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Spectra of Weighted Composition Operators on Analytic Function Spaces

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Abstract. Let E be a complex Banach space with open unit ball B_E. For analytic self-maps φ of B_E with $\varphi(0) = 0$, we investigate the spectra of weighted composition operators uC_φ acting on a large class of spaces of analytic functions. This class contains, for example, weighted Banach spaces of H^∞-type on B_E, weighted Bergman spaces $A^p_\alpha(\mathbb{B}_N)$ and Hardy spaces $H^p(\mathbb{B}_N)$. We present a general approach for deducing new information about the spectrum and for estimating the essential spectral radius of uC_φ.

Mathematics Subject Classification. Primary 47B33; Secondary 47B38.

Keywords. Spectra, essential spectral radius, Hardy spaces, weighted Bergman spaces, weighted composition operator.

1. Introduction

Throughout this article, E stands for a complex Banach space of arbitrary dimension and $B_E = \{x \in E : \|x\| < 1\}$ for its open unit ball. Moreover, let $\varphi : B_E \to B_E$ be an analytic mapping and $u \in H(B_E)$, where $H(B_E)$ is the space of analytic functions on B_E. Recall that a mapping is analytic if it is Fréchet differentiable at every point in its domain. Each such pair (φ, u) induces via composition and multiplication a weighted composition operator $uC_\varphi(f) = u(f \circ \varphi)$ which preserves $H(B_E)$. Our object of study is the operator uC_φ acting on a Banach space, $X(B_E)$, of analytic functions on B_E, specifically, its spectrum $\sigma(uC_\varphi)$. This is a topic of current interest; see for instance [4,5,7,11,13,16,25] and other references quoted below. Very little is known about the spectrum of the composition operator C_φ acting on classical analytic function spaces for a non-univalent symbol φ of the open unit ball $\mathbb{B}_N$ in $\mathbb{C}^N$.

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Some information about the space $X(B_E)$ is unavoidable to obtain interesting results for uC_φ acting on $X(B_E)$. Therefore, the space $X(B_E)$ will always satisfy five conditions stated in Sect. 2 that are fulfilled by very natural and common Banach spaces of analytic functions like the weighted Bergman spaces, $A^p_\alpha(\mathbb{B}_N)$, the Hardy spaces, $H^p(\mathbb{B}_N)$, $1 \le p < \infty$, and, even in the infinite dimensional setting, the weighted spaces of analytic functions $H^\infty_\upsilon(B_E)$ as we show in the examples of the next section.

Our quite general approach allows us to extend earlier results. For instance, the main result (Theorem 15) of Cowen and MacCluer in [9] containing information about the spectrum for (weighted) Hilbert spaces of analytic functions and composition operators with univalent and not unitary on any slice symbol that fixes the origin, is generalized by Corollary 4.14 where it is only required that the norm of the derivative mapping at the fixed point 0 be less than 1 and neither the analytic function spaces are required to be Hilbert. In the context of the weighted spaces of analytic functions $H^\infty_\upsilon(B_E)$ and weighted composition operators, we extend to arbitrary dimensions the description of the spectrum in case $E = \mathbb{C}$ by Aron and Lindström [2], and the one of Yuan and Zhou [27] given for the case $\upsilon \equiv 1$. Also the results for weighted Bergman spaces are new.

The main result (Theorem 4.9) provides conditions for the spectrum to contain a disc centered at 0 and all finite products of eigenvalues of the derivative mapping $\varphi'(0) \in \mathcal{L}(E)$, the Banach algebra of all bounded operators on E. The radius of such disc is closely related to the essential spectral radius of the operator: actually, equal in the case of $H^\infty_\upsilon(\mathbb{B}_N)$ in which we give a complete description of the spectrum. This relationship motivated the estimations in Sect. 3 where we extend Lefèvre's Theorem 2.5 in [20] to weighted spaces of analytic functions using quite different techniques.

Our standing assumptions are $\varphi(0) = 0$ with $\|\varphi'(0)\| < 1$ and that the range of φ is a relatively compact set. The core of our results is Lemma 4.8. It is an elaboration on the nice sharpening in [9] of Kamowitz technique [18] that has been further exploited by many other authors [3,12,15,22,28]. It strongly depends on the existence of (iterated) interpolating sequences that in the infinite dimensional setting was initiated in [12] and developed in [14]. Such suitable interpolating sequences are known to exist when E is a Hilbert space or $E = C_0(\mathcal{X})$, $\mathcal{X}$ a locally compact Hausdorff topological space. Thus, the mentioned standing assumptions suffice to get the results for both the ball and the polydisc.

2. Conditions and Examples

Recall that $H_b(B_E) := \{f : B_E \to \mathbb{C} : f$ analytic and bounded on balls of radius less than $1\}$ is a Fréchet algebra when endowed with the topology of uniform convergence on balls of radius less than 1. By $H^\infty(B_E)$ we denote the subspace of $H_b(B_E)$ of bounded functions endowed with the topology of uniform convergence on B_E.

We deal with a vector space $X(B_E)$ of analytic functions on B_E and a norm on it $\|\cdot\|$ that renders $X(B_E)$ a Banach space. As usual, for each $x \in B_E$, δ_x is the evaluation functional defined by $\delta_x(f) = f(x)$ for all $f \in X(B_E)$. We assume that $X(B_E)$ contains the constant functions, so then all δ_x are non-zero.

The Banach space $X(B_E)$ is assumed to satisfy the following conditions:

(**I**) *For every $x \in B_E$, $\delta_x : X(B_E) \to \mathbb{C}$ is a linear bounded functional, and the closed unit ball $\mathbf{B} = \{f \in X(B_E) : ||f|| \leq 1\}$ of $X(B_E)$ is compact with respect to the compact-open topology τ_0.*

In particular, for each $x \in B_E$ there is a $f_x \in X(B_E)$ with $||f_x|| \leq 1$ such that $||\delta_x||_X = f_x(x)$. Moreover, by the Dixmier–Ng theorem, there is a Banach space ${}^*X(B_E)$ whose dual space is isometrically isomorphic to $X(B_E)$ and, further, the mapping $x \in B_E \mapsto \delta_x \in {}^*X(B_E)$ is holomorphic because it is weakly holomorphic. Actually, ${}^*X(B_E)$ is the subspace of $X(B_E)^*$ of the elements that are τ_0-continuous on bounded sets.

(**II**) *For every $g \in H^\infty(B_E)$* and *$f \in X(B_E)$, the function $fg \in X(B_E)$.*

If both (**I**) and (**II**) hold, the multiplication operator $M_g(f) = fg$ is continuous on $X(B_E)$, thanks to the closed graph theorem. A subsequent application of the closed graph theorem shows the existence of a constant $M_X > 0$ such that $\|M_g\| \leq M_X \|g\|_\infty$.

(**III**) $X(B_E) \subset H_b(B_E)$.

This inclusion mapping is a continuous embedding thanks to the closed graph theorem.

Denote by $P_n f$ the n-th term of the Taylor series at 0 of the analytic function $f \in X(B_E)$. For $m \in \mathbb{N}$, let

$$X_m(B_E) = \{f \in X(B_E) : P_n f = 0 \text{ for } n = 0, 1, \ldots, m-1\}.$$

That is, a function in $X(B_E)$ belongs to $X_m(B_E)$ if the first m terms of its Taylor series at 0 vanish. Equivalently, $f \in X(B_E)$ belongs to $X_m(B_E)$ if, and only if, $\frac{f(x)}{\|x\|^m}$ is bounded in some punctured ball centered at 0.

(**IV**) *For each $m \in \mathbb{N}$ there is a constant $c(m) > 0$ (depending also on the norm of $X(B_E)$) such that for all $x \in B_E$ we have*

$$||\delta_x||_{X_m} \leq c(m)||x||^m||\delta_x||,$$

where $X_m(B_E)$ is endowed with norm of $X(B_E)$ and $\|\delta_x\|_{X_m}$ denotes the norm of δ_x restricted to X_m.

What can we say about this condition in the complex plane? Now, $B_\mathbb{C} = \mathbb{D}$ is the open unit disk. If for a positive integer m we have that $X_m(\mathbb{D}) = z^m X(\mathbb{D})$, then there is a constant $c(m) > 0$ such that

$$|f(z)| \leq c(m)|z|^m \|f\|_X \|\delta_z\| \tag{2.1}$$

for every $f \in X_m(\mathbb{D})$ and $z \in \mathbb{D}$. Indeed, this can be proved following the proof of [3, Proposition 3.3] and for completeness we give the details. The map $f \in X_m(\mathbb{D}) \mapsto f/z^m \in X(\mathbb{D})$ is well defined, linear and continuous by the closed graph theorem. Hence there is $c(m) > 0$ such that $||f/z^m||_X \leq$

$c(m)||f||_{X_m}$ for each $f \in X_m(\mathbb{D})$. Now, for $0 \neq w \in \mathbb{D}$ and $f \in X_m(\mathbb{D})$, we obtain

$$|f(w)| = |w|^m|f(w)/w^m| \leq |w|^m||f/z^m||_X||\delta_w|| \leq c(m)|w|^m||f||_{X_m}||\delta_w||.$$

Notice that from this result, we can obtain Propositions 2 and 11 in [22].

(**V**) For every $0 < r < 1$, consider $K_r(f)(x) = f(rx)$. *The operator* $K_r : X(B_E) \to X(B_E)$ *is well defined and* $\|K_r\| \leq 1$. In case $\dim E < \infty$, the operator K_r is compact.

The operator $uC_\varphi : X(B_E) \to X(B_E)$ will be assumed to be bounded. Since $u = uC_\varphi(1)$, we get that $u \in X(B_E)$.

Notice that whenever $\varphi(B_E)$ is a relatively compact set strictly inside B_E, C_φ is a compact operator: for any net $(f_i) \subset \mathbf{B}$ that we may suppose by (II) to be τ_0-convergent to some $g \in \mathbf{B}$, we have that $(f_i \circ \varphi)$ is uniformly convergent to $g \circ \varphi$ in $H^\infty(B_E)$, hence convergent in $X(B_E)$.

Next, we list a number of spaces satisfying the above conditions to which our main result applies.

2.1. Examples

(a) The weighted space of analytic functions

$$H_v^\infty(B_E) := \left\{ f : B_E \to \mathbb{C} : f \text{ is analytic and } \|f\|_v = \sup_{x \in B_E} v(x)|f(x)| < \infty \right\}$$

is a Banach space when endowed with the $\|\cdot\|_v$ norm. Here, $v : B_E \to (0, \infty)$ is a *weight*, that is, a continuous, bounded and norm non-increasing function, in particular, $v(x) = v(y)$ if $\|x\| = \|y\|$. For example, $v_\alpha(x) = (1 - ||x||^2)^\alpha$ with $\alpha > 0$ is such a weight. Moreover, the associated weight of v is defined by $\tilde{v}(x) = \frac{1}{||\delta_x||}, x \in B_E$. Notice that for the constant weight $v(x) = 1$, $H_v^\infty(B_E) = H^\infty(B_E)$.

Using Montel's theorem [6, Theorem 17.21] it follows that condition (I) holds. Next, we check condition (IV). For given $m \in \mathbb{N}$, we need to show that there exists a constant $c(m)$ depending only on m, so that if $f \in H_{v,m}^\infty(B_E)$ and $x \in B_E$, then

$$|f(x)| \leq c(m)\|x\|^m\|\delta_x\|\|f\|_v.$$

Indeed, for $\xi \in E$ such that $\|\xi\| = 1$, consider the function $f_\xi : \mathbb{D} \to \mathbb{C}$, $f_\xi(z) = f(z\xi)$, where $f \in H_{v,m}^\infty(B_E)$. Moreover, define $w_\xi(z) = v(z\xi)$. By radiality of the weight v, it follows that $w_\xi(z) = w_\xi(|z|)$. Clearly, $f_\xi \in H_{w_\xi,m}^\infty(\mathbb{D})$ and $\|f_\xi\|_{w_\xi} \leq \|f\|_v$. Now since $H_{w_\xi,m}^\infty(\mathbb{D}) = z^m H_{w_\xi}^\infty(\mathbb{D})$, we may apply (2.1) to get for $0 \neq x \in B_E$ that

$$|f(x)| = |f_{\frac{x}{\|x\|}}(\|x\|) \leq c(m)\|\delta_{||x||}\|\|f\|_v\|x\|^m,$$

and since $\|\delta_{||x||}\| \leq \|\delta_x\|$, the statement follows. Also, the other conditions are easily seen to be satisfied.

(b) The standard weighted Bergman space $A_\alpha^p(\mathbb{B}_N)$, $\alpha > -1, p \geq 1$, is the set of all analytic functions on $\mathbb{B}_N$ such that

$$||f||_{A_\alpha^p}^p = \int_{\mathbb{B}_N} |f(z)|^p c_\alpha (1 - |z|^2)^\alpha \mathrm{d}v(z) < \infty,$$

where $\mathrm{d}v(z)$ is the normalized volume measure on $\mathbb{B}_N$ and $c_\alpha = \frac{\Gamma(N+\alpha+1)}{N!\Gamma(\alpha+1)}$. The set of polynomials are dense in $A^p_\alpha(\mathbb{B}_N)$. By [26], for $z \in \mathbb{B}_N$ we have

$$||\delta_z|| = \frac{1}{(1-|z|^2)^{\frac{N+1+\alpha}{p}}}. \tag{2.2}$$

By Montel's theorem and Fatou's lemma it can be seen that condition (I) holds. The only condition that we need to verify is (IV), since the other conditions are clearly valid. For $\xi \in \mathbb{S}_N$, the map $f \in A^p_\alpha(\mathbb{B}_N) \mapsto f_\xi \in A^p_{N+\alpha-1}(\mathbb{D})$ is bounded by Theorem 1.1. in [19], so there is a constant $c(N) > 0$ such that

$$||f_\xi||_{A^p_{N+\alpha-1}(\mathbb{D})} \leq c(N)||f||_{A^p_\alpha(\mathbb{B}_N)}.$$

If $f \in A^p_{\alpha,m}(\mathbb{B}_N)$, then $f_\xi \in A^p_{N+\alpha-1,m}(\mathbb{D}) = z^m A^p_{N+\alpha-1}(\mathbb{D})$, so by (2.1) we obtain for $0 \neq z \in \mathbb{B}_N$ that

$$|f(z)| = |f_{\frac{z}{|z|}}(|z|) \leq c(m)c(N)|z|^m \|f\|_{A^p_\alpha(\mathbb{B}_N)}\|\delta_z\|.$$

(c) The Hardy spaces $H^p(\mathbb{B}_N)$, $1 \leq p < \infty$, are defined by

$$H^p(\mathbb{B}_N) = \left\{ f \in H(\mathbb{B}_N) : ||f||^p_{H^p} = \sup_{0<r<1} \int_{\mathbb{S}_N} |f(r\zeta)|^p \mathrm{d}\sigma(\zeta) < \infty \right\},$$

where $\mathbb{S}_N$ denotes the unit sphere in $\mathbb{C}^N$ and σ is the normalized surface measure on it. The set of polynomials are dense in $H^p(\mathbb{B}_N)$. It is known [29] that for $z \in \mathbb{B}_N$, we have

$$||\delta_z|| = \frac{1}{(1-|z|^2)^{\frac{N}{p}}}. \tag{2.3}$$

For condition (I) we use Montel's theorem and Fatou's lemma. For $\xi \in \mathbb{S}_N$, using Theorem 1.1. in [19], we obtain that the map $f \in H^p(\mathbb{B}_N) \mapsto f_\xi \in A^p_{N-2}(\mathbb{D})$ is bounded. Here, $A^p_{-1}(\mathbb{D}) = H^p(\mathbb{D})$. Therefore, there is a constant $c(N) > 0$ such that

$$||f_\xi||_{A^p_{N-2}(\mathbb{D})} \leq c(N)||f||_{H^p(\mathbb{B}_N)}.$$

For $f \in H^p_m(\mathbb{B}_N)$, then $f_\xi \in A^p_{N-2,m}(\mathbb{D}) = z^m A^p_{N-2}(\mathbb{D})$, so using (2.1) we get for $0 \neq z \in \mathbb{B}_N$ that

$$|f(z)| = |f_{\frac{z}{|z|}}(|z|) \leq c(m)c(N)|z|^m \|f\|_{H^p(\mathbb{B}_N)}\|\delta_z\|.$$

All the other conditions can easily be verified.

(d) The *weighted Hardy spaces of bounded type* $\mathcal{H}(\mathbb{B}_N)$ introduced by Cowen and MacCluer in [9] also satisfy the above five conditions. See the paper to verify it: Condition (I) is recalled in the bottom line of page 227. Condition (II) is [9, Proposition 1]. Condition (IV) follows from their computations in page 227 for the reproducing kernels, that is, the evaluation functionals:

$$\begin{aligned}
\|\delta_z\|^2_{X_m} &= \sum_{s=m}^{\infty} \frac{(N-1+s)!|z|^{2s}}{(N-1)!s!}\frac{1}{\tau(s)^2} \\
&= |z|^{2m}\sum_{j=0}^{\infty}|z|^{2j}\frac{(N-1+j+m)!}{(N-1)!(j+m)!}\frac{1}{\tau(j+m)^2} \\
&\le |z|^{2m}\sum_{j=0}^{\infty}\frac{(N-1+j)!(N-1+j+1)\cdots(N-1+j+m)}{(N-1)!j!(j+1)\cdots(j+m)}\frac{b^{-m}}{\tau(j)^2} \\
&= |z|^{2m}\sum_{j=0}^{\infty}\frac{(N-1+j)!}{(N-1)!j!}\left(\left(1+\frac{N-1}{j+1}\right)\cdots\left(1+\frac{N-1}{j+m}\right)\right)\frac{b^{-m}}{\tau(j)^2} \\
&\le Q(m)|z|^{2m}\|\delta_z\|^2 b^{-m},
\end{aligned}$$

where b is the assumed constant satisfying $\frac{\tau^2(s+1)}{\tau^2(s)} \ge b > 0$ and $Q(m) = \prod_{k=1}^{m}(1+\frac{N-1}{k})$. Notice also that $||\delta_z|| \to \infty$ when $|z| \to 1$ by Proposition 2 in [9] and that the set of polynomials is dense in $\mathcal{H}(\mathbb{B}_N)$.

3. The Essential Spectral Radius

Recall that for the essential spectral radius of an operator T, we have that $r_e(T) = \inf_n \sqrt[n]{\|T^n\|_e}$, and by φ_n we denote the n-fold iterate of φ, so that $\varphi_n = \varphi \circ \varphi \circ \cdots \circ \varphi$ (n times). In this section, we obtain estimates for the essential norm and spectral radius of the weighted composition operator uC_φ.

We first consider the operators $C_\varphi \circ K_r = C_{r\varphi}$ where $0 < r < 1$.

Lemma 3.1. *Suppose that $\varphi(\lambda B_E)$ is a relatively compact subset of E for all $0 < \lambda < 1$. Then every $C_\varphi \circ K_r : \big(H_b(B_E), \tau_0\big) \to H_b(B_E)$ defines a linear continuous mapping and $\{C_\varphi \circ K_r : 0 < r < 1\}$ is an equicontinuous family.*

Proof. The balanced hull, L_λ, of $\varphi(\lambda B_E) \subset \lambda B_E$ is a relatively compact set strictly inside B_E for all $0 < \lambda < 1$.

Since for all $0 < r < 1$,

$$\|(C_\varphi \circ K_r)(f)\|_{\lambda B_E} = \sup_{\|x\|<\lambda} |f(r\varphi(x))| \le \sup_{y \in L_\lambda} |f(y)| = \|f\|_{L_\lambda},$$

$\{C_\varphi \circ K_r : 0 < r < 1\}$ is an equicontinuous family. □

The closed unit ball $\mathbf{B}$ of $X(B_E)$ is a compact subset of $\big(H_b(B_E), \tau_0\big)$ by condition (I).

Lemma 3.2. *Suppose that $\varphi(\lambda B_E)$ is a relatively compact subset of E for all $0 < \lambda < 1$. For every $0 < \lambda < 1$,*

$$\lim_{r\to 1}\sup_{f\in\mathbf{B}}\sup_{\|\varphi(x)\|<\lambda} |f(r\varphi(x)) - f(\varphi(x))| = 0.$$

Proof. Let us see that for all $f \in H_b(B_E)$, $\lim_{r\to 1}(C_\varphi \circ K_r)(f) = C_\varphi(f)$ in $H_b(B_E)$. Fix $\epsilon > 0$ and $0 < \lambda < 1$. The uniform continuity of f in λB_E leads to some $\delta > 0$ such that $|f(u) - f(v)| < \epsilon$ if $u, v \in \lambda B_E$ and $\|u - v\| < \delta$. Therefore, if $r > 1-\delta$, one has $\|r\varphi(x) - \varphi(x)\| < \delta$. So, for all x with $\|\varphi(x)\| < \lambda$, we get $|f(r\varphi(x)) - f(\varphi(x))| < \epsilon$, that is, $\|(C_\varphi \circ K_r)(f) - C_\varphi(f)\|_{\lambda B_E} < \epsilon$.

Therefore, $\{C_\varphi \circ K_r : 0 < r < 1\}$ converges to C_φ for the topology of the pointwise convergence, hence also for the topology of uniform convergence on compact subsets of $\big(H(B_E), \tau_0\big)$ since they coincide on compact subsets due to [24, III. 4.5]. So the statement follows because $\mathbf{B}$ is $\big(H(B_E), \tau_0\big)$-compact. □

Proposition 3.3. *Assume that $\varphi(B_E)$ is a relatively compact subset of E. For the weighted composition operator $uC_\varphi : X(B_E) \to H_v^\infty(B_E)$, we have that*

$$\|uC_\varphi\|_e \leq 2 \lim_{s\to 1} \sup_{\|\varphi(x)\|\geq s} \frac{|u(x)|\|\delta_{\varphi(x)}\|_X}{\|\delta_x\|}.$$

Proof. First of all, notice that for $0 < r < 1$, $\big(C_\varphi \circ K_r\big)(f)(x) = f(r\varphi(x))$, so $C_\varphi \circ K_r : X(B_E) \to H^\infty(B_E)$ is a compact operator as a composition operator whose symbol $r\varphi$ lies in a compact subset of B_E. Since $u \in H_v^\infty(B_E)$, also the operators $uC_\varphi \circ K_r : X(B_E) \to H_v^\infty(B_E)$ are compact.

Next, by using that $v(x) \leq \tilde{v}(x) = \frac{1}{||\delta_x||}$ and $u \in H_{\tilde{v}}^\infty(B_E)$, we estimate $\|uC_\varphi \circ K_r - uC_\varphi\| \leq$

$$\begin{aligned} &\sup_{f\in\mathbf{B}} \sup_{\|\varphi(x)\|<s} \frac{|u(x)|}{\|\delta_x\|}\big|f(r\varphi(x)) - f(\varphi(x))\big| \\ &\quad + \sup_{f\in\mathbf{B}} \sup_{\|\varphi(x)\|\geq s} \frac{|u(x)|}{\|\delta_x\|}\big|f(r\varphi(x)) - f(\varphi(x))\big|. \end{aligned} \tag{3.1}$$

Concerning the first summand, we have

$$\begin{aligned} &\sup_{f\in\mathbf{B}} \sup_{\|\varphi(x)\|<s} \frac{|u(x)|}{\|\delta_x\|}\big|f(r\varphi(x)) - f(\varphi(x))\big| \\ &\quad \leq \|u\| \sup_{f\in\mathbf{B}} \sup_{\|\varphi(x)\|<s} \big|f(r\varphi(x)) - f(\varphi(x))\big|. \end{aligned}$$

So, we may apply Lemma 3.2 to conclude that for fixed $0 < s < 1$, the first summand tends to 0 whenever $r \to 1$.

Concerning the second summand, realize that

$$f(r\varphi(x)) - f(\varphi(x)) = K_r(f)(\varphi(x)) - f(\varphi(x)) = (K_r - Id)\,(f)(\varphi(x)).$$

Hence,

$$\begin{aligned} \frac{1}{\|\delta_x\|}\big|f(r\varphi(x)) - f(\varphi(x))\big| &= \frac{1}{\|\delta_x\|}\|\delta_{\varphi(x)}\|_X \left|\frac{1}{\|\delta_{\varphi(x)}\|_X}\big(K_r - Id\big)(f)(\varphi(x))\right| \\ &\leq \frac{\|\delta_{\varphi(x)}\|_X}{\|\delta_x\|} \cdot 2. \end{aligned}$$

So the second summand is bounded from above by $2 \cdot \sup_{\|\varphi(x)\|\geq s} \frac{|\psi(x)|\|\delta_{\varphi(x)}\|_X}{\|\delta_x\|}$.

Therefore, $\|uC_\varphi\|_e \leq \liminf_{r\to 1} \|uC_\varphi \circ K_r - \psi C_\varphi\| \leq 2 \cdot \sup_{\|\varphi(x)\|\geq s} \frac{|u(x)|\|\delta_{\varphi(x)}\|_X}{\|\delta_x\|}$. □

Proposition 3.4. *Assume that $\varphi(B_E)$ is a relatively compact subset of E. There exists $M_X > 0$ such that for the weighted composition operator $uC_\varphi : H_v^\infty(B_E) \to X(B_E)$, we have*

$$\|uC_\varphi\|_e \geq M_X^{-1} \lim_{s\to 1} \sup_{\|\varphi(x)\|\geq s} \frac{|u(x)|\|\delta_{\varphi(x)}\|}{\|\delta_x\|_X}.$$

Proof. If for some $s < 1$, $\{x : \|\varphi(x)\| > s\} = \emptyset$, then the right hand side is 0, and we are done. So we are left in the case that $\varphi(B_E)$ does not lie strictly inside B_E.

We can find a sequence $(x_n) \in B_E$ such that $\lim_n \|\varphi(x_n)\| = 1$ and $\lim_n \frac{|u(x_n||\delta_{\varphi(x_n)}\|}{\|\delta_{x_n}\|_X} = \lim_{s\to 1} \sup_{\|\varphi(x)\|\geq s} \frac{|u(x)|\|\delta_{\varphi(x)}\|}{\|\delta_x\|_X}$. Without loss of generality, we may assume that also $\big(\varphi(x_n)\big)$ converges to some $a \in B_E$ with $\|a\| = 1$. Let $l \in E^*$, $\|l\| = 1$ such that $\|a\| = l(a)$. Thus, $\lim_n l\big(\varphi(x_n)\big) = 1$. Put $z_n = l\big(\varphi(x_n)\big)$.

As shown in the proof of [17, Theorem 3.1] there is a subsequence of (z_n) that we still denote the same, there are functions $f, g_n \in A(\mathbb{D})$, two sequences of increasing positive integers (n_k) and (m_k), and a sequence of complex numbers (c_k) with $|c_k| < 1$, such that

$$\sum_{k=1}^{\infty} |c_k f^{m_k}(z) g_{n_k}(z)| \leq 1 \quad \text{for all } z,\ |z| \leq 1 \tag{3.2}$$

and

$$c_k f^{m_k}(z_k) g_{n_k}(z_k) > 1 - \left(\frac{1}{2}\right)^k \text{ for all } k. \tag{3.3}$$

By condition (I), for each $k \in \mathbb{N}$ we can also find a function $f_k \in H_v^\infty(B_E)$ such that $\|f_k\| \leq 1$ and that

$$\|\delta_{\varphi(x_k)}\| = f_k(\varphi(x_k)). \tag{3.4}$$

Now, consider $F_k := M_{(c_k f^{m_k} g_{n_k})\circ l}(f_k)$. According to condition (II), the sequence $(F_k) \subset H_v^\infty(B_E)$ and

$$\big\|F_k\big\| \leq \big\|M_{(c_k f^{m_k} g_{n_k})\circ l}\big\|\big\|f_k\big\| \leq M_X\big\| \left((c_k f^{m_k} g_{n_k}) \circ l\right) \big\|_\infty \big\|f_k\big\| \leq M_X.$$

Let $T : H_v^\infty(B_E) \to X(B_E)$ be a compact operator. By (3.2) the map $(\xi_k)_k \mapsto \sum_{k=1}^\infty \xi_k F_k$ is a well-defined, bounded operator from c_0 into $H_v^\infty(B_E)$. Consequently, the sequence (F_k) converges weakly to zero and $||T(F_k)|| \to 0$ in $X(B_E)$. Thus using condition (I), we get

$$\begin{aligned} M_X\ \|uC_\varphi - T\| &\geq \|(uC_\varphi - T)F_k\| \geq \|(uC_\varphi)F_k\| - \|T(F_k)\| \\ &\geq \frac{1}{\|\delta_{x_k}\|_X} |u(x_k)| \cdot \big\|\delta_{\varphi(x_k)}\big\| \left(1 - \left(\frac{1}{2}\right)^k\right) - \|T(F_k)\|, \end{aligned}$$

and we are done. □

The above two results yield an extension of [20, Theorem 2.5] to the weighted spaces case. We are now able to state the following essential spectral radius result.

Corollary 3.5. *Assume that $\varphi(B_E)$ is a relatively compact subset of E. For the weighted composition operator uC_φ acting on $H_v^\infty(B_E)$, we have that*

$$r_e(uC_\varphi) = \liminf_n \sqrt[n]{\lim_{s\to 1} \sup_{\|\varphi_n(x)\|\geq s} \frac{\|\delta_{\varphi_n(x)}\||u(x)\cdot \cdots \cdot u(\varphi_n(x))|}{\|\delta_x\|}}.$$

The proof of the next result is much easier than Proposition 3.4 and the result can be applied to the spaces $H^p(\mathbb{B}_N)$ and $A_\alpha^p(\mathbb{B}_N)$.

Proposition 3.6. *Assume that $||\delta_x|| \to \infty$ when $||x|| \to 1$ and that the continuous polynomials are dense in $X(B_E)$. Then for the weighted composition operator uC_φ acting on $X(B_E)$, we have that*

$$\|uC_\varphi\|_e \geq \lim_{s\to 1} \sup_{\|x\|\geq s} \frac{|u(x)|\|\delta_{\varphi(x)}\|}{\|\delta_x\|}.$$

Proof. Take a sequence $(x_n) \subset B_E$ with $||x_n|| \to 1$ when $n \to \infty$ such that

$$\lim_{s\to 1} \sup_{\|x\|\geq s} \frac{|u(x)|\|\delta_{\varphi(x)}\|}{\|\delta_x\|} = \lim_{n\to\infty} |u(x_n)| \frac{||\delta_{\varphi(x_n)}||}{||\delta_{x_n}||}.$$

Let $l_n := \frac{\delta_{x_n}}{||\delta_{x_n}||} \in X(B_E)^*$, then $l_n \to 0$ weak* in $X(B_E)^*$. Indeed, clearly for any continuous polynomial P, $\lim_n l_n(P) = \lim_n \frac{P(x_n)}{||\delta_{x_n}||} = 0$, and on the closed unit ball of $X(B_E)^*$, the weak*-topology coincides with that of the pointwise convergence on the total subset of the continuous polynomials.

For an arbitrary compact operator $T : X(B_E) \to X(B_E)$, it follows now that

$$||uC_\varphi - T|| \geq \lim_{n\to\infty} (||(uC_\varphi)^*(l_n)|| - ||T^*(l_n)||) = \lim_{n\to\infty} |u(x_n)| \frac{||\delta_{\varphi(x_n)}||}{||\delta_{x_n}||},$$

and the statement follows. □

Now, we can deduce the following lower estimate of the essential spectral radius of uC_φ.

Corollary 3.7. *For the weighted composition operator uC_φ acting on $H^p(\mathbb{B}_N)$ and $A_\alpha^p(\mathbb{B}_N)$, respectively, for $\alpha > -1, p \geq 1$, we have that*

$$r_e(uC_\varphi) \geq \liminf_n \sqrt[n]{\lim_{s\to 1} \sup_{|z|\geq s} \frac{\|\delta_{\varphi_n(z)}\||u(z)\cdot \cdots \cdot u(\varphi_n(z))|}{\|\delta_z\|}}.$$

Here, we recall an upper estimate of the spectral radius of C_φ on $A_\alpha^p(\mathbb{B}_N)$.

Proposition 3.8. *Let $\alpha > -1, p > 1$. Assume that $\varphi(0) = 0$ and that the composition operator C_φ is bounded on $A_\beta^p(\mathbb{B}_N)$ for some $-1 < \beta < \alpha$. Then for C_φ acting on $A_\alpha^p(\mathbb{B}_N)$, we have that*

$$r_e(C_\varphi)^{\frac{N+1+\alpha}{\alpha-\beta}} \leq \liminf_n \sqrt[n]{\lim_{s\to 1} \sup_{|z|\geq s} \frac{\|\delta_{\varphi_n(z)}\|}{\|\delta_z\|}}.$$

In case $N = 1$, *then* $C_\varphi : A^p_\alpha(\mathbb{D}) \to A^p_\alpha(\mathbb{D})$, $\alpha > -1, p > 1$, *is always bounded and*

$$r_e(C_\varphi)^{\frac{2+\alpha}{1+\alpha}} \le \liminf_n \sqrt[n]{\lim_{s\to 1}\sup_{|z|\ge s} \frac{\|\delta_{\varphi_n(z)}\|}{\|\delta_z\|}}.$$

Proof. We will use Corollary 4.7 in [24] which gives that

$$||C_\varphi||_e \le C \limsup_{|z|\to 1} \left(\frac{1-|z|^2}{1-|\varphi(z)|^2}\right)^{\frac{\alpha-\beta}{p}},$$

where C is an absolute constant. Since $||\delta_z|| = \frac{1}{(1-|z|^2)^{\frac{N+1+\alpha}{p}}}$, we conclude that

$$r_e(C_\varphi) \le \liminf_n \sqrt[n]{\lim_{s\to 1}\sup_{|z|\ge s} \left(\frac{\|\delta_{\varphi_n(z)}\|}{\|\delta_z\|}\right)^{\frac{\alpha-\beta}{N+1+\alpha}}},$$

and the first statement follows. For the second statement, we use Corollary 3.9 in [21] (see also page 141 in [10]), that is,

$$||C_\varphi||_e \le C \limsup_{|z|\to 1} \left(\frac{1-|z|^2}{1-|\varphi(z)|^2}\right)^{\frac{\alpha+1}{p}}.$$

□

The next lemma contains useful information about the essential spectral radius of uC_φ.

Lemma 3.9. *Assume that* $\varphi(B_E)$ *is a relatively compact subset of* E *and that* $\varphi(0) = 0$. *Suppose that* $|u(x)| \; ||\delta_x||^{-1} \to 0$ *as* $||x|| \to 1$. *Then,*

$$\lim_{s\to 1}\sup_{\|x\|\ge s} \frac{|u(x)|\|\delta_{\varphi(x)}\|}{\|\delta_x\|} = \lim_{s\to 1}\sup_{\|\varphi(x)\|\ge s} \frac{|u(x)|\|\delta_{\varphi(x)}\|}{\|\delta_x\|}. \tag{3.5}$$

Proof. Since $\|\varphi(x)\| \le \|x\|$, the limit on the right hand side is not greater than the one on the left hand side. There is a sequence $(x_n) \subset B_E$ such that $\|x_n\| \to 1$ and $\limsup_{\|x\|\to 1} \frac{|u(x)|\|\delta_{\varphi(x)}\|}{\|\delta_x\|} = \lim_n \frac{|u(x_n)|\|\delta_{\varphi(x_n)}\|}{\|\delta_{x_n}\|}$. From the bounded sequence $\big(\varphi(x_n)\big)$, we get a convergent subsequence, say to $a \in E$, which we denote the same. If $\|a\| = 1$, we have $\limsup_{\|\varphi(x)\|\to 1} \frac{|u(x)|\|\delta_{\varphi(x)}\|}{\|\delta_x\|} \ge \lim_n \frac{|u(x_n)|\|\delta_{\varphi(x_n)}\|}{\|\delta_{x_n}\|}$ that leads to the equality in (3.5). While if $\|a\| < 1$, then the sequence $\big(\delta_{\varphi(x_n)}\big)$ is, by condition (I), a convergent one in ${}^*X(B_E)$, and hence bounded. Therefore, according to the assumption $\limsup_{\|x\|\to 1} \frac{|u(x)|\|\delta_{\varphi(x)}\|}{\|\delta_x\|} = 0$, (3.5) holds as well. □

Remark 3.10. Equality (3.5) also holds if there is a constant $d > 0$ such that $\|\varphi(x)\| \ge d\|x\|$ for all $x \in B_E$ and $\varphi(0) = 0$. Such is the case of univalent $\varphi : \mathbb{B}_N \to \mathbb{B}_N$ with $\varphi(0) = 0$ and $\|\varphi'(0)\| < 1$, as pointed out in [9, page 239].

4. The Spectrum

Define an operator S on a direct sum of Banach spaces $X = X_1 \oplus \cdots \oplus X_m$. Such an operator leaves invariant each direct subsum $X_k \oplus \cdots \oplus X_m$ if and only if it has a lower triangular matrix representation

$$S = \begin{pmatrix} S_{11} & 0 & 0 & \dots & 0 \\ S_{21} & S_{22} & 0 & \dots & 0 \\ \vdots & & & & \\ \dots & \dots & \dots & S_{m-1,m-1} & 0 \\ S_{m1} & S_{m2} & \dots & S_{m,m-1} & S_{mm} \end{pmatrix},$$

where $S_{jk} : X_j \to X_k$. Recall that an operator S is called a Riesz operator if $r_e(S) = 0$.

Throughout this section, we assume that $\varphi(0) = 0$ unless otherwise stated.

Theorem 4.1. [12, Corollary 2.4] *Let $X = X_1 \oplus \cdots \oplus X_m$ be a direct sum of Banach spaces, and let S be an operator on X with a lower triangular matrix representation. If X is infinite dimensional, and the operators $S_{11}, \dots, S_{m-1,m-1}$ are Riesz operators, then $\sigma(S) = \sigma(S_{11}) \cup \cdots \cup \sigma(S_{mm})$.*

Let $P_k := P(^kE) \subset H^\infty(B_E)$ denote the subspace of homogeneous polynomials of degree k on E. The Taylor series expansion at 0 of each element f in $X(B_E)$ yields a direct sum decomposition of $X(B_E)$,

$$X(B_E) = P_0 \oplus \cdots \oplus P_{m-1} \oplus X_m(B_E),$$

because the mapping $f \in X(B_E) \mapsto P_k(f) \in P_k$ is a continuous projection of $X(B_E)$ thanks to conditions (II) and (III).

Lemma 4.2. *The operator uC_φ leaves invariant the space $X_m(B_E)$.*

Proof. Fix $x \in B_E$. It is easy to see, from the Taylor series expansion of φ at 0, that the function $g : \mathbb{D} \longrightarrow E$ defined by $g(\lambda) = \varphi(\lambda x)$ satisfies $g(\lambda) = \lambda h_x(\lambda)$ for a particular analytic function h_x which depends on x and λ. Set $f \in X_m(B_E)$. We have that

$$\begin{aligned} u(f \circ \varphi)(\lambda x) &= u(\lambda x) \sum_{n \geq m} P_n f(\varphi(\lambda x)) \\ &= u(\lambda x) \sum_{n \geq m} P_n f(\lambda h_x(\lambda)) \\ &= u(\lambda x) \sum_{n \geq m} \lambda^n P_n f(h_x(\lambda)), \end{aligned}$$

and so there is no non-null term of degree less than m in the series expansion of $u(f \circ \varphi)(\lambda x)$. Therefore, if $\sum_n Q_n$ is the Taylor series of $uC_\varphi(f)$, there must be no non-null term of degree less than m in

$$\sum_n Q_n(\lambda x) = \sum_n \lambda^n Q_n(x).$$

□

By Lemma 4.2, the weighted composition operator uC_φ leaves invariant the spaces $X_{k-1}(B_E) = P_k \oplus \ldots \oplus P_{m-1} \oplus X_m(B_E)$, $\quad 0 \le k \le m-1$, and leaves the space $X_m(B_E)$ invariant as well. Consequently, uC_φ has a lower triangular matrix representation

$$uC_\varphi = \begin{pmatrix} C_{11} & 0 & 0 & \ldots & 0 \\ C_{21} & C_{22} & 0 & \ldots & 0 \\ \vdots & & & & \\ \ldots & \ldots & \ldots & C_{m-1,m-1} & 0 \\ C_{m1} & C_{m2} & \ldots & C_{m,m-1} & C_m, \end{pmatrix},$$

where the operator C_m is the restriction of uC_φ to $X_m(B_E)$. By Theorem 4.1, we can determine the spectrum of uC_φ by determining the spectrum of $C_m : X_m(B_E) \to X_m(B_E)$ and the diagonal elements $C_{kk} : P_k \to P_k$ as soon as they are Riesz operators.

Next, we proceed to determine the operators C_{kk}. We use the following result:

Lemma 4.3. (i) *For $v := u - u(0)$, the function $\frac{v(x)}{\|x\|}$ is bounded in some punctured neighborhood of* 0.

(ii) *For every $x \in B_E$, we have $\|\varphi(x) - \varphi'(0)(x)\| \le \frac{\|x\|^2}{1-\|x\|}$.*

Proof. Since $\lim_{x\to 0} \frac{v(x)-u'(0)x}{\|x\|} = \lim_{x\to 0} \frac{u(z)-u(0)-u'(0)x}{\|x\|} = 0$, and $\frac{|v(x)|}{\|x\|} \le \frac{|v(x)-u'(0)x|}{\|x\|} + \|u'(0)\|$, the statement i) follows.
To realize ii), consider the Taylor series of φ, $\sum_{m=1}^\infty P_m\varphi$, and recall that according to Cauchy inequalities, $\|P_m\varphi\| \le 1$. Then,

$$\begin{aligned} \|\varphi(x) - \varphi'(0)(x)\| &\le \sum_{m=2}^\infty \|P_m\varphi(x)\| \le \sum_{m=2}^\infty \|x\|^m \|P_m\varphi\| \\ &\le \sum_{m=2}^\infty \|x\|^m = \frac{\|x\|^2}{1-\|x\|}. \end{aligned}$$

□

Proposition 4.4. *For every $f \in P_k$,*

$$C_{kk}(f)(x) = u(0)\hat{f}\left(\varphi'(0)(x), \ldots, \varphi'(0)(x)\right),$$

where $\hat{f}$ is the k-linear symmetric mapping determining f.

Proof. Denote $R(x) = \varphi(x) - \varphi'(0)(x)$. Then we use the binomial formula

$$\begin{aligned} f(\varphi(x)) &= \hat{f}\left(\varphi'(0)(x) + R(x)\right) \\ &= \sum_{l=0}^k \binom{k}{l} \hat{f}\left(\varphi'(0)(x), \overset{k-l}{\ldots}\, \varphi'(0)(x), R(x), \overset{l}{\ldots}, R(x)\right). \end{aligned}$$

We claim that if $l > 0$, the corresponding term in this sum belongs to X_{k+1} : Indeed,

$$\frac{\hat{f}\left(\varphi'(0)(x), \overset{k-l}{\ldots}, \varphi'(0)(x), R(x), \overset{l}{\ldots}, R(x)\right)}{\|x\|^{k+1}}$$
$$= \hat{f}\left(\varphi'(0)\left(\frac{x}{\|x\|}\right), \overset{k-l}{\ldots}, \varphi'(0)\left(\frac{x}{\|x\|}\right), \frac{R(x)}{\|x\|^2}, \overset{l}{\ldots}, \frac{R(x)}{\|x\|}\right),$$

where all terms are bounded bearing in mind Lemma 4.3. Moreover, for $l = 0$ the term is a k-homogeneous polynomial, and

$$\frac{1}{\|x\|^{k+1}}\left|v(x)\hat{f}\left(\varphi'(0)(x), \overset{k}{\ldots}, \varphi'(0)(x)\right)\right|$$
$$\leq \frac{|v(x)|}{\|x\|}\left|\hat{f}\left(\varphi'(0)\left(\frac{x}{\|x\|}\right), \overset{k}{\ldots}, \varphi'(0)\left(\frac{x}{\|x\|}\right)\right)\right|$$

is bounded in a neighborhood of 0, where $v(x) = u(x) - u(0)$.

Now,

$$\begin{aligned} &u(x)f\left(\varphi(x)\right) \\ &\quad= (u(0) + v(x))\Big(\hat{f}\left(\varphi'(0)(x), \overset{k}{\ldots}, \varphi'(0)(x)\right. \\ &\qquad+ \sum_{l=1}^{k}\hat{f}\left(\varphi'(0)(x), \overset{k-l}{\ldots}, \varphi'(0)(x), R(x), \overset{l}{\ldots}, R(x)\right)\Big) \\ &\quad= u(0)\Big(\hat{f}\left(\varphi'(0)(x), \overset{k}{\ldots}, \varphi'(0)(x)\right) \\ &\qquad+u(0)\left(\sum_{l=1}^{k}\hat{f}\left(\varphi'(0)(x), \overset{k-l}{\ldots}, \varphi'(0)(x), R(x), \overset{l}{\ldots}, R(x)\right)\right) \\ &\qquad+v(x)\Big(\hat{f}\left(\varphi'(0)(x), \overset{k}{\ldots}, \varphi'(0)(x)\right) \\ &\qquad+v(x)\left(\sum_{l=1}^{k}\hat{f}\left(\varphi'(0)(x), \overset{k-l}{\ldots}, \varphi'(0)(x), R(x), \overset{l}{\ldots}, R(x)\right)\right). \end{aligned}$$

Here, the last three terms belong to $X_k(B_E)$, and so their k^{th} term in the Taylor series vanishes. Therefore, $C_{kk}(f) = u(0)\big(\hat{f}\big(\varphi'(0)(x), \overset{k}{\ldots}, \varphi'(0)(x)\big)\big)$. □

Now, we apply Lemma 3.1 in [12] to obtain

Lemma 4.5. $\sigma\big(C_{kk}\big) = \{u(0) \cdot \lambda_1 \cdots \lambda_k : \lambda_j \in \sigma(\varphi'(0)), 1 \leq j \leq k\}$.

Definition 4.6. A finite or infinite sequence $\{z_n\} \subset B_E$ is called an iteration sequence for φ if $\varphi(z_k) = z_{k+1}$ for $k \geq 0$, and a sequence $\{z_n\} \subset B_E$ is called an interpolating sequence for $H^\infty(B_E)$ if for any bounded sequence $\{a_n\} \subset \mathbb{C}$ there exists $f \in H^\infty(B_E)$ such that $f(z_n) = a_n$ for $n \in \mathbb{N}$.

Recall the following Schwarz's lemma type inequality as shown in [12]: *Suppose that* $\varphi : B_E \longrightarrow B_E$ *satisfies* $\varphi(0) = 0$ *and* $\|\varphi'(0)\| < 1$. *Then for each* $s < 1$, *there exists* $a < 1$ *such that*

$$\|\varphi(x)\| \leq a\|x\|, \quad \text{for } x \in E, \quad \|x\| \leq s. \tag{4.1}$$

Hence, given $0 < r < s < 1$, there exists $\varepsilon > 0$ such that

$$\frac{1 - \|\varphi(x)\|}{1 - \|x\|} \geq 1 + \epsilon, \quad x \in B_E, \quad r < \|x\| < s.$$

Lemma 4.7. *Let E be a complex Banach space and let $\varphi : B_E \to B_E$ be analytic such that $\varphi(0) = 0$ and $\|\varphi'(0)\| < 1$. Suppose that there exist $W \subset B_E$, with $\varphi(W) \subset W$, $\delta > 0$ and $\epsilon > 0$ such that*

$$\frac{1 - \|\varphi(x)\|}{1 - \|x\|} \geq 1 + \epsilon, \quad \textit{for all } x \in W \textit{ such that } \|x\| \geq \delta \; . \tag{4.2}$$

Then, there exists a constant $M \geq 1$ which depends only on ϵ, such that any finite iteration sequence $\{x_0, x_1, \ldots, x_N\}$ satisfying $x_0 \in W$ and $\|x_N\| \geq \delta$ is an interpolating sequence for $H^\infty(B_E)$ with interpolation constant not greater than M.

The proof of this lemma can be seen in [15]. It relies on the interpolation result [14, Corollary 8]. We will refer to inequalities of the form (4.2) as Julia-type estimates.

Denote

$$\gamma(uC_\varphi; W) := \liminf_n \sqrt[n]{\lim_{s \to 1} \sup_{\substack{\|\varphi_n(x)\| \geq s \\ x \in W}} \frac{\|\delta_{\varphi_n(x)}\|_X |u(x) \cdot \cdots \cdot u(\varphi_n(x))|}{\|\delta_x\|_X}}.$$

Lemma 4.8. *Consider the weighted composition operator uC_φ acting on $X(B_E)$. Assume that $\varphi(0) = 0$, $\|\varphi'(0)\| < 1$ and that $\varphi(B_E)$ is a relatively compact subset of E. Suppose also that $||\varphi_n|| = 1$ for all $n \in \mathbb{N}$ and that there exists $W \subseteq B_E$ with $\varphi(W) \subseteq W$ and such that a Julia-type estimate holds for some $\epsilon, \delta > 0$. If $\lambda \neq 0$ satisfies $|\lambda| < \gamma(uC_\varphi; W)$, then $\lambda \in \sigma(uC_\varphi)$.*

Proof. We will consider iteration sequences $\{z_k\}_{k=0}^\infty$ such that $z_0 \in W$ and $||z_0|| > \delta$. In view of (4.1), the norms of the elements of any such iteration sequence decrease to 0. We define $N = N(z_0)$ to be the largest integer such that $\|z_N\| > \delta$. The hypothesis guarantees that for all $k \geq 1$, $\varphi_k(B_E)$ is not contained in the ball $\{||z|| \leq \delta\}$. Consequently, we can find z_0 for which $N(z_0)$ is arbitrarily large.

Choose $c < 1$ such that

$$\|\varphi(z)\| \leq c\|z\|, \qquad z \in E, \, \|z\| \leq \sqrt{\delta}.$$

We can assume that $c > \sqrt{\delta}$. By considering separately the cases $\|z_N\| \leq \sqrt{\delta}$ and $\|z_N\| > \sqrt{\delta}$, we see also that $\|z_{N+1}\| \leq c\|z_N\|$. Since $\|z_{n+1}\| \leq c\|z_n\|$ for $n > N + 1$, we obtain by induction that

$$\|z_{N+k}\| \leq c^k \|z_N\|, \qquad k \geq 0.$$

Since $u \in H_b(B_E)$ it holds that $0 < C := \max\{\sup_{\|z\| \leq \delta} |u(z)|, \sup_n | u(z_n)|\} < \infty$. Put $D_k(z_0) = |u(z_0) \cdot \cdots \cdot u(z_{k-1})|$.

For any iteration sequence $(z_k)_{k=0}^\infty$ and $m \in \mathbb{N}$, let us define $L_{\lambda,u}$ on $X_m(B_E)$ by

$$L_{\lambda,u}(f) = f(z_0) + \sum_{k=1}^\infty u(z_0) \overset{k}{\cdots} u(z_{k-1}) f(z_k) \lambda^{-k}.$$

By using condition (IV), we get that $L_{\lambda,u}$ is bounded, because

$$\begin{aligned}
&\left|\sum_{k=N+1}^{\infty} u(z_0)\dots u(z_{k-1})f(z_k)\lambda^{-k}\right| \\
&\quad\le D_N(z_0)\sum_{k=N+1}^{\infty} C^{k-N}|\lambda|^{-k}c(m)\|z_k\|^m\|\delta_{z_k}\|_X\|f\| \\
&\quad\le D_N(z_0)\|f\|c(m)\sum_{k=N+1}^{\infty} C^{k-N}|\lambda|^{-k}\left(c^{k-N}\|z_N\|\right)^m\|\delta_{z_k}\|_X \\
&\quad\le \|f\|D_N(z_0)c(m)\sum_{k=N+1}^{\infty}\|\delta_{z_k}\|_X\|z_N\|^m\left(\frac{C\cdot c^m}{|\lambda|}\right)^{k-N}\frac{1}{|\lambda|^N}.
\end{aligned}$$

Since (z_k) converges to 0 in B_E, also (δ_{z_k}) converges, so

$$\|f\|D_N(z_0)c(m)M_0\|z_N\|^m\frac{1}{|\lambda|^N}\left(\sum_{k=N+1}^{\infty}\left(\frac{C\cdot c^m}{|\lambda|}\right)^{k-N}\right).$$

So there exists m_0 so that if $m \ge m_0$, then $L_{\lambda,u}$ is bounded, i.e, $\frac{Cc^{m_0}}{|\lambda|} < 1$.

Note that $(C_m^* - \lambda I)(L_{\lambda,u}) = -\lambda\delta_{z_0}$, because for any $f \in X_m(B_E)$,

$$\begin{aligned}
&\langle (C_m^* - \lambda I)(L_{\lambda,u}), f\rangle = L_{\lambda,u}(uC_\varphi(f) - \lambda f) \\
&\quad= L_{\lambda,u}(u\cdot f\circ\varphi) - \lambda L_{\lambda,u}(f) = u(z_0)f(\varphi(z_0)) \\
&\qquad+\sum_{k=1}^{\infty} u_{z_0}\cdot\dots\cdot u(z_{k-1})u(z_k)f(\varphi(z_k))\lambda^{-k} \\
&\qquad-\lambda f(z_0) - \sum_{k=1}^{\infty} u(z_0)\cdot\dots\cdot u(z_{k-1})\lambda^{-k+1}f(z_k) = -\lambda f(z_0) \\
&\qquad+\sum_{k=1}^{\infty} u(z_0)\cdot\dots\cdot u(z_k)\lambda^{-k}f(z_{k+1}) \\
&\qquad-\sum_{k=2}^{\infty} u(z_0)\cdot\dots\cdot u(z_{k-1})\lambda^{-k+1}f(z_k) = -\lambda f(z_0).
\end{aligned}$$

Now, we find a suitable lower bound for $\|L_{\lambda,u}\|$. For $0 \le K \le N$, pick $l \in E^*$, $\|l\| = 1$ such that $l(z_K) = \|z_K\|$, and by using Lemma 4.7, pick $f \in H^\infty(B_E)$ with $\|f\|_\infty \le M$ and $f(z_k) = 0$ for all $0 \le k \le N$ except for $k = K$, in which case $f(z_K) = 1$. By (I), there is $f_0 \in X(B_E)$ such that $\|f_0\| \le 1$ and $f_0(z_K) = \|\delta_{z_K}\|_X$. Then by condition (II), the function $g := l^m \cdot f_0 \cdot f \in X_m(B_E)$ and $\|g\| \le M_1$.

We now calculate

$$\begin{aligned}
L_{\lambda,u}(g) &= u(z_0)\cdot\dots\cdot u(z_{K-1})\lambda^{-K}\|z_K\|^m\|\delta_{z_K}\|_X + D_N(z_0) \\
&\qquad\times\sum_{k=N+1}^{\infty} u(z_N)\cdot\dots\cdot u(z_{k-1})\lambda^{-k}f_0(z_k)f(z_k)l^m(z_k).
\end{aligned}$$

We assume that $D_K(z_0) \neq 0$. The first term is bounded above by

$$D_K(z_0)|\lambda|^{-K}\|z_K\|^m\|\delta_{z_K}\|_X,$$

and the second term is bounded above by

$$\begin{aligned}
&D_N(z_0)\sum_{k=N+1}^{\infty} C^{k-N}|\lambda|^{-k}M\|\delta_{z_k}\|_X\|z_k\|^m \\
&\quad \leq D_N(z_0)MM_0\left(\sum_{k=N+1}^{\infty} C^{k-N}|\lambda|^{-k}(c^{k-N}\|z_N\|)^m\right) \\
&\quad = D_N(z_0)MM_0|\lambda|^{-N}\left(\sum_{k=N+1}^{\infty}\left(\frac{C}{|\lambda|}c^m\right)^{k-N}\right)\|z_N\|^m \\
&\quad \leq D_N(z_0)MM_0|\lambda|^{-N}\left(\sum_{k=N+1}^{\infty}\left(\frac{C}{|\lambda|}c^m\right)^{k-N}\right)\|z_K\|^m.
\end{aligned}$$

Thus,

$$\begin{aligned}
|L_{\lambda,u}(g)| &\geq D_K(z_0)|\lambda|^{-K}\|z_K\|^m\|\delta_{z_K}\|_X \\
&\quad -D_N(z_0)|\lambda|^{-(N-K)}MM_0\sum_{k=N+1}^{\infty}\left(\frac{C}{|\lambda|}c^m\right)^{k-N}|\lambda|^{-K}||z_K||^m.
\end{aligned}$$

There is $m_1 \geq m_0$, so that if $m \geq m_1$ we have

$$\sum_{k=N+1}^{\infty}\left(\frac{C}{|\lambda|}c^m\right)^{k-N} < \frac{D_K(z_0)}{D_N(z_0)2MM_0|\lambda|^{K-N}}\|\delta_{z_K}\|_X \quad \text{if } D_N(z_0) \neq 0.$$

So,

$$|L_{\lambda,u}(g)| \geq \frac{1}{2}D_K(z_0)|\lambda|^{-K}\|z_K\|^m\|\delta_{z_K}\|_X \quad \text{regardless the value of } D_N(z_0).$$

Consequently,

$$\frac{1}{2}D_K(z_0)|\lambda|^{-K}\|z_K\|^m\|\delta_{z_K}\|_X \leq \|L_{\lambda,u}\|\|g\|_v \leq \|L_{\lambda,u}\| \cdot M_1,$$

which gives us

$$\|L_{\lambda,u}\| \geq \frac{1}{2}\frac{D_K(z_0)}{M_1}\|z_K\|^m\|\delta_{z_K}\|_X|\lambda|^{-K},$$

in the case $|D_K(z_0)| \neq 0$.

If

$$|\lambda| < \liminf_n \sqrt[n]{\lim_{s\to 1}\sup_{\substack{\|\varphi_n(x)\|\geq s\\ x\in W}}\frac{\|\delta_{\varphi_n(x)}\|_X|u(x)\cdot\dots\cdot u(\varphi_n(x))|}{\|\delta_x\|_X}},$$

we can pick $\mu > 0$ such that

$$|\lambda| < \mu < \liminf_n \sqrt[n]{\lim_{s\to 1}\sup_{\substack{\|\varphi_n(x)\|\geq s\\ x\in W}}\frac{\|\delta_{\varphi_n(x)}\|_X|u(x)\cdot\dots\cdot u(\varphi_n(x))|}{\|\delta_x\|_X}}.$$

Hence, there is n_0 such that if $K \geq n_0$, it holds that

$$\mu^K < \lim_{s\to 1} \sup_{\substack{\|\varphi_K(x)\|\geq s \\ x\in W}} \frac{\|\delta_{\varphi_K(x)}\|_X |u(x)\cdot \cdots \cdot u(\varphi_K(x))|}{\|\delta_x\|_X}.$$

So for every $K \geq n_0$, we can find $x \in W$ with $\|\varphi_K(x)\| > \delta$ such that

$$\mu^K \leq \frac{\|\delta_{\varphi_K(x)}\|_X}{\|\delta_x\|_X} |u(x)| \cdot \cdots \cdot |u(\varphi_K(x))|,$$

for which necessarily $D_K(z_0) \neq 0$.

We consider the iteration sequence $\{\varphi_i(x)\}_{i=0}^{\infty}$ that satisfies indeed the condition $\|\varphi_K(x)\| > \delta$. So for $z_0 = x$, we have $N = N(z_0)$ so that $\|z_N\| > \delta$, $K \leq N$ and $z_K = \varphi_K(x)$. Pick $L_{\lambda,u}$ as above.

Now,

$$\begin{aligned}
\frac{\|(C_m^* - \lambda I)(L_{\lambda,u})\|}{\|L_{\lambda,u}\|} &\leq \frac{|\lambda| \|\delta_{z_0}\|_{X_m}}{\frac{D_K(z_0)}{2M_1}\|\delta_{\varphi_K(x)}\|_X |\lambda|^{-K} \|\varphi_K(x)\|^m} \\
&= \frac{2|\lambda|^{K+1}M_1}{D_K(z_0)} \frac{\|\delta_{z_0}\|_{X_m}}{\|\varphi_K(x)\|^m \|\delta_{\varphi_K(x)}\|_X} \\
&\leq \frac{2|\lambda|^{K+1}M_1}{D_K(z_0)} \frac{\|\delta_{z_0}\|_{X_m}}{\delta^m \|\delta_{\varphi_K(x)}\|_X} \\
&\leq \frac{2|\lambda|^{K+1}M_1 \|\delta_{z_0}\|_X}{\delta^m \|\delta_{\varphi_K(x)}\|_X} \frac{1}{D_K(z_0)}.
\end{aligned}$$

Since

$$\mu^K \leq \frac{\|\delta_{\varphi_K(z_0)}\|_X}{\|\delta_{z_0}\|_X} D_K(z_0),$$

in combination with the above inequality we now get

$$\begin{aligned}
\frac{2|\lambda|^{K+1}M\|\delta_{z_0}\|_X}{\delta^m \|\delta_{\varphi_K(x)}\|_X} \frac{1}{D_K(z_0)} &\leq \frac{2|\lambda|^{K+1}M\|\delta_{z_0}\|_X}{\delta^m \|\delta_{z_K}\|_X} \frac{\|\delta_{\varphi_K(z_0)}\|_X}{\|\delta_{z_0}\|_X} \mu^{-K} \\
&\leq \frac{2M|\lambda|}{\delta^m} \left(\frac{|\lambda|}{\mu}\right)^K.
\end{aligned}$$

By choosing $K \geq n_0$ large enough, we see that $C_m^* - \lambda I$ is not bounded from below and consequently $C_m - \lambda I$ is not invertible, i.e., $\lambda \in \sigma(C_m)$. □

We are now ready to formulate the main result.

Theorem 4.9. *Consider the weighted composition operator uC_φ acting on $X(B_E)$. Assume that $\varphi(0) = 0$, $\|\varphi'(0)\| < 1$ and that $\varphi(B_E)$ is a relatively compact subset of E. Suppose that there exists $W \subseteq B_E$ with $\varphi(W) \subseteq W$ such that a Julia-type estimate holds for some $\epsilon, \delta > 0$. Then,*

$$\begin{aligned}
&\{u(0)\} \cup \{u(0)\lambda_1 \cdots \lambda_k : \lambda_j \in \sigma(\varphi'(0)), \\
&\quad 1 \leq j \leq k,\ k \geq 1\} \cup \{\lambda : |\lambda| \leq \gamma(uC_\varphi; W)\} \subset \sigma(uC_\varphi).
\end{aligned}$$

Proof. First of all, notice that the linear mapping $\varphi'(0) \in \mathcal{L}(E)$ is a compact operator, since by the Cauchy integral formula (see [23, 7.3 Corollary]), $\varphi'(0)(x) = \frac{1}{2\pi i}\int_{|\xi|=1/2} \frac{\varphi(\xi x)}{\xi}\mathrm{d}\xi$ belongs to the closed convex hull of the compact set $\overline{2\varphi(B_E)}$. Therefore, the mappings C_{kk} in Proposition 4.4 are compact.

In case that $||\varphi_n|| = 1$ for all $n \in \mathbb{N}$, the result follows from Theorem 4.1, Lemmas 4.8 and 4.5. If for some $n \in \mathbb{N}, ||\varphi_n|| < 1$, then $\gamma(uC_\varphi; W) = 0$, and the argument is simpler as there is no need of Lemma 4.8. □

From the above result, we get several consequences.

Corollary 4.10. *Let E be a Hilbert space or $E = C_0(\mathcal{X})$, $\mathcal{X}$ a locally compact Hausdorff topological space. Assume that $uC_\varphi : H_v^\infty(B_E) \to H_v^\infty(B_E)$ is bounded with $\varphi(0) = 0$ and $\|\varphi'(0)\| < 1$. Suppose that $\varphi(B_E)$ is a relatively compact subset of E. Then,*

$$\left\{ \lambda \in \mathbb{C} : |\lambda| \leq \liminf_n \sqrt[n]{\lim_{s\to 1} \sup_{\substack{\|\varphi_n(x)\|\geq s \\ x\in\varphi(B_E)}} \frac{\|\delta_{\varphi_n(x)}\||u(x)\cdot \cdots \cdot u(\varphi_n(x))|}{\|\delta_x\|}} \right\}$$
$$\cup \sigma_p(uC_\varphi) \subset \sigma(uC_\varphi).$$

Proof. We need, respectively, [12, Theorem 6.1] and [15, Theorem 2.2]. Each of them guarantees, respectively, that under the current assumptions, $W \equiv \varphi(B_E)$ satisfies the Julia-type estimates (4.2) for some $\epsilon, \delta > 0$. □

One can find plenty of mappings φ which do fulfill the assumptions in Corollary 4.10. Indeed, consider for every pair $(k, m) \in \mathbb{N}\times\mathbb{N}$, $m > 1$, the mapping

$$\varphi^{k,m} : (x_n) \in \ell_2 \mapsto \left(x_1^m, \ldots, x_k^m, x_{k+1}^m, \frac{x_{k+2}^m}{2}, \frac{x_{k+3}^m}{3}, \ldots, \frac{x_{k+i}^m}{i}, \ldots \right) \in \ell_2.$$

Clearly, $\varphi^{k,m}(0) = 0$, $(\varphi^{k,m})'(0) = 0$, since $\varphi^{k,m}$ is an m-homogeneous polynomial and $\varphi^{k,m}(B_{\ell_2}) \subset B_{\ell_2}$ is relatively compact.

Corollary 4.11. *Let B_E be either the n-ball $\mathbb{B}_N$ or the n-polydisc Δ_N. Assume that $uC_\varphi : H_v^\infty(B_E) \to H_v^\infty(B_E)$ is bounded with $\varphi(0) = 0$ and $\|\varphi'(0)\| < 1$. Then,*

$$\{\lambda \in \mathbb{C} : |\lambda| \leq r_e(uC_\varphi)\} \cup \sigma_p(uC_\varphi) = \sigma(uC_\varphi).$$

Proof. Recall that according to Corollary 3.5,

$$r_e(uC_\varphi) = \liminf_n \sqrt[n]{\lim_{s\to 1} \sup_{|\varphi_n(x)|\geq s} \frac{\|\delta_{\varphi_n(x)}\||u(x)\cdot \cdots \cdot u(\varphi_n(x))|}{\|\delta_x\|}}.$$

As in the proof of Corollary 4.10 by using [12, Theorem 6.1] and [15, Theorem 2.2] respectively, $W \equiv B_E$ satisfies the Julia-type estimates (4.2) for some $\epsilon, \delta > 0$. Thus, it is clear from Theorem 4.9 that

$$\{\lambda \in \mathbb{C} : |\lambda| \leq r_e(uC_\varphi)\} \cup \sigma_p(uC_\varphi) \subset \sigma(uC_\varphi).$$

The converse inclusion follows from the fact that if $\lambda \in \sigma(uC_\varphi)$ and $|\lambda| > r_e(uC_\varphi)$, then $\lambda \in \sigma_p(uC_\varphi)$ by Lemma 7.43 and Theorem 7.44 in [1] or Propositions 2.2 and 3.4 in [5]. □

Remark 4.12. Also for the Hardy space $H^\infty(B_E)$, $r_e(uC_\varphi)=\gamma\big(uC_\varphi;\varphi(B_E)\big)$.

Indeed, in this case, we have that $||\delta_x|| = 1$ and that u is bounded by some $M > 0$. Hence,

$$\begin{aligned}&\sup_{||\varphi_n(x)||>s} |u(x)u(\varphi(x))\cdots u(\varphi_n(x)| \\ &\quad \le M \sup_{\substack{\|\varphi_{n-1}(y)\|\ge s\\ y\in\varphi(B_E)}} |u(y)u(\varphi(y))\cdots u(\varphi_{n-1}(y))|,\end{aligned}$$

from where we get that $r_e(uC_\varphi) \le \gamma\big(uC_\varphi;\varphi(B_E)\big)$, as required. This yields the same conclusion as in Corollary 4.11, so we recover the main results concerning the spectrum in [12,15,27].

Corollary 4.13. *Let $p \ge 1$ and $\alpha > -1$. If uC_φ is a bounded operator on $\mathcal{H}(\mathbb{B}_N)$, $A^p_\alpha(\mathbb{B}_N)$ and $H^p(\mathbb{B}_N)$, respectively, with $\varphi(0) = 0$ and $\|\varphi'(0)\| < 1$, then*

$$\begin{aligned}&\{u(0)\} \cup \{u(0)\lambda_1\cdots\lambda_k : \lambda_j \in \sigma(\varphi'(0)), 1 \le j \le k,\ k \ge 1\} \cup \{\lambda : |\lambda| \\ &\quad \le \gamma\left(uC_\varphi;\mathbb{B}_N\right)\} \subset \sigma(uC_\varphi).\end{aligned}$$

Proof. If the range of some iterated of φ lies strictly inside $\mathbb{B}_N$, we have $\gamma\big(uC_\varphi;\mathbb{B}_N\big) = 0$. If that wasn't the case, then $||\varphi_n|| = 1$ for all $n \in \mathbb{N}$ and as in the proof of Corollary 4.11, $\mathbb{B}_N$ satisfies the Julia-type estimates (4.2) for some $\epsilon, \delta > 0$. So we may apply Theorem 4.9. □

Corollary 4.14. *Let $p \ge 1$ and $\alpha > -1$. If C_φ is a bounded operator on $\mathcal{H}(\mathbb{B}_N)$, $A^p_\alpha(\mathbb{B}_N)$ and $H^p(\mathbb{B}_N)$, respectively with $\varphi(0) = 0$ and $\|\varphi'(0)\| < 1$, then*

$$\begin{aligned}&\{1\} \cup\ \{\lambda_1\cdots\lambda_k : \lambda_j \in \sigma(\varphi'(0)), 1 \le j \le k,\ k \ge 1\} \cup \{\lambda : |\lambda| \\ &\quad \le \gamma_0\left(C_\varphi;\mathbb{B}_N\right)\} \subset \sigma(C_\varphi),\end{aligned}$$

where

$$\gamma_0(C_\varphi;\mathbb{B}_N) = \liminf_n \sqrt[n]{\lim_{s\to 1}\sup_{|z|\ge s} \frac{\|\delta_{\varphi_n(z)}\|}{\|\delta_z\|}}.$$

Proof. Notice that Lemma 3.9 applies since $\lim_{\|z\|\to 1}\|\delta_z\| = \infty$ by Proposition 2 in [9], (2.2) and (2.3), respectively. Thus, $\gamma(C_\varphi;\mathbb{B}_N) = \gamma_0(C_\varphi;\mathbb{B}_N)$. Now the statement follows from Corollary 4.13. □

This corollary yields [9, Theorem 15] because any map φ with $\varphi(0) = 0$ not unitary on any slice does satisfy $\|\varphi'(0)\| < 1$, as shown in the proof of [9, Lemma 14].

Corollary 4.15. *Assume that $\varphi(0) = 0$, $||\varphi'(0)|| < 1$ and that the composition operator C_φ is bounded on $A^p_\beta(\mathbb{B}_N)$ for some $-1 < \beta < \alpha$ and $p > 1$. Then for C_φ acting on $A^p_\alpha(\mathbb{B}_N)$, we have*

$$\{1\} \cup \{\lambda_1 \cdots \lambda_k : \lambda_j \in \sigma(\varphi'(0)), 1 \le j \le k,\ k \ge 1\} \\ \cup \left\{\lambda \in \mathbb{C} : |\lambda| \le r_e(C_\varphi)^{\frac{N+1+\alpha}{\alpha-\beta}}\right\} \subset \sigma(C_\varphi),$$

and, when $N = 1$*, then* $C_\varphi : A^p_\alpha(\mathbb{D}) \to A^p_\alpha(\mathbb{D})$, $\alpha > -1, p > 1$, *is always bounded and*

$$\{\varphi'(0)^n : n \ge 0\} \cup \left\{\lambda \in \mathbb{C} : |\lambda| \le r_e(C_\varphi)^{\frac{2+\alpha}{1+\alpha}}\right\} \subset \sigma(C_\varphi).$$

Proof. Both statements follow from Corollary 4.14 and Proposition 3.8. □

Remark 4.16. For every bounded operator $T : E \to E$, it holds by the general argument used in the proof of Corollary 4.11 that $\sigma(T) \subset \sigma_p(T) \cup \{\lambda \in \mathbb{C} : |\lambda| \le r_e(T)\}$. Therefore for $C_\varphi : A^p_\alpha(\mathbb{D}) \to A^p_\alpha(\mathbb{D})$, $\alpha > -1, p > 1$, with $\varphi(0) = 0$ and $|\varphi'(0)| < 1$, we obtain that

$$\{\varphi'(0)^n : n \ge 0\} \cup \left\{\lambda \in \mathbb{C} : |\lambda| \le r_e(C_\varphi)^{\frac{2+\alpha}{1+\alpha}}\right\} \\ \subset \sigma(C_\varphi) \subset \{\varphi'(0)^n : n \ge 0\} \cup \{\lambda \in \mathbb{C} : |\lambda| \le r_e(C_\varphi)\}.$$

The univalent case with $\alpha = 0$ was studied in [22]. In fact, it follows also for univalent symbol φ with the above assumptions and $\alpha > -1, p > 1$, that $\sigma(C_\varphi) = \{\varphi'(0)^n : n \ge 0\} \cup \{\lambda \in \mathbb{C} : |\lambda| \le r_e(C_\varphi)\}$. Indeed, in this case the essential spectral radius $r_e(C_\varphi)$ can be calculated using that the generalized Nevalinna counting function $N_{\varphi,2+\alpha}(z) = \left(\log \frac{1}{|\varphi^{-1}(z)|}\right)^{2+\alpha}$.

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