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Generalized Riesz Systems and Quasi Bases in Hilbert Space

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Abstract. The purpose of this article is twofold. First of all, the notion of $(\mathcal{D}, \mathcal{E})$-quasi basis is introduced for a pair $(\mathcal{D}, \mathcal{E})$ of dense subspaces of Hilbert spaces. This consists of two biorthogonal sequences $\{\varphi_n\}$ and $\{\psi_n\}$, such that $\sum_{n=0}^{\infty} \langle x, \varphi_n\rangle \langle \psi_n, y\rangle = \langle x, y\rangle$ for all $x \in \mathcal{D}$ and $y \in \mathcal{E}$. Second, it is shown that if biorthogonal sequences $\{\varphi_n\}$ and $\{\psi_n\}$ form a $(\mathcal{D}, \mathcal{E})$-quasi basis, then they are generalized Riesz systems. The latter play an interesting role for the construction of non-self-adjoint Hamiltonians and other physically relevant operators.

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1. Introduction

A sequence $\{\varphi_n\}$ in a Hilbert space $\mathcal{H}$ is called a generalized Riesz system if there exist an orthonormal basis (from now on, ONB) $\mathcal{F}_e = \{e_n\}$ in $\mathcal{H}$ and a densely defined closed operator T in $\mathcal{H}$ with densely defined inverse, such that $\mathcal{F}_e \subset D(T) \cap D((T^{-1})^*)$ and $Te_n = \varphi_n$, $n = 0, 1, \ldots$. In this case, $(\mathcal{F}_e, T)$ is called a constructing pair for $\{\varphi_n\}$, [4,7,8]. Then, if we put $\psi_n := (T^{-1})^* e_n$, $n = 0, 1, \ldots$, $\mathcal{F}_\varphi := \{\varphi_n\}$ and $\mathcal{F}_\psi := \{\psi_n\}$ are biorthogonal sequences in $\mathcal{H}$, that is, $\langle \varphi_n, \psi_m\rangle = \delta_{nm}$, $n, m = 0, 1, \ldots$.

The notion of generalized Riesz system is useful to investigate non-self-adjoint Hamiltonians constructed from $\mathcal{F}_\varphi$ and $\mathcal{F}_\psi$. More precisely, let $\mathcal{F}_\varphi$ be a generalized Riesz system with a constructing pair $(\mathcal{F}_e, T)$ and define ψ_n as above. Then, we consider the operators:

$$H_\varphi^{\boldsymbol{\alpha}} := T H_e^{\boldsymbol{\alpha}} T^{-1}, \; A_\varphi^{\boldsymbol{\alpha}} := T A_e^{\boldsymbol{\alpha}} T^{-1} \;\text{ and }\; B_\varphi^{\boldsymbol{\alpha}} := T B_e^{\boldsymbol{\alpha}} T^{-1},$$

together with

$$H_\psi^{\boldsymbol{\alpha}} := (T^*)^{-1} H_e^{\boldsymbol{\alpha}} T^*, \; A_\psi^{\boldsymbol{\alpha}} := (T^*)^{-1} A_e^{\boldsymbol{\alpha}} T^* \;\text{ and }\; B_\psi^{\boldsymbol{\alpha}} := (T^{-1})^* B_e^{\boldsymbol{\alpha}} T^*,$$

where $\boldsymbol{\alpha} = \{\alpha_n\} \subset \mathbb{C}$. Here:

$$H_e^{\boldsymbol{\alpha}} := \sum_{n=0}^{\infty} \alpha_n e_n \otimes \bar{e}_n, \; A_e^{\boldsymbol{\alpha}} := \sum_{n=0}^{\infty} \alpha_{n+1} e_n \otimes \bar{e}_{n+1}, \; B_e^{\boldsymbol{\alpha}} := \sum_{n=0}^{\infty} \alpha_{n+1} e_{n+1} \otimes \bar{e}_n$$

are a self-adjoint Hamiltonian, the lowering operator, and the raising operator for $\{e_n\}$, respectively (if, $x, y, z \in \mathcal{H}$, $(y \otimes \bar{z})x := \langle x, z \rangle\, y$).

Since $H^{\boldsymbol{\alpha}}_{\varphi}\varphi_n = \alpha_n \varphi_n$, $A^{\boldsymbol{\alpha}}_{\varphi}\varphi_n = \alpha_n \varphi_{n-1}$ (0 if $n = 0$) and $B^{\boldsymbol{\alpha}}_{\varphi}\varphi_n = \alpha_{n+1}\varphi_{n+1}$, $n = 0, 1, \ldots$, it seems natural to call the operators $H^{\boldsymbol{\alpha}}_{\varphi}$, $A^{\boldsymbol{\alpha}}_{\varphi}$ and $B^{\boldsymbol{\alpha}}_{\varphi}$ the non-self-adjoint Hamiltonian, and the generalized lowering and raising operators for $\{\varphi_n\}$, respectively. Similarly, since $H^{\boldsymbol{\alpha}}_{\psi}\psi_n = \alpha_n\psi_n$, $A^{\boldsymbol{\alpha}}_{\psi}\psi_n = \alpha_n\psi_{n-1}$ (0 if $n = 0$) and $B^{\boldsymbol{\alpha}}_{\psi}\psi_n = \alpha_{n+1}\psi_{n+1}$, the operators $H^{\boldsymbol{\alpha}}_{\psi}$, $A^{\boldsymbol{\alpha}}_{\psi}$, $B^{\boldsymbol{\alpha}}_{\psi}$ are called the non-self-adjoint Hamiltonian, generalized lowering operator, and raising operator for $\{\psi_n\}$ respectively.

Then, it is interesting to understand under what conditions biorthogonal sequences $\mathcal{F}_\varphi$ and $\mathcal{F}_\psi$ are generalized Riesz system, which is what we will discuss in this paper.

Studies on this subject have been undertaken in Refs. [6–9]. Here, we want to explore this question in a more general framework.

Let D_φ and D_ψ be the linear spans of the biorthogonal sequences $\mathcal{F}_\varphi$ and $\mathcal{F}_\psi$, respectively, and define the subspaces $D(\varphi)$ and $D(\psi)$ in $\mathcal{H}$ by:

$$D(\varphi) = \left\{ x \in \mathcal{H}; \sum_{n=0}^{\infty} |\, \langle x, \varphi_n \rangle\, |^2 < \infty \right\},$$

$$D(\psi) = \left\{ x \in \mathcal{H}; \sum_{n=0}^{\infty} |\, \langle x, \psi_n \rangle\, |^2 < \infty \right\}.$$

Clearly, $D_\psi \subset D(\varphi)$ and $D_\varphi \subset D(\psi)$. In Ref. [6], one of us has shown that if both D_φ and D_ψ are dense in $\mathcal{H}$ (this case is called regular), then $\mathcal{F}_\varphi$ and $\mathcal{F}_\psi$ are generalized Riesz systems. After that, in Ref. [7], it was proved that, if either D_φ and $D(\varphi)$, or D_ψ and $D(\psi)$, are dense in $\mathcal{H}$ (the case is called semiregular), again, $\mathcal{F}_\varphi$ and $\mathcal{F}_\psi$ are generalized Riesz systems. Hence, we will consider under what conditions $\mathcal{F}_\varphi$ and $\mathcal{F}_\psi$ are generalized Riesz systems when none of the above conditions is satisfied. In Ref. [4], we have proved that this holds under the assumptions that $\mathcal{F}_\varphi$ and $\mathcal{F}_\psi$ are biorthogonal and, at the same time, $\mathcal{D}$-quasi bases, in the sense that:

$$\sum_{n=0}^{\infty} \langle x, \varphi_n \rangle \langle \psi_n, y \rangle = \langle x, y \rangle\,, \quad \forall x, y \in \mathcal{D},$$

where $\mathcal{D}$ is a dense subspace in $\mathcal{H}$, such that $\mathcal{F}_\varphi \cup \mathcal{F}_\psi \subset \mathcal{D} \subset D(\varphi) \cap D(\psi)$, with some additional assumptions. In this paper, we shall show that this result holds in a more general case. In Sect. 3, we define the notion of $(\mathcal{D}, \mathcal{E})$-quasi bases which is a generalization of $\mathcal{D}$-quasi bases as follows:

$$\sum_{n=0}^{\infty} \langle x, \varphi_n \rangle \langle \psi_n, y \rangle = \langle x, y \rangle\,, \quad \forall x \in \mathcal{D},\, y \in \mathcal{E},$$

where $\mathcal{D}$ and $\mathcal{E}$ are dense subspaces in $\mathcal{H}$, such that $D_\psi \subset \mathcal{D} \subset D(\varphi)$ and $D_\varphi \subset \mathcal{E} \subset D(\psi)$, and we show in Theorem 3.2 that, under this condition, $\mathcal{F}_\varphi$ and $\mathcal{F}_\psi$ are generalized Riesz systems.

In Sect. 4, we shall investigate non-self-adjoint Hamiltonians, generalized lowering and raising operators constructed from $(\mathcal{D}, \mathcal{E})$-quasi bases. This analysis can be relevant for concrete physical applications, and extends what already deduced, for instance, in Refs. [2,3,6].

2. Preliminaries

In this section, we review some results on generalized Riesz systems needed in the rest of the paper. By Lemma 3.2, [7], we have the following.

Lemma 2.1. *Let $\{\varphi_n\}$ be a generalized Riesz basis with a constructing pair $(\mathcal{F}_e, T)$. Then, we have the following statements.*

(1) T^* has a densely defined inverse and $(T^*)^{-1} = (T^{-1})^*$.
(2) Let $\psi_n := (T^{-1})^* e_n$, $n = 0, 1, \dots$. Then, $\{\varphi_n\}$ and $\{\psi_n\}$ are biorthogonal and $(T^{-1})^*$ is a densely defined closed operator in $\mathcal{H}$ with densely defined inverse T^*. Hence, $\{\psi_n\}$ is a generalized Riesz basis with a constructing pair $(\mathcal{F}_e, (T^{-1})^*)$.
(3) $D(\varphi) \cap D(\psi)$ is dense in $\mathcal{H}$.

Next, for any ONB $\{e_n\}$ in $\mathcal{H}$ and a sequence $\{\varphi_n\}$ in $\mathcal{H}$, we introduce the operators $T^0_{\varphi,e}$, $T_{\varphi,e}$ and $T_{e,\varphi}$ as follows:

$$T^0_{\varphi,e} := \text{the linear operator defined by } T^0_{\varphi,e} e_n = \varphi_n,\ n = 0, 1, \dots,$$

$$T_{\varphi,e} := \sum_{n=0}^{\infty} \varphi_n \otimes \bar{e}_n,$$

$$T_{e,\varphi} := \sum_{n=0}^{\infty} e_n \otimes \bar{\varphi}_n.$$

Similarly, we can introduce, for the set $\{\psi_n\}$ in Lemma 2.1, the operators $T^0_{\psi,e}$, $T_{\psi,e}$, and $T_{e,\psi}$. These operators had a role in Ref. [7] and will also be relevant here. By Lemmas 2.1, 2.2 in Ref. [7], we get the following.

Lemma 2.2. *(1) $T_{\varphi,e}$ is a densely defined linear operator in $\mathcal{H}$, such that:*

$$T_{\varphi,e} \supseteq T^0_{\varphi,e} \text{ and } T^0_{\varphi,e} e_n = T_{\varphi,e} e_n = \varphi_n,\ n = 0, 1, \dots.$$

(2) $D(T_{e,\varphi}) = D(\varphi)$ and $(T^0_{\varphi,e})^* = T^*_{\varphi,e} = T_{e,\varphi}$.
(3) $T^0_{\varphi,e}$ is closable if and only if $T_{\varphi,e}$ is closable if and only if $D(\varphi)$ is dense in $\mathcal{H}$. If this holds, then:

$$\bar{T}^0_{\varphi,e} = \bar{T}_{\varphi,e} = (T_{e,\varphi})^*. \tag{1}$$

Furthermore, by Lemmas 2.3 and 2.4 in Ref. [7], we have:

Lemma 2.3. *Let $\mathcal{F}_\varphi$ and $\mathcal{F}_\psi$ be biorthogonal sequences in $\mathcal{H}$. Suppose that $D(\varphi)$ is dense in $\mathcal{H}$. Then, we have the following:*

(1) *$\bar{T}_{\varphi,e}$ has an inverse and $\bar{T}^{-1}_{\varphi,e} \subseteq T_{e,\psi} = (T_{\psi,e})^*$.*
(2) *The following (i), (ii), and (iii) are equivalent:*
   (i) *D_ϕ is dense in $\mathcal{H}$.*
   (ii) *$\bar{T}_{\varphi,e}$ has a densely defined inverse.*
   (iii) *$T^*_{\varphi,e} (= T_{e,\varphi})$ has a densely defined inverse.*
   If this holds, then $T^{-1}_{e,\varphi} = (\bar{T}^{-1}_{\varphi,e})^*$.

(3) *For the operators $T_{\psi,\boldsymbol{e}}$ and $T_{\boldsymbol{e},\psi}$, the same results as in* (1) *and* (2) *hold.*

By [7], Theorem 3.4, we also get

Theorem 2.4. *Let $\mathcal{F}_\varphi$ and $\mathcal{F}_\psi$ be biorthogonal sequences in $\mathcal{H}$, and let $\mathcal{F}_e$ be an arbitrary ONB in $\mathcal{H}$. Then, the following statements hold:*

(1) Suppose that both D_φ and D_ψ are dense in $\mathcal{H}$. Then, $\mathcal{F}_\varphi$ (resp. $\mathcal{F}_\psi$) is a generalized Riesz basis with constructing pairs $(\mathcal{F}_e, \bar{T}_{\phi,\boldsymbol{e}})$ and $(\mathcal{F}_e, T^{-1}_{\boldsymbol{e},\psi})$ (resp. $(\mathcal{F}_e, \bar{T}_{\psi,\boldsymbol{e}})$ and $(\mathcal{F}_e, T^{-1}_{\boldsymbol{e},\phi})$), and $\bar{T}_{\phi,\boldsymbol{e}}$ (resp. $\bar{T}_{\psi,\boldsymbol{e}}$) is the minimum among constructing operators of the generalized Riesz basis $\mathcal{F}_\varphi$ (resp. $\mathcal{F}_\psi$), and $T^{-1}_{\boldsymbol{e},\psi}$ (resp. $T^{-1}_{\boldsymbol{e},\phi}$) is the maximum among constructing operators of $\mathcal{F}_\varphi$ (resp. $\mathcal{F}_\psi$). Furthermore, any closed operator T (resp. K) satisfying $\bar{T}_{\phi,\boldsymbol{e}} \subset T \subset T^{-1}_{\boldsymbol{e},\psi}$ (resp. $\bar{T}_{\psi,\boldsymbol{e}} \subset K \subset T^{-1}_{\boldsymbol{e},\phi}$) is a constructing operator for $\mathcal{F}_\varphi$ (resp. $\mathcal{F}_\psi$).

(2) Suppose that $D(\phi)$ and D_ϕ are dense in $\mathcal{H}$. Then, $\mathcal{F}_\varphi$ (resp. $\mathcal{F}_\psi$) is a generalized Riesz basis with a constructing pair $(\mathcal{F}_e, \bar{T}_{\phi,\boldsymbol{e}})$ (resp. $(\mathcal{F}_e, T^{-1}_{\boldsymbol{e},\phi})$) and the constructing operator $\bar{T}_{\phi,\boldsymbol{e}}$ (resp. $T^{-1}_{\boldsymbol{e},\phi}$) is the minimum (resp. the maximum) among constructing operators of $\mathcal{F}_\varphi$ (resp. $\mathcal{F}_\psi$).

(3) Suppose that $D(\psi)$ and D_ψ are dense in $\mathcal{H}$. Then, $\mathcal{F}_\psi$ (resp. $\mathcal{F}_\varphi$) is a generalized Riesz basis with a constructing pair $(\mathcal{F}_e, \bar{T}_{\psi,\boldsymbol{e}})$ (resp. $(\mathcal{F}_e, T^{-1}_{\boldsymbol{e},\psi})$) and the constructing operator $\bar{T}_{\psi,\boldsymbol{e}}$ (resp. $T^{-1}_{\boldsymbol{e},\psi}$) is the minimum (resp. the maximum) among constructing operators of $\mathcal{F}_\psi$ (resp. $\mathcal{F}_\varphi$).

Theorem 2.4 shows how the problem stated in Introduction (under what conditions biorthogonal sequences $\mathcal{F}_\varphi$ and $\mathcal{F}_\psi$ are generalized Riesz systems) can be solved in the case when either D_φ and $D(\psi)$ or D_ψ and $D(\varphi)$ are dense in $\mathcal{H}$. However, this problem has not been solved completely in case that both D_φ and D_ψ are not dense in $\mathcal{H}$, which is what is interesting for us here. We will see how the operators $T_{\varphi,\boldsymbol{e}}$, $T_{\boldsymbol{e},\varphi}$, $T_{\psi,\boldsymbol{e}}$ and $T_{\boldsymbol{e},\psi}$ will be relevant in our analysis, together with the $(\mathcal{D},\mathcal{E})$-quasi bases, we will define in the next section. This result is a generalization of the one obtained in Ref. [4].

3. $(\mathcal{D},\mathcal{E})$-Quasi Bases

In this section, we extend the notion of $\mathcal{D}$-quasi bases by introducing a second dense subset $\mathcal{E}$ of the Hilbert space $\mathcal{H}$, and we relate these new families of vectors to generalized Riesz systems.

Definition 3.1. Let $\mathcal{F}_\varphi$ and $\mathcal{F}_\psi$ be biorthogonal sequences in $\mathcal{H}$, and let $\mathcal{D}$ and $\mathcal{E}$ be dense subspaces, such that $D_\psi \subseteq \mathcal{D} \subseteq D(\varphi)$ and $D_\varphi \subseteq \mathcal{E} \subseteq D(\psi)$. Then, $(\{\varphi_n\}, \{\psi_n\})$ is said to be a $(\mathcal{D},\mathcal{E})$-quasi basis if:

$$\sum_{k=0}^{\infty} \langle x, \varphi_k \rangle \langle \psi_k, y \rangle = \langle x, y \rangle$$

for all $x \in \mathcal{D}$ and $y \in \mathcal{E}$.

It is clear that any $(\mathcal{D},\mathcal{D})$-quasi basis is a $\mathcal{D}$-quasi basis in the sense of [1].

Example 1. A very simple example of a $(\mathcal{D},\mathcal{E})$-quasi basis can be constructed as follows. Let $\{e_n\}$ be an ONB for $\mathcal{H}$. Let α_n an unbounded sequence of positive real numbers having 0 as limit point. To be more concrete, let us take:

$$\alpha_n = \begin{cases} \frac{1}{n} & \text{if } n \text{ is even} \\ n & \text{if } n \text{ is odd.} \end{cases}$$

Let $Tx = \sum_{n=1}^{\infty} \alpha_n \langle x, e_n\rangle e_n$ be defined on the domain:

$$D(T) = \left\{ x \in \mathcal{H} : \sum_{k=0}^{\infty} (2k+1)^2 |(x, e_{2k+1})|^2 < \infty \right\}.$$

The operator T is unbounded, self-adjoint, invertible with inverse T^{-1} is defined as $T^{-1}y = \sum_{n=1}^{\infty} \alpha_n^{-1} \langle x, e_n\rangle e_n$ on the domain:

$$D(T^{-1}) = \left\{ y \in \mathcal{H} : \sum_{k=1}^{\infty} (2k)^2 |(y, e_{2k})|^2 < \infty \right\}.$$

Both $D(T)$ and $D(T^{-1})$ are dense subspaces of $\mathcal{H}$ and they are different as one can easily check. Let us set $\varphi_n = Te_n$ and $\psi_n = T^{-1}e_n$, $n \in \mathbb{N}$. The $\varphi_n = \alpha_n e_n$, while $\psi_n = T^{-1}e_n = \alpha_n^{-1} e_n$. Moreover $D(\varphi) = D(T)$, $D(\psi) = D(T^{-1})$. Then, we have:

$$\sum_{n=0}^{\infty} \langle x, \varphi_n\rangle \langle \psi_n, y\rangle = \sum_{n=0}^{\infty} \langle x, \alpha_n e_n\rangle \left\langle \alpha_n^{-1} e_n, y\right\rangle = \langle x, y\rangle.$$

Thus, $(\mathcal{F}_\varphi, \mathcal{F}_\psi)$ is a $(D(\varphi), D(\psi))$-quasi basis.

Example 2. Let $H_0 = p^2 + x^2$ be (twice) the self-adjoint Hamiltonian of a one-dimensional harmonic oscillator. We consider H_0 to be the closure of the operator acting in the same way on the Schwartz space $\mathcal{S}(\mathbb{R})$, and $T = \mathbb{1} + p^2$, which is an unbounded self-adjoint operator defined on $D(T) = W^{2,2}(\mathbb{R})$, the Sobolev space of functions having first and second order weak derivatives in $L^2(\mathbb{R})$. The operator $T = H_0 + \mathbb{1} - x^2$ is unbounded, invertible with bounded inverse T^{-1}. The eigensystem of H_0 is well known:

$$H_0 e_n(x) = (2n+1)e_n(x),\ e_n(x) = \frac{1}{\sqrt{2^n n! \pi^{1/2}}} H_n(x)\, e^{-x^2/2},$$

$n \geq 0$, where $H_n(x)$ is the nth Hermite polynomial. Moreover:

$$H_0 f = \sum_{n=0}^{\infty} (2n+1)(e_n \otimes \bar{e}_n) f = \sum_{n=0}^{\infty} (2n+1)(f, e_n) e_n, \quad \forall f \in \mathcal{S}(\mathbb{R}). \tag{2}$$

It is easy to see that $e_n(x) \in D(T)$, so that we can define $\varphi_n(x) = (Te_n)(x)$ and $\psi_n(x) = (T^{-1}e_n)(x)$. We get:

$$\varphi_n(x) = (2 + 2n - x^2) e_n(x),\ \psi_n(x) = \frac{1}{2} \int_{\mathbb{R}} e^{-|x-y|}\, e_n(y)\, \mathrm{d}y.$$

These functions are, respectively, eigenvectors of $H = TH_0T^{-1}$ and $H^\dagger$, with eigenvalue $2n + 1$. Some computations show that, for instance:

$$H = H_0 - 2\left(\mathbb{1} + 2x\frac{\mathrm{d}}{\mathrm{d}x}\right) G\star .$$

Here, $G(x)$ is the Green function of T, $G(x) = \frac{1}{2}e^{-|x|}$, and $(G \star f)(x) = \int_{\mathbb{R}} G(x-y)f(y)\mathrm{d}y$, for all $f(x) \in L^2(\mathbb{R})$. Of course, we can rewrite H as follows: $H = H_0 - 2(\mathbb{1} + 2ixp)\, G\star$, which is manifestly non-self-adjoint.

The sets $\mathcal{F}_\varphi$ and $\mathcal{F}_\psi$ are biorthogonal and form a $(D(T), \mathcal{H})$-quasi basis, since:

$$\sum_{k=0}^{\infty} \langle f, \varphi_k\rangle \langle \psi_k, g\rangle = \langle f, g\rangle,$$

for all $f(x) \in D(T)$ and $g(x) \in L^2(\mathbb{R})$.

Let $\mathcal{F}_\varphi$ and $\mathcal{F}_\psi$ be biorthogonal sequences. Suppose that $\mathcal{F}_\varphi$ is a generalized Riesz system with constructing pair $(\mathcal{F}_{\boldsymbol{e}}, T)$. We put $\psi_n^T := (T^{-1})^* e_n$, $n = 0, 1, \ldots$. Then, $\mathcal{F}_\psi$ and $\mathcal{F}_\psi^T := \{\psi_n^T\}$ are biorthogonal sequences, but $\mathcal{F}_\psi$ does not necessarily coincide with $\mathcal{F}_\psi^T$. For this reason, we will call the constructing pair $(\mathcal{F}_{\boldsymbol{e}}, T)$ natural for the biorthogonal sequences $\mathcal{F}_\varphi$ and $\mathcal{F}_\psi$ if $\mathcal{F}_\psi = \mathcal{F}_\psi^T$. If D_φ is dense in $\mathcal{H}$, then $(\mathcal{F}_e, T)$ is automatically natural for $\mathcal{F}_\varphi$ and $\mathcal{F}_\psi$.

The next theorem, which is the main result of this paper, shows that the notion of $(\mathcal{D}, \mathcal{E})$-quasi basis is intimately linked to that of generalized Riesz system.

Theorem 3.2. *Let $(\mathcal{F}_\varphi, \mathcal{F}_\psi)$ be a biorthogonal pair and $\mathcal{D}$ and $\mathcal{E}$ be dense subspaces in $\mathcal{H}$, such that $D_\psi \subseteq \mathcal{D} \subseteq D(\varphi)$ and $D_\varphi \subseteq \mathcal{E} \subseteq D(\psi)$. Then, the following statements are equivalent:*

(i) $(\mathcal{F}_\varphi, \mathcal{F}_\psi)$ is a $(\mathcal{D}, \mathcal{E})$-quasi basis.

(ii) For any ONB $\mathcal{F}_e = \{e_n\}$ in $\mathcal{H}$, $\mathcal{F}_\varphi$ is a generalized Riesz system with a natural constructing pair $(\mathcal{F}_e, T)$ satisfying $D(T^*) \supseteq \mathcal{D}$ and $D(T^{-1}) \supseteq \mathcal{E}$.

(iii) For any ONB $\mathcal{F}_e = \{e_n\}$ in $\mathcal{H}$, $\mathcal{F}_\psi$ is a generalized Riesz system with a natural constructing pair $(\mathcal{F}_e, K)$ satisfying $D(K^*) \supseteq \mathcal{E}$ and $D(K^{-1}) \supseteq \mathcal{D}$.

If the statement (i) holds, then we can take $(\overline{T_{\boldsymbol{e},\psi}\lceil_{\mathcal{E}}})^{-1}$ and $(\overline{T_{\boldsymbol{e},\varphi}\lceil_{\mathcal{D}}})^{-1}$ as T and K in (ii) and (iii), respectively. If D_φ is not dense in $\mathcal{H}$, then $T_{\boldsymbol{e},\psi}$ does not have an inverse, but $T_{\boldsymbol{e},\psi}\lceil_{\mathcal{E}}$ has an inverse.

Proof. Take arbitrary $x \in \mathcal{D}$ and $y \in \mathcal{E}$. Since $x \in D(T_{\boldsymbol{e},\varphi}) = D(\varphi)$ and $y \in D(T_{\boldsymbol{e},\psi}) = D(\psi)$, we have:

$$\begin{aligned}\langle x, y\rangle &= \sum_{n=0}^{\infty} \langle x, \varphi_n\rangle \langle \psi_n, y\rangle = \sum_{n=0}^{\infty} \langle x, T_{\varphi,\boldsymbol{e}} e_n\rangle \langle T_{\psi,\boldsymbol{e}} e_n, y\rangle \\ &= \sum_{n=0}^{\infty} \langle T_{\boldsymbol{e},\varphi} x, e_n\rangle \langle e_n, T_{\boldsymbol{e},\psi} y\rangle = \langle T_{\boldsymbol{e},\varphi} x, T_{\boldsymbol{e},\psi} y\rangle,\end{aligned}$$

which implies that:

$$(\overline{T_{e,\psi}\lceil_{\mathcal{E}}})^{-1} \subseteq (T_{e,\varphi}\lceil_{\mathcal{D}})^* \text{ and } (\overline{T_{e,\varphi}\lceil_{\mathcal{D}}})^{-1} \subseteq (T_{e,\psi}\lceil_{\mathcal{E}})^*. \tag{3}$$

Now, we put $T := (\overline{T_{e,\psi}\lceil_{\mathcal{E}}})^{-1}$. Since $D(T) = \overline{T_{e,\psi}\lceil_{\mathcal{E}}}D(\overline{T_{e,\psi}\lceil_{\mathcal{E}}}) \supseteq \overline{T_{e,\psi}\lceil_{\mathcal{E}}}\mathcal{E} \supseteq \overline{T_{e,\psi}\lceil_{\mathcal{E}}}D_\varphi = D_{e}$ and $D((T^{-1})^*) = D((\overline{T_{e,\psi}\lceil_{\mathcal{E}}})^*) \supseteq D((\overline{T_{e,\varphi}\lceil_{\mathcal{D}}})^{-1}) = \overline{T_{e,\varphi}\lceil_{\mathcal{D}}}D(\overline{T_{e,\varphi}\lceil_{\mathcal{D}}}) \supseteq \overline{T_{e,\varphi}\lceil_{\mathcal{D}}}D_\psi = D_{e}$, it follows that T is a densely defined closed operator in $\mathcal{H}$ with densely defined inverse, such that $\boldsymbol{e} \subseteq D(T) \cap D((T^{-1})^*)$. Furthermore, we have:

$$\begin{aligned} Te_n &= (\overline{T_{e,\psi}\lceil_{\mathcal{E}}})^{-1}\overline{T_{e,\psi}\lceil_{\mathcal{E}}}\varphi_n = \varphi_n, \\ (T^{-1})^* e_n &= (\overline{T_{e,\psi}\lceil_{\mathcal{E}}})^* e_n = T_{\psi,e}e_n = \psi_n, \quad n = 0,1,\dots \end{aligned}$$

Thus, $\mathcal{F}_\varphi$ is a generalized Riesz system with a natural constructing pair $(\mathcal{F}_e, T)$. Furthermore, we have $D(T^{-1}) = D(\overline{T_{e,\psi}\lceil_{\mathcal{E}}}) \supseteq \mathcal{E}$ and by (2) $D(T^*) \supseteq D(\overline{T_{e,\varphi}\lceil_{\mathcal{D}}}) \supseteq \mathcal{D}$. Thus, (i) $\Rightarrow$ (ii).
In a similar way, setting $K = (\overline{T_{e,\varphi}\lceil_{\mathcal{D}}})^{-1}$, we can show that $\mathcal{F}_\psi$ is a generalized Riesz system for a natural constructing pair $(\mathcal{F}_e, K)$ satisfying $D(K^*) \supseteq \mathcal{E}$ and $D(K^{-1}) \supseteq \mathcal{D}$. Thus, (i) implies (iii).
(ii) $\Rightarrow$ (i) Take arbitrary $x \in \mathcal{D}$ and $y \in \mathcal{E}$. Since:

$$\begin{aligned} \sum_{k=0}^{\infty} \langle x, \varphi_k\rangle \langle \psi_k, y\rangle &= \sum_{k=0}^{\infty} \langle x, Te_n\rangle \left\langle (T^{-1})^* e_n, y\right\rangle \\ &= \sum_{k=0}^{\infty} \langle T^* x, e_n\rangle \left\langle e_n, T^{-1}y\right\rangle = \left\langle T^*x, T^{-1}y\right\rangle = \langle x, y\rangle, \end{aligned}$$

it follows that $(\mathcal{F}_\varphi, \mathcal{F}_\psi)$ is a $(\mathcal{D}, \mathcal{E})$-quasi basis. Similarly, we can show (iii) $\Rightarrow$ (i). This completes the proof. $\square$

For $\mathcal{D}$-quasi basis, we have the following:

Corollary 3.3. *Let $\mathcal{F}_\varphi$ and $\mathcal{F}_\psi$ be biorthogonal sequences and $\mathcal{D}$ be a dense subspace in $\mathcal{H}$, such that $D_\varphi \cup D_\psi \subseteq \mathcal{D} \subseteq D(\varphi) \cap D(\psi)$. Then, the following statements are equivalent:*

- *(i) $(\mathcal{F}_\varphi, \mathcal{F}_\psi)$ is a $\mathcal{D}$-quasi basis.*
- *(ii) For any ONB $\mathcal{F}_e = \{e_n\}$ in $\mathcal{H}$, $\mathcal{F}_\varphi$ is a generalized Riesz system with a natural constructing pair $(\mathcal{F}_e, T)$ satisfying $D(T^*) \cap D(T^{-1}) \supseteq \mathcal{D}$.*
- *(iii) For any ONB $\mathcal{F}_e = \{e_n\}$ in $\mathcal{H}$, $\mathcal{F}_\psi$ is a generalized Riesz system with a natural constructing pair $(\mathcal{F}_e, K)$ satisfying $D(K^*) \cap D(K^{-1}) \supseteq \mathcal{D}$.*

If (i) holds, then we can take $(\overline{T_{e,\psi}\lceil_{\mathcal{D}}})^{-1}$ and $(\overline{T_{e,\varphi}\lceil_{\mathcal{D}}})^{-1}$ as T in (ii) and K in (iii), respectively.

By Theorem 3.2, if $(\mathcal{F}_\varphi, \mathcal{F}_\psi)$ is a $(\mathcal{D}, \mathcal{E})$-quasi basis, then, for any ONB $\mathcal{F}_e = \{e_n\}$, $(\overline{T_{e,\psi}\lceil_{\mathcal{E}}})^{-1}$ and $(\overline{T_{e,\varphi}\lceil_{\mathcal{D}}})^*$ are constructing operators for the generalized Riesz system $\mathcal{F}_\varphi$, such that $(\overline{T_{e,\psi}\lceil_{\mathcal{E}}})^{-1} \subseteq (\overline{T_{e,\varphi}\lceil_{\mathcal{D}}})^*$, and $(\overline{T_{e,\varphi}\lceil_{\mathcal{D}}})^{-1}$ and $(\overline{T_{e,\psi}\lceil_{\mathcal{E}}})^*$ are constructing operators for the generalized Riesz system $\mathcal{F}_\psi$, such that $(\overline{T_{e,\varphi}\lceil_{\mathcal{D}}})^{-1} \subseteq (\overline{T_{e,\psi}\lceil_{\mathcal{E}}})^*$.

Remark. For a biorthogonal pair $(\mathcal{F}_\varphi, \mathcal{F}_\psi)$, it is clear that $D_\psi \subseteq D(\varphi)$ and $D_\varphi \subseteq D(\psi)$. What is not clear is whether $D_\varphi \subseteq D(\varphi)$ and $D_\psi \subseteq D(\psi)$. For this reason, it may be more convenient to work, in some concrete cases, with $(\mathcal{D}, \mathcal{E})$-quasi bases rather than with $\mathcal{D}$-quasi bases.

Let $\mathcal{F}_\varphi$ be a generalized Riesz system with constructing pair $(\mathcal{F}_e, T)$. We discuss now when there exists a sequence $\mathcal{F}_\psi$ in $\mathcal{H}$ and subspaces $\mathcal{D}$ and $\mathcal{E}$ in $\mathcal{H}$, such that $\mathcal{F}_\varphi$ and $\mathcal{F}_\psi$ are biorthogonal and define a $(\mathcal{D}, \mathcal{E})$-quasi basis:

Proposition 3.4. *Let $\mathcal{F}_\varphi$ be a generalized Riesz system with a constructing pair $(\mathcal{F}_e, T)$. Then, $(\mathcal{F}_\varphi, \mathcal{F}_\psi^T)$ is a $(D(T^*), D(T^{-1}))$-quasi basis and $T = \left(T_{\boldsymbol{e},\psi^T}\lceil_{D(T^{-1})}\right)^{-1}$, $(T^{-1})^* = \left(T_{\boldsymbol{e},\varphi}\lceil_{D(T^*)}\right)^{-1}$.*

Proof. It is clear that $(\mathcal{F}_\varphi, \mathcal{F}_\psi^T)$ is a $(D(T^*), D(T^{-1}))$-quasi basis. Furthermore, since $Te_n = \varphi_n$, $n = 0, 1, \ldots$, we have:

$$T_{\varphi,\boldsymbol{e}} \subseteq T,$$

which implies that:

$$T^* \subseteq T_{\boldsymbol{e},\varphi}.$$

Hence, we have:

$$T^* = T_{\boldsymbol{e},\varphi}\lceil_{D(T^*)}.$$

Thus, we have:

$$(T^*)^{-1} = \left(T_{\boldsymbol{e},\varphi}\lceil_{D(T^*)}\right)^{-1}.$$

Since $(T^{-1})^* e_n = \psi_n^T$, $n = 0, 1, \ldots$, we can similarly show $T = \left(T_{\boldsymbol{e},\psi^T}\lceil_{D(T^{-1})}\right)^{-1}$. This completes the proof. □

Next, we consider when there exists a subspace $\mathcal{D}$ in $\mathcal{H}$, such that $(\mathcal{F}_\varphi, \mathcal{F}_\psi^T)$ is $\mathcal{D}$-quasi basis.

Proposition 3.5. *Let $\mathcal{F}_\varphi$ be a generalized Riesz system with constructing pair $(\mathcal{F}_e, T)$. Suppose that $\mathcal{F}_e \subset D(T^*T) \cap D(T^{-1}(T^{-1})^*)$. Then, $(\mathcal{F}_\varphi, \mathcal{F}_\psi^T)$ is a $(D(T^*) \cap D(T^{-1}))$-quasi basis and $T = \left(\overline{T_{\boldsymbol{e},\psi^T}\lceil_{D(T^*)\cap D(T^{-1})}}\right)^{-1}$, $(T^{-1})^* = \left(\overline{T_{\boldsymbol{e},\varphi}\lceil_{D(T^*)\cap D(T^{-1})}}\right)^{-1}$.*

Proof. We denote for simplicity ψ^T by ψ. At first, we show that $D(T^{-1}) \cap D(T^*)$ is a core for T^{-1}. Take an arbitrary $x \in D(T)$. Let $|T| = \int_0^\infty \lambda \mathrm{d}E_T(\lambda)$ be the spectral resolution of the absolute $|T| := (T^*T)^{1/2}$ of T. Then, we have $TE_T(n)x \in D(T^*) \cap D(T^{-1})$, $n = 0, 1, \ldots$ and $\lim_{n\to\infty} TE_T(n)x = Tx$. Furthermore, take an arbitrary $y \in D(T^{-1})$. Then, $y = Tx$ for some $x \in D(T)$ and we have $\lim_{n\to\infty} TE_T(n)x = Tx = y$ and $\lim_{n\to\infty} T^{-1}(TE_T(n)x) = \lim_{n\to\infty} E_T(n)x = x = T^{-1}y$. Thus, $D(T^{-1}) \cap D(T^*)$ is a core for T^{-1}.

At second, we show that $D(T^{-1}) \cap D(T^*)$ is a core for T^*. Take an arbitrary $y \in D(T^*)$. Let $|T^*| = \int_0^\infty \lambda \mathrm{d}E_{T^*}(\lambda)$ be the spectral resolution of the absolute $|T^*| := (TT^*)^{1/2}$ of T^*. Then, it follows that $E_{T^*}(n)y =$

$T(T^*|T^*|^{-2}E_{T^*}(n)y) \in D(T^{-1}) \cap D(T^*)$, $n = 0, 1, \ldots$, $\lim_{n\to\infty} E_{T^*}(n)y = y$ and $\lim_{n\to\infty} T^* E_{T^*}(n)y = T^*y$. Thus, $D(T^{-1}) \cap D(T^*)$ is a core for T^*.

At third, we show that $D_\varphi \subseteq D(T^{-1}) \cap D(T^*) \subseteq D(\varphi) \cap D(\psi)$ and $D_\psi \subseteq D(T^{-1}) \cap D(T^*) \subseteq D(\varphi) \cap D(\psi)$. It is clear that $\varphi_n = Te_n \in D(T^{-1})$. Furthermore, since $\mathcal{F}_e \subseteq D(T^*T)$, we have:

$$\langle Tx, \varphi_n \rangle = \langle Tx, Te_n \rangle = \langle x, T^*Te_n \rangle$$

for all $x \in D(T)$. Hence, we have $\varphi_n \in D(T^*)$. Thus $D_\varphi \subseteq D(T^{-1}) \cap D(T^*)$. And since $\psi_n = (T^{-1})^* e_n (= (T^*)^{-1} e_n)$, we have $\psi_n \in D(T^*)$. Furthermore, since $\mathcal{F}_e \subseteq D(T^{-1}(T^{-1})^*)$, we have:

$$\left\langle (T^{-1})^* y, \psi_n \right\rangle = \left\langle (T^{-1})^* y, (T^{-1})^* e_n \right\rangle = \left\langle y, T^{-1}(T^{-1})^* e_n \right\rangle$$

for all $y \in D((T^{-1})^*)$. Hence, we have $\psi_n \in D(T^{-1})$. Thus $D_\psi \subseteq D(T^{-1}) \cap D(T^*)$. We show $D(T^{-1}) \cap D(T^*) \subseteq D(\varphi) \cap D(\psi)$. Indeed, take an arbitrary $y \in D(T^{-1}) \cap D(T^*)$. Since

$$\sum_{k=0}^{\infty} |\langle y, \varphi_k \rangle|^2 = \sum_{k=0}^{\infty} |\langle y, Te_k \rangle|^2 = \sum_{k=0}^{\infty} |\langle T^*y, e_k \rangle|^2 = \|T^*y\|^2$$

and

$$\sum_{k=0}^{\infty} |\langle y, \psi_k \rangle|^2 = \sum_{k=0}^{\infty} |\left\langle T^{-1}y, e_k \right\rangle|^2 = \|T^{-1}y\|^2,$$

we have $y \in D(\varphi) \cap D(\psi)$.

Finally, we show that $(\mathcal{F}_\varphi, \mathcal{F}_\psi^T)$ is a $(D(T^*) \cap D(T^{-1}))$-quasi basis and $T = \left(\overline{T_{e,\psi}\lceil_{D(T^*)\cap D(T^{-1})}}\right)^{-1}$, $(T^{-1})^* = \left(\overline{T_{e,\varphi}\lceil_{D(T^*)\cap D(T^{-1})}}\right)^{-1}$. Since

$$\begin{aligned}\sum_{k=0}^{\infty} \langle x, \varphi_k \rangle \langle \psi_k, y \rangle &= \sum_{k=0}^{\infty} \langle x, Te_k \rangle \left\langle (T^{-1})^* e_k, y \right\rangle \\ &= \sum_{k=0}^{\infty} \langle T^*x, e_k \rangle \left\langle e_k, T^{-1}y \right\rangle \\ &= \left\langle T^*x, T^{-1}y \right\rangle \\ &= \langle x, y \rangle\end{aligned}$$

for all $x, y \in D(T^*) \cap D(T^{-1})$, it follows that $(\mathcal{F}_\varphi, \mathcal{F}_\psi^T)$ is a $(D(T^*) \cap D(T^{-1}))$-quasi basis. Furthermore, since $T^{-1} \subseteq T_{e,\psi}$ and $D(T^{-1}) \cap D(T^*)$ is a core for T^{-1}, we have:

$$T^{-1} = \overline{T^{-1}\lceil_{D(T^*)\cap D(T^{-1})}} = \overline{T_{e,\psi}\lceil_{D(T^*)\cap D(T^{-1})}},$$

which implies that $T = (\overline{T_{e,\psi}\lceil_{D(T^*)\cap D(T^{-1})}})^{-1}$. Furthermore, since $T_{\varphi,e} \subseteq T$ and $D(T^{-1}) \cap D(T^*)$ is a core for T^*, we have:

$$T^* = \overline{T^*\lceil_{D(T^*)\cap D(T^{-1})}} = \overline{T_{e,\varphi}\lceil_{D(T^*)\cap D(T^{-1})}},$$

which implies that $(T^*)^{-1} = (\overline{T_{e,\varphi}\lceil_{D(T^*)\cap D(T^{-1})}})^{-1}$. This completes the proof. $\square$

4. Physical Operators Constructed from $(\mathcal{D}, \mathcal{E})$-Quasi Bases

In this section, extending what was discussed recently for instance in Refs. [2,3,6], we investigate some physical operators constructed from $(\mathcal{D}, \mathcal{E})$-quasi bases. Let $(\mathcal{F}_\varphi, \mathcal{F}_\psi)$ be a $(\mathcal{D}, \mathcal{E})$-quasi basis. As shown in Theorem 3.2, F_φ is a generalized Riesz system with constructing pairs $(\mathcal{F}_e, (\overline{T_{\boldsymbol{e},\psi}\lceil_\mathcal{E}})^{-1})$ and $(\mathcal{F}_e, (T_{\boldsymbol{e},\varphi}\lceil_\mathcal{D})^*)$ for any ONB $\mathcal{F}_e = \{e_n\}$, such that $(\overline{T_{\boldsymbol{e},\psi}\lceil_\mathcal{E}})^{-1} \subseteq (T_{\boldsymbol{e},\varphi}\lceil_\mathcal{D})^*$, and $\{\psi_n\}$ is a generalized Riesz system with constructing pairs $(\mathcal{F}_e, (\overline{T_{\boldsymbol{e},\varphi}\lceil_\mathcal{D}})^{-1})$ and $(\mathcal{F}_e, (T_{\boldsymbol{e},\psi\lceil_\mathcal{D}})^*)$, such that $(\overline{T_{\boldsymbol{e},\varphi}\lceil_\mathcal{D}})^{-1} \subseteq (T_{\boldsymbol{e},\psi}\lceil_\mathcal{E})^*$. Here, we put, to keep the notation simple:

$$T = (\overline{T_{\boldsymbol{e},\psi}\lceil_\mathcal{E}})^{-1} \quad \text{or} \quad (T_{\boldsymbol{e},\varphi}\lceil_\mathcal{D})^*,$$
$$K = (\overline{T_{\boldsymbol{e},\varphi}\lceil_\mathcal{D}})^{-1} \quad \text{or} \quad (T_{\boldsymbol{e},\psi}\lceil_\mathcal{E})^*.$$

For a generalized Riesz system $\mathcal{F}_\varphi$ with constructing pair $(\mathcal{F}_e, T)$, we can define a non-self-adjoint Hamiltonian $H_\varphi^{\boldsymbol{\alpha}} := TH_{\boldsymbol{e}}^{\boldsymbol{\alpha}}T^{-1}$, a generalized lowering operator $A_\varphi^{\boldsymbol{\alpha}} := TA_{\boldsymbol{e}}^{\boldsymbol{\alpha}}T^{-1}$, and a generalized raising operator $B_\varphi^{\boldsymbol{\alpha}} := TB_{\boldsymbol{e}}^{\boldsymbol{\alpha}}T^{-1}$. Similarly, for a generalized Riesz system $\{\psi_n\}$ with a constructing pair $(\mathcal{F}_e, K)$, we define a non-self-adjoint Hamiltonian $H_\psi^{\boldsymbol{\alpha}} := KH_{\boldsymbol{e}}^{\boldsymbol{\alpha}}K^{-1}$, a generalized lowering operator $A_\psi^{\boldsymbol{\alpha}} := KA_{\boldsymbol{e}}^{\boldsymbol{\alpha}}K^{-1}$, and a generalized raising operator $B_\psi^{\boldsymbol{\alpha}} := KB_{\boldsymbol{e}}^{\boldsymbol{\alpha}}K^{-1}$. However, we do not know whether these operators are even densely defined or not. Suppose that D_φ is dense in $\mathcal{H}$. Then, since $H_\varphi^{\boldsymbol{\alpha}}\varphi_n = \alpha_n\varphi_n$, $A_\varphi^{\boldsymbol{\alpha}}\varphi_n = \alpha_n\varphi_{n-1}$ (0 if $n = 0$) and $B_\varphi^{\boldsymbol{\alpha}}\varphi_n = \alpha_{n+1}\varphi_{n+1}$, it is clear that $H_\varphi^{\boldsymbol{\alpha}}$, $A_\varphi^{\boldsymbol{\alpha}}$ and $B_\varphi^{\boldsymbol{\alpha}}$ are densely defined, but since D_ψ is not necessarily dense in $\mathcal{H}$, the operators $H_\psi^{\boldsymbol{\alpha}}$, $A_\psi^{\boldsymbol{\alpha}}$, and $B_\psi^{\boldsymbol{\alpha}}$ need not being densely defined. Therefore, we first investigate when $\mathcal{D}_\varphi$ or $\mathcal{D}_\psi$ are dense in $\mathcal{H}$ under the assumption that $(\mathcal{F}_\varphi, \mathcal{F}_\psi)$ is a $(\mathcal{D}, \mathcal{E})$-quasi basis.

Before going forth, we shortly discuss an example which is the leading model for the objects which we are dealing with and which allows an explicit computation of all involved operators.

Example 3. Let $H_0 = p^2 + x^2$ be the self-adjoint Hamiltonian introduced in Example 2 above, and let T be the following multiplication operator: $(Tf)(x) = (1 + x^2)f(x)$, for all functions $f(x) \in D(T) = \{g(x) \in \mathcal{L}^2(\mathbb{R}) : (1 + x^2)g(x) \in \mathcal{L}^2(\mathbb{R})\}$. T is an unbounded self-adjoint operator, invertible with bounded inverse T^{-1}.

As seen in (2), H_0 has the form $H_{\boldsymbol{e}}^{\boldsymbol{\alpha}}$ where $\boldsymbol{\alpha} = \{2n + 1, n \in \mathbb{N}\}$ and $\{e_n\}$ is the orthonormal basis constructed from the Hermite polynomials. To simplify notations, we will omit here explicit reference to $\boldsymbol{\alpha}$.

If we identify K with T^{-1}, straightforward computations show that:

$$H_\varphi = p^2 + V_\varphi(x) + \frac{4ix}{1 + x^2}\,p, \quad H_\psi = p^2 + V_\psi(x) - \frac{4ix}{1 + x^2}\,p,$$

where $V_\varphi(x) = x^2 + 2\frac{(1-3x^2)}{(1+x^2)^2}$ and $V_\psi(x) = x^2 - \frac{2}{1+x^2}$. Notice that, because of the relation between T and K, $H_\varphi = H_\psi^*$, even if this is not evident from our explicit formulas. From a physical point of view both H_φ and H_ψ can be seen as a modified version of the harmonic oscillator where an extra potential is added, going to zero as x^{-2}, and the manifestly non-self-adjoint terms $\pm\frac{4ix}{1+x^2}\,p$ appear. These Hamiltonians can be factorized as follows: $H_\varphi = 2B_\varphi A_\varphi + \mathbb{1}$ and $H_\psi = 2B_\psi A_\psi + \mathbb{1}$, where

$$A_\varphi = \frac{1}{\sqrt{2}}\left(x - \frac{2x}{1+x^2} + ip\right),\ B_\varphi = \frac{1}{\sqrt{2}}\left(x + \frac{2x}{1+x^2} - ip\right),$$

while

$$A_\psi = \frac{1}{\sqrt{2}}\left(x + \frac{2x}{1+x^2} + ip\right),\ B_\psi = \frac{1}{\sqrt{2}}\left(x - \frac{2x}{1+x^2} - ip\right).$$

All these operators formally collapse to $c = \frac{1}{\sqrt{2}}(x+ip)$ or to $c^\dagger = \frac{1}{\sqrt{2}}(x-ip)$ for large x. It is also interesting to observe that $B_\varphi = A_\psi^*$ and $A_\varphi = B_\psi^*$

The two vacua of A_φ and A_ψ, corresponding to the lower eigenvectors of H_φ and H_ψ respectively, can be easily obtained by solving the differential equations $A_\varphi\varphi_0(x) = 0$ and $A_\psi\psi_0(x) = 0$. The solutions we find in this way coincide with those we find introducing:

$$\varphi_n(x) = (Te_n)(x) = \frac{1}{\sqrt{2^n\, n!\, \pi^{1/2}}}(1+x^2)H_n(x)e^{-x^2/2},$$

and

$$\varphi_n(x) = (Ke_n)(x) = \frac{1}{\sqrt{2^n\, n!\, \pi^{1/2}}}\frac{H_n(x)}{1+x^2}\, e^{-x^2/2},$$

see Example 2. Incidentally, it is clear that $e_n(x) \in D(T)$. Of course, $e_n(x) \in D(K)$, since $D(K) = \mathcal{L}^2(\mathbb{R})$.

The last point we want to consider here concerns the density of $\mathcal{D}_\varphi$ and $\mathcal{D}_\psi$ in $\mathcal{L}^2(\mathbb{R})$. More concretely, we will check that $\mathcal{F}_\varphi$ is total in $D(T)$ and that $\mathcal{F}_\psi$ is total in $D(K) = \mathcal{L}^2(\mathbb{R})$. In fact, let $f(x) \in D(T)$ be such that $\langle f, \varphi_n\rangle = 0$ for all n. Hence, $0 = \langle f, \varphi_n\rangle = \langle Tf, e_n\rangle$, so that $Tf = 0$ and, since $Tf \in D(K)$, $f(x) = 0$ a.e. in $\mathbb{R}$. Similarly, we can prove that, if $g(x) \in \mathcal{L}^2(\mathbb{R})$ is such that $\langle g, \psi_n\rangle = 0$ for all n, then $g(x) = 0$ a.e. in $\mathbb{R}$.

We come now back to investigate more general situations.

Proposition 4.1. *Suppose that $(\mathcal{F}_\varphi, \mathcal{F}_\psi)$ is a $(\mathcal{D}, \mathcal{E})$-quasi basis. Then, we have the following statements.*

(1) $D_\varphi^\perp \subseteq D(\varphi)$, where $D_\varphi^\perp$ is an orthogonal complement of D_φ in $\mathcal{H}$.

(2) If $\mathcal{D} \cap D_\varphi^\perp$ is dense in $D_\varphi^\perp$, then D_φ is dense in $\mathcal{H}$.

Similar results hold for $\mathcal{F}_\psi$.

Proof. (1) For $x \in D_\varphi^\perp$, we have:

$$\langle T_{\varphi,\boldsymbol{e}} e_n, x\rangle = \langle \varphi_n, x\rangle = 0,$$

for any ONB $\mathcal{F}_{\boldsymbol{e}}$ in $\mathcal{H}$ and $n = 0, 1, \ldots$. Since $\mathcal{F}_e$ is a core for $\bar{T}_{\varphi,\boldsymbol{e}}$ by Lemma 2.2, we have $x \in D(T_{\varphi,\boldsymbol{e}}^*) = D(T_{\boldsymbol{e},\varphi}) = D(\varphi)$.

(2) For any $x \in D_\varphi^\perp$, there exists a sequence $\{x_n\} \subseteq \mathcal{D} \cap D_\varphi^\perp$, such that $\lim_{n\to\infty} x_n = x$. Since $(\mathcal{F}_\varphi, \mathcal{F}_\psi)$ is a $(\mathcal{D}, \mathcal{E})$-quasi basis, we have:

$$\begin{aligned}\langle x, y\rangle &= \lim_{n\to\infty}\langle x_n, y\rangle \\ &= \lim_{n\to\infty}\sum_{k=0}^{\infty}\langle x_n, \varphi_k\rangle\,\langle\psi_k, y\rangle = 0\end{aligned}$$

for all $y \in \mathcal{E}$. Hence, we have $x = 0$. Thus, D_φ is dense in $\mathcal{H}$. □

Proposition 4.2. *Let $(\mathcal{F}_\varphi, \mathcal{F}_\psi)$ be a biorthogonal pair, such that $D(\varphi)$ and $D(\psi)$ are dense in $\mathcal{H}$. Then, we have the following:*

(1) $(\mathcal{F}_\varphi, \mathcal{F}_\psi)$ is a $(D(\varphi), \mathcal{E})$-quasi basis for some dense subspace $\mathcal{E}$ in $\mathcal{H}$, such that $D_\varphi \subseteq \mathcal{E} \subseteq D(\psi)$ if and only if D_φ is dense in $\mathcal{H}$. If this is true, $(\mathcal{F}_\varphi, \mathcal{F}_\psi)$ is a $(D(\varphi), D_\varphi)$-quasi basis.

(2) $(\mathcal{F}_\varphi, \mathcal{F}_\psi)$ is a $(\mathcal{D}, D(\psi))$-quasi basis for some dense subspace $\mathcal{D}$ in $\mathcal{H}$, such that $D_\psi \subseteq \mathcal{D} \subseteq D(\varphi)$ if and only if D_ψ is dense in $\mathcal{H}$. If this is true, $(\mathcal{F}_\varphi, \mathcal{F}_\psi)$ is a $(D_\psi, D(\psi))$-quasi basis.

Proof. (1) Suppose that $(\mathcal{F}_\varphi, \mathcal{F}_\psi)$ is a $(D(\varphi), \mathcal{E})$-quasi basis for some dense subspace $\mathcal{E}$ in $\mathcal{H}$, such that $D_\varphi \subseteq \mathcal{E} \subseteq D(\psi)$. Take an arbitrary $x \in D_\varphi^\perp$. By Proposition 4.1, (1) we have $x \in D(\varphi)$. Since $(\{\varphi_n\}, \{\psi_n\})$ is a $(D(\varphi), \mathcal{E})$-quasi basis, we have:

$$\langle x, y\rangle = \sum_{k=0}^{\infty} \langle x, \varphi_k\rangle \langle \psi_k, y\rangle = 0$$

for all $y \in \mathcal{E}$, which implies that $x = 0$. Hence, D_φ is dense in $\mathcal{H}$. Conversely, suppose that $\mathcal{D}_\varphi$ is dense in $\mathcal{H}$. Then, we show that $(\mathcal{F}_\varphi, \mathcal{F}_\psi)$ is a $(D(\varphi), D_\varphi)$-quasi basis. Indeed, take arbitrary $x \in D(\varphi)$ and $y \in D_\varphi$. Then, $y = \sum_{j=0}^{n} \alpha_j \varphi_j$ for some $\alpha_j \in \mathbb{C}$, $j = 0, 1, \ldots, n$, and we have:

$$\begin{aligned}
\sum_{k=0}^{\infty} \langle x, \varphi_k\rangle \langle \psi_k, y\rangle &= \sum_{k=0}^{\infty} \langle x, T_{\varphi,\boldsymbol{e}} e_k\rangle \langle T_{\psi,\boldsymbol{e}} e_k, y\rangle \\
&= \langle T_{\boldsymbol{e},\varphi} x, T_{\boldsymbol{e},\psi} y\rangle \\
&= \sum_{j=0}^{n} \bar{\alpha}_j \langle T_{\boldsymbol{e},\varphi} x, T_{\boldsymbol{e},\psi} \varphi_j\rangle \\
&= \sum_{j=0}^{n} \bar{\alpha}_j \langle x, T_{\varphi,\boldsymbol{e}} e_j\rangle \\
&= \left\langle x, \sum_{j=0}^{n} \alpha_j \varphi_j \right\rangle \\
&= \langle x, y\rangle .
\end{aligned}$$

(2) This is shown similarly to (1). □

Suppose that $(\mathcal{F}_\varphi, \mathcal{F}_\psi)$ is a $(\mathcal{D}, \mathcal{E})$-quasi basis. Let $\boldsymbol{r} := \{r_n\} \subset \mathbb{R}$; $1 \le r_n$, $n = 0, 1, \ldots$ and we put:

$$\begin{aligned}
\varphi_r &:= \{r_n \varphi_n\}, \\
\psi_{\frac{1}{r}} &:= \left\{ \frac{1}{r_n} \psi_n \right\}.
\end{aligned}$$

Then, $(\varphi_r, \psi_{\frac{1}{r}})$ is a biorthogonal pair satisfying:

$$\begin{aligned} D_{\psi_r} &= D_\psi \subseteq D(\varphi_r) \subseteq D(\varphi), \\ D_{\varphi_r} &= D_\varphi \subseteq \mathcal{E} \subseteq D(\psi) \subseteq D(\psi_{\frac{1}{r}}), \end{aligned}$$

where

$$D(\varphi_r) := \left\{ x \in \mathcal{H}; \sum_{k=0}^{\infty} r_k^2 |\langle x, \varphi_k \rangle|^2 < \infty \right\} \text{ and}$$

$$D(\psi_{\frac{1}{r}}) := \left\{ x \in \mathcal{H}; \sum_{k=0}^{\infty} \frac{1}{r_k^2} |\langle x, \psi_k \rangle|^2 < \infty \right\}.$$

Then, we have the following:

Proposition 4.3. *Suppose that $(\mathcal{F}_\varphi, \mathcal{F}_\psi)$ is a $(\mathcal{D}, \mathcal{E})$-quasi basis and there exists a sequence $\boldsymbol{r} := \{r_n\} \subset \mathbb{R}$, such that $1 \leq r_n$, $n = 0, 1, \ldots$ and $D(\varphi_r) \subseteq \mathcal{D}$ and $D(\varphi_r)$ is dense in $\mathcal{H}$. Then, D_φ is dense in $\mathcal{H}$ and $(\mathcal{F}_\varphi, \mathcal{F}_\psi)$ is a $(D(\varphi), D_\varphi)$-quasi basis.*

Proof. Since $D(\varphi_r) \subseteq \mathcal{D}$, it follows that $(\varphi_r, \psi_{\frac{1}{r}})$ is a $(D(\varphi_r), \mathcal{E})$-quasi basis, which implies by Proposition 4.2 that $D_{\varphi_r} = D_\varphi$ is dense in $\mathcal{H}$. □

We next consider the case that D_φ and D_ψ are not necessarily dense in $\mathcal{H}$.

Proposition 4.4. *Suppose that $(\mathcal{F}_\varphi, \mathcal{F}_\psi)$ is a $(\mathcal{D}, \mathcal{E})$-quasi basis. Then, there exists an ONB $\mathcal{F}_f := \{f_n\}$ in $\mathcal{H}$, such that $\overline{T_{\boldsymbol{f},\varphi} \lceil_{\mathcal{D}}}$ is a positive self-adjoint operator in $\mathcal{H}$ and $(\mathcal{F}_{\boldsymbol{f}}, \overline{T_{\boldsymbol{f},\varphi} \lceil_{\mathcal{D}}})$ is a constructing pair for the generalized Riesz system $\mathcal{F}_\varphi$. Furthermore, $(\mathcal{F}_{\boldsymbol{f}}, (\overline{T_{\boldsymbol{f},\varphi} \lceil_{\mathcal{D}}})^{-1})$ is a constructing pair for the generalized Riesz system $\mathcal{F}_\psi$.*

Proof. By Theorem 3.2, $(\overline{T_{\boldsymbol{e},\varphi} \lceil_{\mathcal{D}}})^*$ is a constructing operator for the generalized Riesz system $\mathcal{F}_\varphi$ and any ONB $\mathcal{F}_e = \{e_n\}$ in $\mathcal{H}$. Let $\overline{T_{\boldsymbol{e},\varphi} \lceil_{\mathcal{D}}} = U |\overline{T_{\boldsymbol{e},\varphi} \lceil_{\mathcal{D}}}|$ be the polar decomposition of $\overline{T_{\boldsymbol{e},\varphi} \lceil_{\mathcal{D}}}$. Since $\overline{T_{\boldsymbol{e},\varphi} \lceil_{\mathcal{D}}}$ has a densely defined inverse, U is a unitary operator on $\mathcal{H}$. Here, we put $f_n = U^* e_n$, $n = 0, 1, \ldots$. Then, it follows that $\{f_n\}$ is an ONB in $\mathcal{H}$ and:

$$|\overline{T_{\boldsymbol{e},\varphi} \lceil_{\mathcal{D}}}| f_n = |\overline{T_{\boldsymbol{e},\varphi} \lceil_{\mathcal{D}}}| U^* e_n = (T_{\boldsymbol{e},\varphi} \lceil_{\mathcal{D}})^* e_n = \varphi_n, \quad n = 0, 1, \ldots,$$

which implies that $(\mathcal{F}_{\boldsymbol{f}}, |\overline{T_{\boldsymbol{e},\varphi} \lceil_{\mathcal{D}}}|)$ is a constructing pair for $\mathcal{F}_\varphi$. Hence:

$$T_{\varphi,\boldsymbol{f}} \subseteq |\overline{T_{\boldsymbol{e},\varphi} \lceil_{\mathcal{D}}}| \subseteq T_{\boldsymbol{f},\varphi},$$

and so $\overline{T_{\boldsymbol{f},\varphi} \lceil_{\mathcal{D}}} = |\overline{T_{\boldsymbol{e},\varphi} \lceil_{\mathcal{D}}}|$. This completes the proof. □

Similarly, we have the following.

Proposition 4.5. *Suppose that $(\mathcal{F}_\varphi, \mathcal{F}_\psi)$ is a $(\mathcal{D}, \mathcal{E})$-quasi basis. Then, there exists an ONB $\mathcal{F}_g := \{g_n\}$ in $\mathcal{H}$, such that $\overline{T_{\boldsymbol{g},\psi} \lceil_{\mathcal{E}}}$ is a positive self-adjoint operator in $\mathcal{H}$ and $(\mathcal{F}_{\boldsymbol{g}}, \overline{T_{\boldsymbol{g},\psi} \lceil_{\mathcal{E}}})$ is a constructing pair for the generalized Riesz system $\mathcal{F}_\psi$. Furthermore, $(\mathcal{F}_{\boldsymbol{g}}, (\overline{T_{\boldsymbol{g},\psi} \lceil_{\mathcal{E}}})^{-1})$ is a constructing pair for the generalized Riesz system $\mathcal{F}_\varphi$.*

We now consider a CCR-algebra-like structure for non-self-adjoint Hamiltonians, and generalized lowering and raising operators by taking a good domain for their operators. For that, the notion of unbounded operator algebras is relevant [5,10,11]. Let $\mathcal{D}$ be a dense subspace in a Hilbert space $\mathcal{H}$. We denote by $\mathcal{L}(\mathcal{D})$ the set of all linear operators from $\mathcal{D}$ to $\mathcal{D}$. Then, $\mathcal{L}(\mathcal{D})$ is an algebra equipped with the usual operations: $X+Y$, αX and XY.

Theorem 4.6. *Suppose that $(\mathcal{F}_\varphi, \mathcal{F}_\psi)$ is a $(\mathcal{D}, \mathcal{E})$-quasi basis, and $\mathcal{F}_{\boldsymbol{f}} = \{f_n\}$ and $\mathcal{F}_{\boldsymbol{g}} = \{g_n\}$ in Proposition 4.4 and Proposition 4.5. Here, we denote by T_φ the constructing operator $\overline{T_{\boldsymbol{f},\varphi}\lceil_{\mathcal{D}}}$ of $\mathcal{F}_\varphi$ and T_ψ the constructing operator $\overline{T_{\boldsymbol{g},\psi}\lceil_{\mathcal{E}}}$ of $\mathcal{F}_\psi$. Then, we have the following:*

(1) If $H_{\boldsymbol{f}}^{\boldsymbol{\alpha}}\mathcal{D} \subseteq \mathcal{D}$ for some $\boldsymbol{\alpha} = \{\alpha_n\} \subset \mathbb{C}$, then the linear span of $T_\varphi\mathcal{D}$ is dense in $\mathcal{H}$ and the non-self-adjoint Hamiltonian $T_\varphi H_{\boldsymbol{f}}^{\boldsymbol{\alpha}} T_\varphi^{-1}$ for $\mathcal{F}_\varphi$ is contained in $\mathcal{L}(T_\varphi\mathcal{D})$.

(2) If $H_{\boldsymbol{g}}^{\boldsymbol{\alpha}}\mathcal{E} \subseteq \mathcal{E}$ for some $\boldsymbol{\alpha} = \{\alpha_n\} \subset \mathbb{C}$, then the linear span of $T_\psi\mathcal{E}$ is dense in $\mathcal{H}$ and the non-self-adjoint Hamiltonian $T_\psi^{-1} H_{\boldsymbol{g}}^{\boldsymbol{\alpha}} T_\psi$ for $\mathcal{F}_\psi$ is contained in $\mathcal{L}(T_\psi\mathcal{E})$.

Here, $H_{\boldsymbol{f}}^{\boldsymbol{\alpha}}$ and $H_{\boldsymbol{g}}^{\boldsymbol{\alpha}}$ are the standard Hamiltonians for the ONB $\mathcal{F}_{\boldsymbol{f}}$ and $\mathcal{F}_{\boldsymbol{g}}$, respectively.

Proof. (1) Since $\mathcal{D}$ is a core for T_φ and T_φ has the inverse, $T_\varphi\mathcal{D}$ is dense in $\mathcal{H}$. By assumption, it is clear that $T_\varphi H_{\boldsymbol{f}}^{\boldsymbol{\alpha}} T_\varphi^{-1} \in \mathcal{L}(T_\varphi\mathcal{D})$.

(2) This is shown similarly to (1).

□

Next, to consider the generalized lowering and raising operators defined by $(\mathcal{D}, \mathcal{E})$-quasi bases, we assume that:

$$0 \le \alpha_0 < \alpha_n < \alpha_{n+1} \text{ and } \alpha_{n+1} \le \alpha_n + r,\ n = 1, \ldots, \text{ for some } r > 0. \quad (4)$$

Then, we have the following.

Theorem 4.7. *Suppose that $(\mathcal{F}_\varphi, \mathcal{F}_\psi)$ is a $(\mathcal{D}, \mathcal{E})$-quasi basis, and T_φ, T_ψ, $\mathcal{F}_{\boldsymbol{f}} = \{f_n\}$ and $\mathcal{F}_{\boldsymbol{g}} = \{g_n\}$ as in Theorem 4.6. Then. we have the following statements.*

(1) Suppose that $D^\infty(H_{\boldsymbol{f}}^{\boldsymbol{\alpha}}) := \cap_{n\in N} D((H_{\boldsymbol{f}}^{\boldsymbol{\alpha}})^n) \subseteq \mathcal{D}$ and $T_{\boldsymbol{f},\varphi} D^\infty(H_{\boldsymbol{f}}^{\boldsymbol{\alpha}})$ is dense in $\mathcal{H}$. Then, $(\mathcal{F}_{\boldsymbol{f}}, T_\varphi^0 := \overline{T_{\boldsymbol{f},\varphi}\lceil_{D^\infty(H_{\boldsymbol{f}}^{\alpha})}})$ is a constructing pair for $\mathcal{F}_\varphi$ and the non-self-adjoint Hamiltonian $H_\varphi^0 := T_\varphi^0 H_{\boldsymbol{f}}^{\boldsymbol{\alpha}} (T_\varphi^0)^{-1}$ for $\mathcal{F}_\varphi$, the generalized lowering operator $A_\varphi^0 := T_\varphi^0 A_{\boldsymbol{f}}^{\boldsymbol{\alpha}} (T_\varphi^0)^{-1}$ for $\mathcal{F}_\varphi$, and the generalized raising operator $B_\varphi^0 := T_\varphi^0 B_{\boldsymbol{f}}^{\boldsymbol{\alpha}} (T_\varphi^0)^{-1}$ for $\mathcal{F}_\varphi$ are contained in $\mathcal{L}(T_\varphi^0 D^\infty(H_{\boldsymbol{f}}^{\boldsymbol{\alpha}}))$.

(2) Suppose that $D^\infty(H_{\boldsymbol{g}}^{\boldsymbol{\alpha}}) \subseteq \mathcal{E}$ and $T_{\boldsymbol{g},\psi} D^\infty(H_{\boldsymbol{g}}^{\boldsymbol{\alpha}})$ is dense in $\mathcal{H}$. Then, $(\mathcal{F}_{\boldsymbol{g}}, T_\psi^0 := \overline{T_{\boldsymbol{g},\psi}\lceil_{D^\infty(H_{\boldsymbol{g}}^{\alpha})}})$ is a constructing pair for $\mathcal{F}_\psi$ and the non-self-adjoint Hamiltonian $H_\psi^0 := T_\psi^0 H_{\boldsymbol{g}}^{\boldsymbol{\alpha}} (T_\psi^0)^{-1}$ for $\mathcal{F}_\psi$, the generalized lowering operator $A_\psi^0 := T_\psi^0 A_{\boldsymbol{g}}^{\boldsymbol{\alpha}} (T_\psi^0)^{-1}$ for $\mathcal{F}_\psi$, and the generalized raising operator $B_\psi^0 := T_\psi^0 B_{\boldsymbol{g}}^{\boldsymbol{\alpha}} (T_\psi^0)^{-1}$ for $\mathcal{F}_\psi$ are contained in $\mathcal{L}(T_\psi^0 D^\infty(H_{\boldsymbol{g}}^{\boldsymbol{\alpha}}))$.

Proof. At first, we show that $(\mathcal{F}_{\boldsymbol{f}}, T_\varphi^0)$ is a constructing pair for $\mathcal{F}_\varphi$. Since $D(T_\varphi^0) \supseteq D^\infty(H_{\boldsymbol{f}}^{\boldsymbol{\alpha}}) \supseteq \mathcal{F}_{\boldsymbol{f}}$, T_φ^0 is a densely defined closed operator in $\mathcal{H}$. Furthermore, since $T_\varphi^0 \subseteq T_\varphi = \overline{T_{\boldsymbol{f},\varphi}\lceil_{\mathcal{D}}}$ and T_φ has the inverse, T_φ^0 has the inverse. By assumption, we have:

$$D((T_\varphi^0)^{-1}) \supseteq T_\varphi^0 D(T_\varphi^0) \supseteq T_\varphi^0 D^\infty(H_{\boldsymbol{f}}^{\boldsymbol{\alpha}}) = T_{\boldsymbol{f},\varphi} D^\infty(H_{\boldsymbol{f}}^{\boldsymbol{\alpha}}),$$

which implies that T_φ^0 has a densely defined inverse. Furthermore, we have the following:

$$T_\varphi^0 f_n = T_\varphi f_n = \varphi_n, \quad n = 0, 1, \ldots.$$

Hence, we have $(\mathcal{F}_\varphi, T_\varphi^0)$ is a constructing pair for $\mathcal{F}_\varphi$.

Next, we consider the non-self-adjoint Hamiltonian H_φ^0 for $\mathcal{F}_\varphi$, the generalized lowering operator A_φ^0 for $\mathcal{F}_\varphi$, and the generalized raising operator for B_φ^0 for $\mathcal{F}_\varphi$. Since we have:

$$\begin{aligned}
(H_{\boldsymbol{f}}^{\boldsymbol{\alpha}})^n x &= \sum_{k=0}^\infty \alpha_k^n \langle x, f_k\rangle f_k, \quad x \in D((H_{\boldsymbol{f}}^{\boldsymbol{\alpha}})^n),\\
(A_{\boldsymbol{f}}^{\boldsymbol{\alpha}})^n x &= \sum_{k=0}^\infty \alpha_{k+1}\alpha_{k+2}\cdots\alpha_{k+n} \langle x, f_{k+1}\rangle f_k, \quad x \in D((A_{\boldsymbol{f}}^{\boldsymbol{\alpha}})^n),\\
(B_{\boldsymbol{f}}^{\boldsymbol{\alpha}})^n x &= \sum_{k=0}^\infty \alpha_{k+1}\alpha_{k+2}\cdots\alpha_{k+n} \langle x, f_k\rangle f_{k+1}, \quad x \in D((B_{\boldsymbol{f}}^{\boldsymbol{\alpha}})^n),
\end{aligned}$$

it follows that:

$$\begin{aligned}
x \in D((H_{\boldsymbol{f}}^{\boldsymbol{\alpha}})^n) \quad &\text{iff} \quad \sum_{k=0}^\infty \alpha_k^{2n} |\langle x, f_k\rangle|^2 < \infty,\\
x \in D((B_{\boldsymbol{f}}^{\boldsymbol{\alpha}})^n) \quad &\text{iff} \quad \sum_{k=0}^\infty (\alpha_{k+1}\cdots\alpha_{k+n})^2 |\langle x, f_{k+1}\rangle|^2 < \infty,\\
x \in D((B_{\boldsymbol{f}}^{\boldsymbol{\alpha}})^n) \quad &\text{iff} \quad \sum_{k=0}^\infty (\alpha_{k+1}\cdots\alpha_{k+n})^2 |\langle x, f_k\rangle|^2 < \infty.
\end{aligned}$$

By (4), we have:

$$\begin{aligned}
\sum_{k=0}^\infty \alpha_{k+1}^{2n} |\langle x, f_{k+1}\rangle|^2 &\leq \sum_{k=0}^\infty (\alpha_{k+1}\cdots\alpha_{k+n})^2 |\langle x, f_{k+1}\rangle|^2\\
&\leq \sum_{k=0}^\infty (\alpha_k + (n-1)r)^{2n} |\langle x, f_k\rangle|^2,
\end{aligned}$$

and

$$\begin{aligned}
\sum_{k=0}^\infty \alpha_k^{2n} |\langle x, f_k\rangle|^2 &\leq \sum_{k=0}^\infty (\alpha_{k+1}\cdots\alpha_{k+n})^2 |\langle x, f_k\rangle|^2\\
&\leq \sum_{k=0}^\infty (\alpha_k + nr)^{2n} |\langle x, f_k\rangle|^2.
\end{aligned}$$

Hence, it follows that $x \in D((H_{\boldsymbol{f}}^{\boldsymbol{\alpha}})^n)$ iff $x \in D((A_{\boldsymbol{f}}^{\boldsymbol{\alpha}})^n)$ iff $x \in D((B_{\boldsymbol{f}}^{\boldsymbol{\alpha}})^n)$, which implies that $D^\infty(H_{\boldsymbol{f}}^{\boldsymbol{\alpha}}) = D^\infty(A_{\boldsymbol{f}}^{\boldsymbol{\alpha}}) = D^\infty(B_{\boldsymbol{f}}^{\boldsymbol{\alpha}})$. Furthermore, it is clear that H_φ^0, A_φ^0, $B_\varphi^0 \in \mathcal{L}(T_\varphi^0 D^\infty(H_{\boldsymbol{f}}^{\boldsymbol{\alpha}}))$. This completes the proof.

(2) This is shown similarly to (1). $\square$

Conclusions

This paper continues our (joint, and separate) analysis of biorthogonal sets of vectors of different nature, and their interest in quantum mechanics. In particular, we have shown that the extension of the notion of $\mathcal{D}$-quasi basis can be technically useful and may be of some interest in applications. However, more should be done, mainly on this aspect, and we plan to focus more on physics in a future paper.

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