_\Omega \nabla\eta\nabla w \mathrm{d}x - \int_\Omega f(x, \eta)w\mathrm{d}x = 0$$

for any $w \in W$, then $\eta \in L^\infty(\Omega)$. Moreover, there exists $K^* > 0$ such that $\|\eta\|_\infty \leq K^*$ with K^* depending on τ.

Lemma 2.9. *If $N = 2, 3$, for any $M > 0$, then there exist $r_0 > 0$ and $C > 0$ such that for any $\eta \in E \cap L^\infty(\Omega)$, with $\|\eta\|_\infty \leq M$, $\|\eta - u_0\| < r_0$, we have*

$$\langle I^{''}(\eta)\varphi, \varphi\rangle \geq C\|\varphi\|^2_{u_0}$$

for any $\varphi \in W$.

Lemma 2.10. *There exists $\delta > 0$ such that for any $w \in W\backslash\{0\}$, with $\|w\| \leq \delta$, we have*

$$I(u_0 + w) > I(u_0).$$

Next, we give three auxiliary results to prove our main results in this paper.

Lemma 2.11. *There exist $r \in (0, \delta)$ and $\rho \in (0, r)$ such that for any $v \in V \cap \bar{B}_\rho(0)$ there exists one and only one $\bar{w} \in W \cap B_r(0)$ such that for any $z \in W \cap \bar{B}_r(0)$ we have*

$$I(v + \bar{w} + u_0) \leq I(v + z + u_0).$$

Moreover, $\bar{w}$ is the only element of $W \cap \bar{B}_r(0)$ such that

$$\langle I'(u_0 + v + \bar{w}), z\rangle = 0, \quad \forall z \in W.$$

Furthermore, u_0 is the only critical point of $B_r(u)$ and $B_r(u) \subset I^{c+1}$, where $c = I(u_0)$.

Proof. The proof of this result essentially derives from [7]. For convenience, we prove it. We first consider the case $N = 2, 3$. Since u_0 is an isolated critical point of I and I is continuous, we can fix $0 < \tau < \delta$ such that u_0 is the only critical point of I in $B_\tau(u_0)$ and $B_\tau(u_0) \subset I^{c+1}$. From Lemma 2.8, if $\eta \in B_\tau(u_0)$ is a solution of $\langle I'(\eta), w\rangle = 0$ for any $w \in W$, then $\|\eta\|_\infty \leq M$, where $M > 0$ is a positive constant, depending on τ. Now, by Lemma 2.9, in correspondence of $2M$, there exists $r_0 \in [0, \tau]$ such that the conclusion of Lemma 2.9 holds.

Now let $r \in [0, \frac{r_0}{3}]$. Since I is sequentially low semicontinuous with respect to the weakly topology of E. Therefore let us fix $v \in B_r(0) \cap V$; there exists a minimum point $\bar{w} \in W \cap \bar{B}_r(0)$ of the function $w \in W \cap \bar{B}_r(0) \mapsto I(u_0 + v + w)$.

We shall prove that there exists $\rho \in [0, r]$ such that for any $v \in V \cap \bar{B}_\rho(0)$ we have

$$\inf\{I(u_0 + v + w) : \ w \in W, \ \|w\| = r\} > I(u_0 + v). \tag{2.23}$$

Arguing by contradiction, we assume that there exist a sequence $\{w_n\}$ in $W \cap \partial B_r(0)$ and a sequence $\{v_n\}$ in V with $\|v_n\| \to 0$ such that

$$I(u_0 + v_n + w_n) \leq I(u_0 + v_n). \tag{2.24}$$

Since $\{w_n\}$ is bounded, there exists $\tilde{w} \in W$ such that $\{w_n\}$ weakly converges to $\tilde{w}$ in E. From Lemma 2.10, 0 is unique minimum point of the function $w \in W \cap \bar{B}_r(0) \mapsto I(u_0 + w)$, therefore, we get

$$I(u_0) \leq I(u_0 + \tilde{w}). \tag{2.25}$$

From (2.24) and (2.25), we can conclude that

$$I(u_0) = I(u_0 + \tilde{w}) = \lim_{n \to +\infty} I(u_0 + v_n + w_n). \tag{2.26}$$

Thus, we have $w_n \to w$ in E. It follows that $\|w\| = r$ which leads to a contradiction.

As a consequence, we infer that there exists $\rho \in [0, r]$ such that for any $v \in V \cap \bar{B}_\rho(0)$, (2.23) holds. Therefore, we have that for any $v \in V \cap \bar{B}_\rho(0)$ the minimum point $\bar{w}$ belongs to $W \cap B_r(0)$ and then $\langle I'(u_0 + v + \bar{w}), z\rangle = 0$ for any $z \in W$.

At last, by Lemmas 2.8, 2.9, similar to the last proof of Lemma 4.6 in [7], we also can prove that $\bar{w}$ is the only element of $W \cap \bar{B}_r(0)$ such that

$$\langle I'(u_0 + v + \bar{w}), z\rangle = 0 \quad \forall z \in W.$$

In the case $N = 1$ the proof is easier and the thesis immediately follows by Lemma 2.7, arguing as before. □

Now we can introduce that map $\psi : V \cap \bar{B}_\rho(0) \to W \cap \bar{B}_r(0)$ defined by $\psi(v) = \bar{w}$ and the function $\varphi^* : V \cap \bar{B}_\rho(0) \to \mathbb{R}$ defined by $\varphi^*(v) = I(u_0 + v + \psi(v))$, which is a continuous map by [7]. Moreover, we have that

Lemma 2.12. *For any $v \in V \cap \bar{B}_\rho(0), z \in V, w \in V$, we have*

$$\begin{aligned} &\psi \textit{ is } C^1, \psi(0) = 0, \psi'(0) = 0, \\ &\langle \varphi^{*\prime}(v), z\rangle = \langle I'(u_0 + v + \psi(v)), z\rangle, \\ &\langle \varphi^{*\prime\prime}(v)z, w\rangle = \langle I''(u_0 + v + \psi(v))(z + \psi'(v)(z)), w\rangle. \end{aligned}$$

Proof. The proof of this lemma is essentially equal to the proof of Lemma 2.2 in [8]. We omit it here. □

Lemma 2.13. *If (f_2) holds, then*

$$C_q(I, u_0) = C_q(\varphi^*, 0), \quad q \in Z.$$

Proof. By the crucial Lemma 2.11, we know that the proof of this lemma is essentially equal to the proof of two formulas (5.4) and (5.5) in [7]. We omit it here. □

3. Proof of the Main Results

Proof. (Proof of Theorem 1.1.) By Lemmas 2.1, 2.2 and the mountain pass theorem, the functional I_+ has a critical point u_1 satisfying $I_+(u_1) \geq \beta$. Since $I_+(0) = 0$, $u_1 \neq 0$ and by the maximum principle, we get $u_1 > 0$. Hence u_1 is a positive solution of the problem (1.1) and satisfies

$$C_1(I_+, u_1) \neq 0,\ \ u_1 > 0. \tag{3.1}$$

By (f_2), the functional I_+ is C^{2-0}. Now, we claim that

$$C_q(I_+, u_1) = \delta_{q,1} Z. \tag{3.2}$$

Using (2.21), for the isolated critical point u_1 we can define $V = H^- \oplus H^0 \subset H_{u_1}$, and it follows from Lemma 2.13 that there exists

$$\varphi^* : V \cap \bar{B}_\rho(0) \to \mathbb{R}$$

such that

$$C_q(I_+, u_1) = C_q(\varphi^*, 0), \quad q = 0, 1, 2, \cdots, \tag{3.3}$$

and

$$C_1(\varphi^*, 0) = C_q(I_+, u_1) \neq 0. \tag{3.4}$$

Set $m = \dim H^-$ and $n = \dim H^0$, we know that $m \leq 1$.

If $n = 0$, then 0 is a non-degenerate critical point of φ^*, and

$$C_q(\varphi^*, 0) = \delta_{q,m} Z,$$

which implies that (3.2) holds.

If $n \neq 0$, then 0 is a degenerate critical point of φ^*, and from the Shifting theorem (see [5]), we have

$$C_q(\varphi^*, 0) = C_{q-m}(\tilde{\varphi^*}, 0),\ \ q = 0, 1, 2, \cdots, \tag{3.5}$$

where $\tilde{\varphi^*}(u) = \varphi^* \mid_{H^0}$.

Case 1. If $m = 1$, then $C_0(\tilde{\varphi^*}, 0) \neq 0$, which is equivalent to 0 being an isolated local minimum of $\tilde{\varphi^*}$, so

$$C_q(\tilde{\varphi^*}, 0) = \delta_{q,0} Z,$$

then (3.2) holds.

Case 2. If $m = 0$, then (3.5) implies that

$$C_q(\varphi^*, 0) = C_q(\tilde{\varphi^*}, 0), \quad q = 0, 1, 2, \ldots. \tag{3.6}$$

Next, we show $n = 1$. For $\ker \varphi^{*\prime\prime}(0)$ to be nontrivial it amounts to saying that 1 is the first eigenvalue of the following linear eigenvalue problem

$$\begin{cases} -\mathrm{div}[(a + b\int_\Omega |\nabla u_1|^2 \mathrm{d}x)\nabla u(x) + 2b(\int_\Omega \nabla u_1 \nabla u \mathrm{d}x)\nabla u_1] = \lambda f'(x, u_1)u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

From [11, Sect. 6.1], the first eigenvalue 1 is simple, then $n = 1$. Thus, by Theorem 2.7 in [19], we have

$$C_q(I_+, u_1) = C_q(\varphi^*, 0) = C_q(\tilde{\varphi^*}, 0) = \delta_{q,1} Z.$$

Now, we claim that $C_1(I, u_1) = C_1(I_+, u_1)$. Set for all $(t, u) \in [0, 1] \times E$,

$$h_+(t, u) = (1 - t)I(u) + tI_+(u).$$

Then, for all $t \in [0,1]$, $h_+(t,\cdot) \in C^1(E)$ and $u_1 \in K(h_+(t,\cdot))$. We claim that u_1 is an isolated critical point of $h_+(t,\cdot)$, uniformly with respect to $t \in [0,1]$. Arguing by contradiction, assume that there exist sequences (t_n) in $[0,1]$ and $\{u_n\}$ in $E \setminus \{u_1\}$, respectively, such that $u_n \in K(h_+(t_n,\cdot))$ for all integer $n \geq 1$ and $u_n \to u_1$ in E. Thus, for all $n \geq 1$, u_n solves the problem

$$\begin{cases} -(a+b\|u_n\|^2)\Delta u_n = (1-t_n)f(x,u_n) + t_n f_+(x,u_n) & \text{in } \Omega \\ u_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.7}$$

By (f_1) and the result of regularity in [1], the sequence $\{u_n\}$ is bounded in $C_0^1(\Omega)$ and $u_n(x) > 0$. Thus (3.7) reduces to

$$\begin{cases} -(a+b\|u_n\|^2)\Delta u_n = f_+(x,u_n) & \text{in } \Omega \\ u_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.8}$$

i.e., $u_n \in K(I_+)$. This leads to a contradiction. Thus, we have

$$C_q(I,u_1) = \delta_{q,1}Z. \tag{3.9}$$

Similarly, we can obtain another negative critical point u_2 of I satisfying

$$C_q(I,u_2) = \delta_{q,1}Z. \tag{3.10}$$

Since $f_0 < a\lambda_1$, the zero function is a local minimizer of I, then

$$C_q(I,0) = \delta_{q,0}Z. \tag{3.11}$$

On the other hand, by Lemmas 2.3, 2.4 and Proposition 2.6, we have

$$C_k(I,\infty) \ncong 0. \tag{3.12}$$

Hence I has a critical point u_3 satisfying

$$C_k(I,u_3) \ncong 0. \tag{3.13}$$

Since $k \geq 2$, it follows from (3.9)–(3.13) that u_1, u_2 and u_3 are three different nontrivial solutions of the problem (1.1). □

Proof. (Proof of Theorem 1.2.) By Lemmas 2.3, 2.5 and Proposition 2.6, we can prove the conclusion (3.12). The other proof is similar to that of Theorem 1.1. □

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