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Existence and Regularity Results for Some Elliptic Equations with Degenerate Coercivity and Singular Quadratic Lower-Order Terms

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Abstract. In this paper, we study the existence and regularity results for some elliptic equations with degenerate coercivity and singular quadratic lower-order terms with natural growth with respect to the gradient. The model problem is

$$\begin{cases} -\mathrm{div}\left(\frac{\nabla u}{(1+|u|)^{\gamma}}\right) + \frac{|\nabla u|^2}{u^{\theta}} = f + u^r & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{0.1}$$

where Ω is a bounded open subset in $\mathbb{R}^N$, $0 < \theta < 1$, $\gamma > 0$ and $0 < r < 2-\theta$. We will prove existence results for solutions under various assumptions on the summability of the source f.

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1. Introduction

This paper will deal with the following problem

$$\begin{cases} -\mathrm{div}\,(M(x,u)\nabla u) + b(x)\frac{|\nabla u|^2}{u^{\theta}} = \lambda u^r + f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded open subset in $\mathbb{R}^N (N > 2)$, and $M : \Omega \times \mathbb{R} \to \mathbb{R}^{N^2}$ is symmetric Carathéodory matrix function satisfying for almost every $x \in \Omega$, for every $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$, and for some real number $\gamma > 0$

$$|M(x,s)| \leq \beta, \quad M(x,s)\xi \cdot \xi \geq \frac{\alpha}{(a(x)+|s|)^{\gamma}}|\xi|^2, \tag{1.2}$$

where $\alpha > 0$, $\beta > 0$ and $a(x)$ is measurable function verifying for some positive numbers ζ, ρ the condition

$$0 < \zeta \leq a(x) \leq \rho. \tag{1.3}$$

We furthermore suppose that

$$0 < \theta < 1, \quad 0 < r < 2 - \theta, \quad \lambda \geq 0, \tag{1.4}$$

$$f \geq 0, \quad f \not\equiv 0, \tag{1.5}$$

$$f \in L^m(\Omega), \quad m \geq 1, \tag{1.6}$$

and that $b(x)$ is measurable function satisfying for some positive numbers μ, ν the condition

$$0 < \mu \leq b(x) \leq \nu. \tag{1.7}$$

When the singular lower-order term does not appear in (1.1) (i.e., $b(x) \equiv 0$), and the nonlinear right-hand term is not present (i.e., $\lambda = 0$), the existence and regularity of solutions to problem (1.1) are proved in [9] under the hypothesis $M(x,s) = a(x,s)I_{N\times N}$, where $a : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying the following condition:

$$\frac{\alpha}{(1+|s|)^\gamma} \leq a(x,s) \leq \beta, \text{ with } 0 \leq \gamma \leq 1.$$

The extension of this work to nonlinear case is investigated in [5]. Other authors studied the regularizing effects of some lower-order terms, see, among others, [8,15,17]. If $\lambda = 1$, $0 \leq \gamma < 1$, and $0 \leq r < 1 - \gamma$, the problem (1.1), have been treated in [24], under the hypothesis

$$|M(x,t) - M(x,s)| \leq L(t-s), \quad \text{for a.e. } x \in \Omega \text{ and for every } s,t \in \mathbb{R},$$

where $L : \mathbb{R} \to \mathbb{R}$ is a non-decreasing function, such that $L(0) = 0$, and $\int_{0^+} \frac{dt}{L(t)} = +\infty$. Existence and regularity results for the problem (1.1) have been obtained in [16] provided $\lambda = 0$, and $M(x,s) = \frac{a(x)}{(1+|s|)^\gamma} I_{N\times N}$, where $a : \Omega \longrightarrow \mathbb{R}$ is a measurable function such that $\alpha \leq a(x) \leq \beta$ a.e. $x \in \Omega$, for some positive constants α and β. In the coercive case (i.e., $\gamma = 0$), the problem (1.1) is studied recently by many researchers under various assumptions on θ, λ, f, and the singular lower-order term. Starting from the classical reference [6], where the author considered the problem (1.1), under the conditions $\lambda = 0$, with a singular quadratic lower-order term has the form $\frac{Q(x,u)\nabla u \cdot \nabla u}{u^\theta}$, where $Q : \Omega \times \mathbb{R} \to \mathbb{R}^{N^2}$ is symmetric Carathéodory matrix function satisfying

$$a|\xi|^2 \leq Q(x,s)\xi\xi \leq b|\xi|^2, \text{ a.e. } x \in \Omega, \text{ for every}(s,\, \xi) \in \mathbb{R} \times \mathbb{R}^N. \tag{1.8}$$

In [1], the authors showed the existence of positive solutions for $\theta < 2$ and non-existence for $\theta \geq 2$. When $\lambda = 1$, and $M(x,s) = A(x)$, in [14], existence and regularity results for the problem (1.1) were proved. For a deeper insight on the subject of elliptic problems with singular quadratic lower-order terms, we refer the readers to [2–4,11,18–21,23,25] and references therein.

In the study of problem (1.1), there are two difficulties, the first one is the fact that, due to hypothesis (1.2), the differential operator $A(u) = -\text{div}\,(M(x,u)\nabla u)$ though well defined between $H_0^1(\Omega)$ and its dual $H^{-1}(\Omega)$, but it fails to be coercive on $H_0^1(\Omega)$ when u is unbounded. Due to the lack of coercivity, the classical theory for elliptic operators acting between spaces in duality (see [22]) can not be applied even if the data f are sufficiently regular (see [27]). The second difficulty comes from the lower-order term: the quadratic dependence with respect to the gradient and the singular dependence with respect to u. We overcome these difficulties by replacing operator A by another one defined by means of truncations, and approximating the singular term by nonsingular one in such a way that the corresponding approximated problems have finite energy solutions.

2. Statement of Main Results

The first result deals with a given f which yields unbounded solutions in energy space $H_0^1(\Omega)$.

Theorem 2.1. *Let us assume that* (1.2)–(1.5), *and* (1.7) *hold true and that* $f \in L^m(\Omega)$, *with*

$$\frac{2N}{2N-\theta(N-2)} \leq m < \frac{N}{2}. \tag{2.1}$$

Then, there exists at least a solution u *of* (1.1), *i.e., a function* $u \in H_0^1(\Omega) \cap L^{(2-\theta)m^{**}}(\Omega)$ *such that* $u > 0$ *in* Ω, $\frac{|\nabla u|^2}{u^\theta}$ *is in* $L^1(\Omega)$, *and*

$$\int_\Omega M(x,u)\nabla u \cdot \nabla \phi + \int_\Omega b(x)\frac{|\nabla u|^2}{u^\theta}\phi = \int_\Omega (\lambda u^r + f)\,\phi, \tag{2.2}$$

for every ϕ *in* $H_0^1(\Omega) \cap L^\infty(\Omega)$.

The next result considers the case where f has a high summability.

Theorem 2.2. *Suppose that assumptions* (1.2)–(1.5), *and* (1.7) *hold, and furthermore suppose that* $f \in L^m(\Omega)$, *with* $m \geq \frac{N}{2}$. *Then, there exists at least a solution* u *of* (1.1), *i.e., a function* $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ *such that* $u > 0$ *in* Ω, $\frac{|\nabla u|^2}{u^\theta}$ *is in* $L^1(\Omega)$, *and*

$$\int_\Omega M(x,u)\nabla u \cdot \nabla \phi + \int_\Omega b(x)\frac{|\nabla u|^2}{u^\theta}\phi = \int_\Omega (\lambda u^r + f)\,\phi, \tag{2.3}$$

for every ϕ *in* $H_0^1(\Omega) \cap L^\infty(\Omega)$.

The next result deals with the case when the summability of f gives the existence of an infinite energy solution, belonging to $u \in W_0^{1,q}(\Omega)$, with $1 < q < 2$.

Theorem 2.3. *Let us assume that* (1.2)–(1.5), *and* (1.7) *hold true and that* $f \in L^m(\Omega)$, *with*

$$1 < m < \frac{2N}{2N-\theta(N-2)}. \tag{2.4}$$

Then, there exists at least a solution u *of* (1.1), *verifying* $u \in W_0^{1,q}(\Omega) \cap L^{(2-\theta)m^{**}}(\Omega)$, *with* $q = \frac{Nm(2-\theta)}{N-m\theta}$, *in the sense that* $u > 0$ *in* Ω, $\frac{|\nabla u|^2}{u^\theta}$ *belongs to* $L^1(\Omega)$, *and*

$$\int_\Omega M(x,u)\nabla u \cdot \nabla \phi + \int_\Omega b(x)\frac{|\nabla u|^2}{u^\theta}\phi = \int_\Omega (\lambda u^r + f)\,\phi, \tag{2.5}$$

for every $\phi \in \mathcal{C}_0^1(\Omega)$.

The last result deals with the case where the source f belongs to $L^1(\Omega)$.

Theorem 2.4. *If hypotheses* (1.2)–(1.5)*, and* (1.7) *hold and* $f \in L^1(\Omega)$*, then there exists at least a solution* u *of* (1.1)*, satisfying* $u \in W_0^{1,\delta}(\Omega)$*, with* $\delta = \frac{N(2-\theta)}{N-\theta}$*, in the sense that* $u > 0$ *in* Ω*,* $\frac{|\nabla u|^2}{u^\theta}$ *belongs to* $L^1(\Omega)$*, and*

$$\int_\Omega M(x,u)\nabla u \cdot \nabla \phi + \int_\Omega b(x)\frac{|\nabla u|^2}{u^\theta}\phi = \int_\Omega (\lambda u^r + f)\,\phi, \tag{2.6}$$

for every $\phi \in \mathcal{C}_0^1(\Omega)$.

Remark 2.5. Notice that the results of previous theorems do not depend on γ and are similar to those obtained in the coercive case (i.e., $\gamma = 0$), see [14], while, in [9] under the hypotheses $\lambda = 0$, $0 \le \gamma < 1$ and $b(x) \equiv 0$ (i.e., the lower-order term does not exist), the authors proved that

1. if $f \in L^m(\Omega)$ with $\frac{2N}{N+2-\gamma(N-2)} \le m < \frac{N}{2}$, then the problem (1.1) admits a solution u belonging to $H_0^1(\Omega) \cap L^{m^{**}(1-\gamma)}(\Omega)$.
2. if $f \in L^m(\Omega)$ with $\frac{N}{N+1-\gamma(N-1)} < m < \frac{2N}{N+2-\gamma(N-2)}$, then the problem (1.1) admits a solution u belonging to $W_0^{1,q}(\Omega)$, with $q = \frac{Nm(1-\gamma)}{N-m(1+\gamma)} < 2$.
3. if $f \in L^m(\Omega)$ with $1 \le m \le \max\left[1, \frac{N}{N+1-\gamma(N-1)}\right]$, then the problem (1.1) admits only an entropy solution u beloging to Marcinkiewicz space $M^{m^{**}(1-\gamma)}(\Omega)$ with $|\nabla u| \in M^{\frac{Nm(1-\gamma)}{N-m(1+\gamma)}}(\Omega)$.

If we compare these results with those of previous theorems, we can easily see that the singular lower-order term improves the regularity of solutions of problem (1.1).

Remark 2.6. In the case where $\gamma = 0$, f belongs to $L^1(\Omega)$ and the lower-order term does not exist (i.e., $b(x) \equiv 0$), the solution u of problem (1.1) belongs only to $W_0^{1,s}(\Omega)$ for every $s < \frac{N}{N-1}$, see [10,26]. Once again, the lower-order term improves the regularity of solutions of problem (1.1), since $\frac{N}{N-1} < \frac{N(2-\theta)}{N-\theta}$ (due to the fact that $0 < \theta < 1$). In [24], under the conditions $b(x) \equiv 0$, $0 \le \gamma < 1$ and $0 \le r < 1-\gamma$, the authors proved only the existence of renormalized solutions for the problem (1.1).

To prove our main results, we will use a standard approximation procedure similarly to [6,13,14,16]. First, we approximate the problem (1.1) by a sequence of non-degenerate and non-singular quasilinear quadratic problems. Then, we prove both a priori estimates and convergence results on the sequence of approximating solutions. Next, by the strong maximum principle, we prove that the weak limit of the approximate solutions is strictly positive in Ω. In the end, we pass to the limit in the approximate problems.

3. The Approximated Problem

Hereafter, we denote by T_k the truncation function at the level $k > 0$, defined by $T_k(s) = \max\{-k, \min\{s,k\}\}$ for every $s \in \mathbb{R}$.

Let $0 < \varepsilon < 1$, we approximate the problem (1.1) by the following non-degenerate and non-singular problem

$$\begin{cases} -\mathrm{div}\left(M(x,T_{\frac{1}{\varepsilon}}(u_\varepsilon))\nabla u_\varepsilon\right) + b(x)\frac{u_\varepsilon|\nabla u_\varepsilon|^2}{(|u_\varepsilon|+\varepsilon)^{\theta+1}} = \frac{\lambda|u_\varepsilon|^r}{1+\varepsilon|u_\varepsilon|^r} + f_\varepsilon & \text{in } \Omega, \\ u_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

where $f_\varepsilon = T_{\frac{1}{\varepsilon}}(f)$. The problem (3.1) admits at least one solution $u_\varepsilon \in H_0^1(\Omega) \cap L^\infty(\Omega)$ by [13, Theorem 2]. Due to the fact that $f_\varepsilon \geq 0$ (since $f \geq 0$), and that the quadratic lower-order term has the same sign of the solution, it is easy to prove by taking u_ε^- as test function in the weak formulation of problem (3.1) that $u_\varepsilon \geq 0$. Therefore, u_ε solves

$$\begin{cases} -\mathrm{div}\left(M(x,T_{\frac{1}{\varepsilon}}(u_\varepsilon))\nabla u_\varepsilon\right) + b(x)\frac{u_\varepsilon|\nabla u_\varepsilon|^2}{(u_\varepsilon+\varepsilon)^{\theta+1}} = \frac{\lambda u_\varepsilon^r}{1+\varepsilon u_\varepsilon^r} + f_\varepsilon & \text{in } \Omega, \\ u_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.2}$$

in the sense that u_ε satisfies

$$\int_\Omega M(x,T_{\frac{1}{\varepsilon}}(u_\varepsilon))\nabla u_\varepsilon \cdot \nabla\phi + \int_\Omega b(x)\frac{u_\varepsilon|\nabla u_\varepsilon|^2}{(u_\varepsilon+\varepsilon)^{\theta+1}}\,\phi = \int_\Omega \frac{\lambda u_\varepsilon^r}{1+\varepsilon u_\varepsilon^r}\,\phi + \int_\Omega f_\varepsilon\,\phi,$$

for every ϕ in $H_0^1(\Omega) \cap L^\infty(\Omega)$.

4. A Priori Estimates

We are now going to prove some a priori estimates on the sequence of approximated solutions u_ε. The following lemma gives a control of the lower-order term.

Lemma 4.1. *Let u_ε be the solutions to problems (3.2). Then it results*

$$\int_\Omega b(x)\frac{u_\varepsilon|\nabla u_\varepsilon|^2}{(u_\varepsilon+\varepsilon)^{\theta+1}} \leq \lambda\int_\Omega u_\varepsilon^r + \int_\Omega f. \tag{4.1}$$

Proof. Following [11,14], for any fixed $h > 0$, let us consider $\frac{T_h(u_\varepsilon)}{h}$ as a test function in the approximated problem (3.2). Dropping the nonnegative first term, we obtain

$$\int_\Omega b(x)\frac{u_\varepsilon|\nabla u_\varepsilon|^2}{(u_\varepsilon+\varepsilon)^{\theta+1}}\frac{T_h(u_\varepsilon)}{h} \leq \lambda\int_\Omega u_\varepsilon^r\,\frac{T_h(u_\varepsilon)}{h} + \int_\Omega f_\varepsilon\frac{T_h(u_\varepsilon)}{h}. \tag{4.2}$$

Using the fact that $f_\varepsilon \leq f$ and $\frac{T_h(u_\varepsilon)}{h} \leq 1$, then

$$\int_\Omega b(x)\frac{u_\varepsilon|\nabla u_\varepsilon|^2}{(u_\varepsilon+\varepsilon)^{\theta+1}}\frac{T_h(u_\varepsilon)}{h} \leq \lambda\int_\Omega u_\varepsilon^r + \int_\Omega f. \tag{4.3}$$

Letting h tend to 0, we deduce (4.1) by Fatou's Lemma. □

In the sequel, we will need the following lemma

Lemma 4.2. *Let $\eta > 0$ and let $0 < \varepsilon < 1$; then there exists $C_0 > 0$ such that*

$$\frac{\alpha\eta(t+\varepsilon)^{\theta-1}}{(\rho+t)^{\gamma}} + \frac{\mu t}{t+\varepsilon} \geq C_0.$$

for every $t \geq 0$.

Proof. Clearly, if $t \geq \varepsilon$ we have $\frac{\mu t}{t+\varepsilon} \geq \frac{\mu}{2}$, while if $t < \varepsilon$ we have $\frac{\alpha\eta(t+\varepsilon)^{\theta-1}}{(\rho+t)^{\gamma}} \geq \frac{\alpha\eta}{(\rho+\varepsilon)^{\gamma}(2\varepsilon)^{1-\theta}} \geq \frac{\alpha\eta}{2^{1-\theta}(\rho+1)^{\gamma}}$, since $\varepsilon < 1$; therefore, the claim is proved. □

Lemma 4.3. *Assume that m satisfies (2.1), let f belongs to $L^m(\Omega)$, and let u_ε be a solution of (3.2). Then, the sequence u_ε is bounded in $H_0^1(\Omega) \cap L^{m^{**}(2-\theta)}(\Omega)$.*

Proof. Choosing now $\eta = \frac{N(m-1)(2-\theta)}{N-2m} = \frac{m^{**}(2-\theta)}{m'}$. Note that by (2.1) and (1.4), we have $\eta > 0$. Testing (3.2) with $(u_\varepsilon+\varepsilon)^{\eta} - \varepsilon^{\eta}$, we get

$$\eta \int_\Omega M(x, T_{\frac{1}{\varepsilon}}(u_\varepsilon))\nabla u_\varepsilon \cdot \nabla u_\varepsilon (u_\varepsilon+\varepsilon)^{\eta-1} + \int_\Omega b(x)\frac{u_\varepsilon(u_\varepsilon+\varepsilon)^{\eta}|\nabla u_\varepsilon|^2}{(u_\varepsilon+\varepsilon)^{\theta+1}}$$
$$= \varepsilon^{\eta}\int_\Omega b(x)\frac{u_\varepsilon|\nabla u_\varepsilon|^2}{(u_\varepsilon+\varepsilon)^{\theta+1}} + \int_\Omega \left[\frac{\lambda u_\varepsilon^r}{1+\varepsilon u_\varepsilon} + f_\varepsilon\right]\left[(u_\varepsilon+\varepsilon)^{\eta} - \varepsilon^{\eta}\right].$$

Using (1.2), (1.3), (1.7), and dropping the nonpositive term on the right-hand side, we get

$$\int_\Omega |\nabla u_\varepsilon|^2 (u_\varepsilon+\varepsilon)^{\eta-\theta}\left[\frac{\alpha\eta(u_\varepsilon+\varepsilon)^{\theta-1}}{(\rho+u_\varepsilon)^{\gamma}} + \frac{\mu u_\varepsilon}{u_\varepsilon+\varepsilon}\right]$$
$$\leq \varepsilon^{\eta}\int_\Omega b(x)\frac{u_\varepsilon|\nabla u_\varepsilon|^2}{(u_\varepsilon+\varepsilon)^{\theta+1}} + \int_\Omega (\lambda u_\varepsilon^r + f_\varepsilon)(u_\varepsilon+\varepsilon)^{\eta}.$$

Recalling Lemma 4.2, we have

$$C_0\int_\Omega |\nabla u_\varepsilon|^2(u_\varepsilon+\varepsilon)^{\eta-\theta} \leq \varepsilon^{\eta}\int_\Omega b(x)\frac{u_\varepsilon|\nabla u_\varepsilon|^2}{(u_\varepsilon+\varepsilon)^{\theta+1}} + \int_\Omega (\lambda u_\varepsilon^r + f_\varepsilon)(u_\varepsilon+\varepsilon)^{\eta}.$$

Using (4.1), we get

$$C_0\int_\Omega |\nabla u_\varepsilon|^2(u_\varepsilon+\varepsilon)^{\eta-\theta} \leq \varepsilon^{\eta}\int_\Omega (\lambda u_\varepsilon^r + f_\varepsilon) + \int_\Omega \lambda u_\varepsilon^r(u_\varepsilon+\varepsilon)^{\eta} + \int_\Omega f_\varepsilon(u_\varepsilon+\varepsilon)^{\eta}.$$

Using the fact that $u_\varepsilon^r \leq (u_\varepsilon+\varepsilon)^r$, $0 < \varepsilon^{\eta} < (u_\varepsilon+\varepsilon)^{\eta}$ (since $u_\varepsilon \geq 0$, $0 < \varepsilon < 1$, $r > 0$, and $\eta > 0$) and that $f_\varepsilon \leq f$, we obtain

$$C_0\int_\Omega |\nabla u_\varepsilon|^2(u_\varepsilon+\varepsilon)^{\eta-\theta} \leq 2\lambda\int_\Omega (u_\varepsilon+\varepsilon)^{\eta+r} + 2\int_\Omega f(u_\varepsilon+\varepsilon)^{\eta}. \tag{4.4}$$

Observe that the first term that appears in the left-hand side of the previous inequality can be rewritten as

$$C_1\int_\Omega \left|\nabla\left[(u_\varepsilon+\varepsilon)^{\frac{\eta-\theta+2}{2}} - \varepsilon^{\frac{\eta-\theta+2}{2}}\right]\right|^2, \tag{4.5}$$

(4.4) and (4.5) imply

$$C_1 \int_\Omega \left| \nabla \left[(u_\varepsilon + \varepsilon)^{\frac{\eta-\theta+2}{2}} - \varepsilon^{\frac{\eta-\theta+2}{2}} \right] \right|^2 \le 2\lambda \int_\Omega (u_\varepsilon + \varepsilon)^{\eta+r} + 2 \int_\Omega f (u_\varepsilon + \varepsilon)^\eta. \tag{4.6}$$

Using Sobolev's inequality (on the left-hand side), and Hölder's inequality (on the right-hand side), we obtain

$$\left(\int_\Omega \left| (u_\varepsilon + \varepsilon)^{\frac{\eta-\theta+2}{2}} - \varepsilon^{\frac{\eta-\theta+2}{2}} \right|^{2^*} \right)^{\frac{2}{2^*}} \le C_2 \int_\Omega (u_\varepsilon + \varepsilon)^{\eta+r} + C_3 \|f\|_{L^m(\Omega)} \left[\int_\Omega (u_\varepsilon + \varepsilon)^{\eta m'} \right]^{\frac{1}{m'}}.$$

Since $|(t+\varepsilon)^s - \varepsilon^s|^{2^*} \ge C_4 (t+\varepsilon)^{2^* s} - C_4$, for every $t \ge 0$ (and for suitable constant C_4 independent on ε) we then have

$$\left(\int_\Omega \left[C_4 (u_\varepsilon + \varepsilon)^{\frac{2^*(\eta-\theta+2)}{2}} - C_4 \right] \right)^{\frac{2}{2^*}} \le C_2 \int_\Omega (u_\varepsilon + \varepsilon)^{\eta+r} + C_3 \|f\|_{L^m(\Omega)} \left[\int_\Omega (u_\varepsilon + \varepsilon)^{\eta m'} \right]^{\frac{1}{m'}}. \tag{4.7}$$

Thanks to the choice of η, we have $\frac{2^*(\eta-\theta+2)}{2} = \eta m' = (2-\theta) m^{**}$. Since $2-\theta > r$, we have $1 < \frac{2^*}{2} = \frac{(2-\theta)m^{**}}{\eta+2-\theta} < \frac{(2-\theta)m^{**}}{\eta+r}$. Thus, using Hölder inequality in the first term of the right-hand side of (4.7), we have

$$\left(\int_\Omega \left[(u_\varepsilon + \varepsilon)^{(2-\theta)m^{**}} - 1 \right] \right)^{\frac{2}{2^*}} \le C_5 \left(\int_\Omega (u_\varepsilon + \varepsilon)^{(2-\theta)m^{**}} \right)^{\frac{\eta+r}{(2-\theta)m^{**}}} + C_6 \|f\|_{L^m(\Omega)} \left[\int_\Omega (u_\varepsilon + \varepsilon)^{(2-\theta)m^{**}} \right]^{\frac{1}{m'}}. \tag{4.8}$$

Now we point out that $\frac{2}{2^*} > \frac{1}{m'}$, since $m < \frac{N}{2}$, and that $\frac{2}{2^*} > \frac{\eta+r}{(2-\theta)m^{**}}$, since $2-\theta > r$. Therefore, from (4.8), it follows the boundedness of the sequence u_ε in $L^{(2-\theta)m^{**}}(\Omega)$, which implies that the right-hand side of (4.4) is bounded. Thus, from (4.4) and the fact that $\eta \ge \theta$ (since $m \ge \frac{2N}{2N-\theta(N-2)}$), it follows that

$$\int_{\{u_\varepsilon \ge 1\}} |\nabla u_\varepsilon|^2 \le \int_\Omega |\nabla u_\varepsilon|^2 (u_\varepsilon + \varepsilon)^{\eta-\theta} \le C. \tag{4.9}$$

On the other hand, the use of $T_1(u_\varepsilon)$ as test function in (3.2) yields

$$\int_\Omega M(x, T_{\frac{1}{\varepsilon}}(u_\varepsilon)) \nabla u_\varepsilon \cdot \nabla T_1(u_\varepsilon) + \int_\Omega b(x) \frac{u_\varepsilon |\nabla u_\varepsilon|^2}{(u_\varepsilon + \varepsilon)^{\theta+1}} T_1(u_\varepsilon) = \int_\Omega \frac{\lambda u_\varepsilon^r}{1 + \varepsilon u_\varepsilon^r} T_1(u_\varepsilon) + \int_\Omega f_\varepsilon T_1(u_\varepsilon).$$

Dropping the nonnegative lower-order term, using (1.2), (1.3), and the boundedness of the sequence u_ε in $L^{(2-\theta)m^{**}}(\Omega)$ (recall that $r < (2-\theta)m^{**}$), we obtain

$$\frac{\alpha}{(1+\rho)^\gamma}\int_{\{u_\varepsilon<1\}} |\nabla T_1(u_\varepsilon)|^2 \le \lambda \int_\Omega u_\varepsilon^r + \int_\Omega f \le C_4. \tag{4.10}$$

From (4.9) and (4.10), we deduce that the sequence u_ε is bounded in $H_0^1(\Omega)$. □

Lemma 4.4. *Assume that* $m \ge \frac{N}{2}$, *let* f *belongs to* $L^m(\Omega)$, *and let* u_ε *be a solution of problem* (3.2). *Then, the sequence* u_ε *is bounded in* $H_0^1(\Omega) \cap L^\infty(\Omega)$.

Proof. Since $2-\theta > 0$, then there exists $\varrho > 1$ such that

$$\frac{Nr}{2(2-\theta+r)} < \varrho < \frac{N}{2}. \tag{4.11}$$

Since f belongs also to $L^\varrho(\Omega)$, by Lemma 4.3 the sequence u_ε is bounded in $L^{(2-\theta)\varrho^{**}}(\Omega)$. From (4.11), we have $\frac{(2-\theta)\varrho^{**}}{r} > \frac{N}{2}$. Hence, the right-hand side of (3.2) is bounded in $L^s(\Omega)$, with $s > \frac{N}{2}$. Let $k > 0$, let us define for $t \ge 0$, the functions

$$G_k(t) = t - T_k(t), \quad H(t) = \int_0^t \frac{\mathrm{d}\tau}{(\rho+\tau)^\gamma}.$$

Note that the function H is well defined since $\rho > 0$. Taking $G_k(H(u_\varepsilon))$ as test function in (3.2), we get

$$\begin{aligned}&\int_{\{H(u_\varepsilon)>k\}} M(x, T_{\frac{1}{\varepsilon}}(u_\varepsilon))\nabla u_\varepsilon \cdot \nabla H(u_\varepsilon)\\ &\quad + \int_{\{H(u_\varepsilon)>k\}} b(x)\frac{u_\varepsilon|\nabla u_\varepsilon|^2}{(u_\varepsilon+\varepsilon)^{\theta+1}} G_k(H(u_\varepsilon))\\ &\quad = \int_\Omega \left[\frac{\lambda u_\varepsilon^r}{1+\varepsilon u_\varepsilon^r} + f_\varepsilon\right] G_k(H(u_\varepsilon)).\end{aligned}$$

Using (1.2), (1.3), the fact that $f_\varepsilon \le f$, and dropping the nonnegative lower-order term, we obtain

$$\alpha \int_\Omega |\nabla G_k(H(u_\varepsilon))|^2 \le \int_\Omega (\lambda u_\varepsilon^r + f) G_k(H(u_\varepsilon)). \tag{4.12}$$

Since the right-hand side of (4.12) is bounded in $L^s(\Omega)$, with $s > \frac{N}{2}$, the inequality (4.12) is exactly the starting point of Stampacchia's L^∞-regularity proof (see [28]), so that there exists a constant c_1 independent of ε such that $0 \le H(u_\varepsilon) \le c_1$. Therefore, the strict monotonicity of H implies the boundedness of the sequence u_ε in $L^\infty(\Omega)$. The estimate of the sequence u_ε in $H_0^1(\Omega)$ is now very easy. In fact, by taking u_ε as test function in (3.2), we get

$$\int_\Omega M(x,T_{\frac{1}{\varepsilon}}(u_\varepsilon))\nabla u_\varepsilon\cdot\nabla u_\varepsilon+\int_\Omega b(x)\frac{|u_\varepsilon|^2|\nabla u_\varepsilon|^2}{(u_\varepsilon+\varepsilon)^{\theta+1}}$$
$$=\int_\Omega\left(\frac{\lambda u_\varepsilon^r}{1+\varepsilon u_\varepsilon^r}+f_\varepsilon\right)u_\varepsilon.$$

Using (1.2), (1.3), the boundedness of the sequence u_ε in $L^\infty(\Omega)$, and dropping the nonnegative lower-order term, we obtain

$$\int_\Omega|\nabla u_\varepsilon|^2\le C\left(\|u_\varepsilon\|^r_{L^\infty(\Omega)}+\|f\|_{L^m(\Omega)}\right),$$

so that the sequence u_ε is bounded in $H_0^1(\Omega)$. □

Lemma 4.5. *Assume that m satisfies (2.4), let f belongs to $L^m(\Omega)$, and let u_ε be a solution of (3.2). Then, the sequence u_ε is bounded in $W_0^{1,q}(\Omega)\cap L^{m^{**}(2-\theta)}(\Omega)$, where $q=\frac{Nm(2-\theta)}{N-m\theta}$. Furthermore, the sequence $T_k(u_\varepsilon)$ is bounded in $H_0^1(\Omega)$ for every $k>0$.*

Proof. The proof is identical to the one of Lemma 4.3 up to the a priori estimate of u_ε in $L^{m^{**}(2-\theta)}(\Omega)$, since the assumption $m>1$ implies that $\eta>0$. From (4.4), and the fact that the sequence u_ε is bounded in $L^{m^{**}(2-\theta)}(\Omega)$, we obtain

$$\int_\Omega\frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon+\varepsilon)^{\theta-\eta}}\le C, \tag{4.13}$$

where C is a positive constant independent of ε. Thanks to (2.4) and the choice of η as in the proof of Lemma 4.3, it is easy to check that $\theta-\eta>0$, and that $1<q=\frac{Nm(2-\theta)}{N-m\theta}<2$. Therefore, by Hölder's inequality, we obtain

$$\begin{aligned}\int_\Omega|\nabla u_\varepsilon|^q&=\int_\Omega\frac{|\nabla u_\varepsilon|^q}{(u_\varepsilon+\varepsilon)^{\frac{q(\theta-\eta)}{2}}}(u_\varepsilon+\varepsilon)^{\frac{q(\theta-\eta)}{2}}\\&\le\left[\int_\Omega\frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon+\varepsilon)^{\theta-\eta}}\right]^{\frac{q}{2}}\left[\int_\Omega(u_\varepsilon+\varepsilon)^{\frac{q(\theta-\eta)}{2-q}}\right]^{\frac{2-q}{2}}\\&\le C_1\left[\int_\Omega(u_\varepsilon+\varepsilon)^{\frac{q(\theta-\eta)}{2-q}}\right]^{\frac{2-q}{2}}.\end{aligned} \tag{4.14}$$

Sobolev inequality on the left-hand side, we get

$$\left[\int_\Omega|u_\varepsilon|^{q^*}\right]^{\frac{q}{q^*}}\le C_2\left[\int_\Omega(u_\varepsilon+\varepsilon)^{\frac{q(\theta-\eta)}{2-q}}\right]^{\frac{2-q}{2}}. \tag{4.15}$$

The choice of q, implies that $q^*=\frac{q(\theta-\eta)}{2-q}$. Therefore, we have

$$\left[\int_\Omega|u_\varepsilon|^{q^*}\right]^{\frac{q}{q^*}}\le C_3\left[\int_\Omega(u_\varepsilon+\varepsilon)^{q^*}\right]^{\frac{\theta-\eta}{q^*}}+C_4. \tag{4.16}$$

Since $\theta-\eta<1<q$, then from (4.16), we deduce that the sequence u_ε is bounded in $L^{q^*}(\Omega)$. Going back to (4.14), this in turn implies that the

sequence u_ε is bounded in $W_0^{1,q}(\Omega)$. Moreover, taking $T_k(u_\varepsilon)$ as test function in (3.2) yields

$$\int_\Omega M(x, T_{\frac{1}{\varepsilon}}(u_\varepsilon))\nabla u_\varepsilon \cdot \nabla T_k(u_\varepsilon) + \int_\Omega b(x)\frac{u_\varepsilon|\nabla u_\varepsilon|^2}{(u_\varepsilon+\varepsilon)^{\theta+1}}\, T_k(u_\varepsilon) = \int_\Omega \frac{\lambda u_\varepsilon^r}{1+\varepsilon u_\varepsilon^r}\, T_k(u_\varepsilon) + \int_\Omega f_\varepsilon\, T_k(u_\varepsilon).$$

Using (1.2), (1.3), the boundedness of the sequence u_ε in $L^{(2-\theta)m^{**}}(\Omega)$ (recall that $r < (2-\theta)m^{**}$), $f_\varepsilon \le f$, and dropping the nonnegative lower-order term, we obtain

$$\frac{\alpha}{(\rho+k)^\gamma}\int_\Omega |\nabla T_k(u_\varepsilon)|^2 \le \int_\Omega [\lambda u_\varepsilon^r + f]\, T_k(u_\varepsilon) \le C,$$

so that the sequence $T_k(u_\varepsilon)$ is bounded in $H_0^1(\Omega)$ for every $k > 0$. □

Lemma 4.6. *Let f belongs to $L^1(\Omega)$, and let u_ε be a solution of (3.2). Then the sequence u_ε is bounded in $W_0^{1,\delta}(\Omega)$, where $\delta = \frac{N(2-\theta)}{N-\theta}$. Moreover, the sequence $T_k(u_\varepsilon)$ is bounded in $H_0^1(\Omega)$ for every $k > 0$.*

Proof. In this proof, C denotes a generic constant independent of ε, whose value might change from line to line. Going back to (4.1), and using (1.7), we have

$$\frac{\mu}{2^{\theta+1}}\int_{\{u_\varepsilon\ge 1\}}\frac{|\nabla u_\varepsilon|^2}{u_\varepsilon^\theta} \le \mu\int_{\{u_\varepsilon\ge 1\}} b(x)\frac{u_\varepsilon|\nabla u_\varepsilon|^2}{(u_\varepsilon+\varepsilon)^{\theta+1}} \le \lambda\|u_\varepsilon^r\|_{L^1(\Omega)} + \|f\|_{L^1(\Omega)}. \tag{4.17}$$

Let s any positive real number such that $1 < s < 2$. Using Hölder's inequality, we obtain

$$\int_\Omega |\nabla G_1(u_\varepsilon)|^s \le \int_{\{u_\varepsilon\ge 1\}}\frac{|\nabla G_1(u_\varepsilon)|^s}{u_\varepsilon^{\frac{\theta s}{2}}}u_\varepsilon^{\frac{\theta s}{2}} \le \left[\int_{\{u_\varepsilon\ge 1\}}\frac{|\nabla u_\varepsilon|^2}{u_\varepsilon^\theta}\right]^{\frac{s}{2}}\left[\int_{\{u_\varepsilon\ge 1\}} u_\varepsilon^{\frac{\theta s}{2-s}}\right]^{\frac{2-s}{2}}. \tag{4.18}$$

Setting

$$L = \lambda\|u_\varepsilon^r\|_{L^1(\Omega)} + \|f\|_{L^1(\Omega)}. \tag{4.19}$$

Choosing now $s = 2-\theta$, then we have $1 < s < 2$. Therefore, using (4.17)–(4.19), we get

$$\int_\Omega |\nabla G_1(u_\varepsilon)|^s \le CL^{\frac{s}{2}}\left[\int_{\{u_\varepsilon\ge 1\}} u_\varepsilon^s\right]^{\frac{\theta}{2}} \le CL^{\frac{s}{2}}\left(\int_{\{u_\varepsilon\ge 1\}} [G_1(u_\varepsilon)+1]^s\right)^{\frac{\theta}{2}} \le C\left[L^{\frac{s}{2}}\left(\int_\Omega G_1(u_\varepsilon)^s\right)^{\frac{\theta}{2}} + L^{\frac{s}{2}}\right]. \tag{4.20}$$

Using Poincaré's inequality on the left-hand side of (4.20), Young's inequality on the right-hand side, we obtain

$$\int_\Omega G_1(u_\varepsilon)^s \leq C\left[L + L^{\frac{s}{2}}\right]. \tag{4.21}$$

Using Minkowski's inequality, the fact that $|T_1(u_\varepsilon)| \leq 1$), and the convexity of the real function $t \mapsto t^s$ (since $s > 1$), we get

$$\int_\Omega u_\varepsilon^s \leq C\left[1 + \int_\Omega G_1(u_\varepsilon)^s\right]. \tag{4.22}$$

From (4.21) and (4.22), it follows that

$$\int_\Omega u_\varepsilon^s \leq C\left[L + L^{\frac{s}{2}} + 1\right]. \tag{4.23}$$

Since $r < s$ (by (1.4)), then, using Hölder's inequality, we get

$$\int_\Omega u_\varepsilon^r \leq C\left[\int_\Omega u_\varepsilon^s\right]^{\frac{r}{s}}. \tag{4.24}$$

From (4.19), (4.23), and (4.24), it follows that

$$L - \|f\|_{L^1(\Omega)} \leq C\left[L^{\frac{r}{s}} + L^{\frac{r}{2}} + 1\right] \tag{4.25}$$

Since $r < s < 2$, then we deduce from the last inequality that $L \leq C$. Therefore, by (4.19), the sequence u_ε^r is bounded in $L^1(\Omega)$. Choosing now $\delta = \frac{N(2-\theta)}{N-\theta}$. Since $0 < \theta < 1$, then we have $1 < \delta < 2$. Taking $s = \delta$ in (4.18) and using the boundedness of sequence u_ε^r in $L^1(\Omega)$, we obtain

$$\int_\Omega |\nabla G_1(u_\varepsilon)|^\delta \leq \int_{\{u_\varepsilon \geq 1\}} \frac{|\nabla G_1(u_\varepsilon)|^\delta}{u_\varepsilon^{\frac{\theta\delta}{2}}} u_\varepsilon^{\frac{\theta\delta}{2}} \leq C\left[\int_{\{u_\varepsilon \geq 1\}} u_\varepsilon^{\frac{\delta\theta}{2-\delta}}\right]^{\frac{2-\delta}{2}}. \tag{4.26}$$

The choice of δ implies that $\delta^* = \frac{\delta\theta}{2-\delta}$. By Sobolev's inequality on the first term of (4.26), we get

$$\left[\int_\Omega G_1(u_\varepsilon)^{\delta^*}\right]^{\frac{\delta}{\delta^*}} \leq C\left[\int_{\{u_\varepsilon \geq 1\}} u_\varepsilon^{\delta^*}\right]^{\frac{\theta}{\delta^*}} \leq C\left[\int_\Omega G_1(u_\varepsilon)^{\delta^*}\right]^{\frac{\theta}{\delta^*}} + C. \tag{4.27}$$

Since $\theta < 1 < \delta$, the inequality (4.27) implies that $G_1(u_\varepsilon)$, hence u_ε, is bounded in $L^{\delta^*}(\Omega)$. From (4.26), it follows the boundedness of $G_1(u_\varepsilon)$ in $W_0^{1,\delta}(\Omega)$. Using $T_1(u_\varepsilon)$ as test function in (3.2), we deduce that $T_1(u_\varepsilon)$ is bounded in $H_0^1(\Omega)$, hence in $W_0^{1,\delta}(\Omega)$. Since $u_\varepsilon = G_1(u_\varepsilon) + T_1(u_\varepsilon)$, then we deduce that u_ε is bounded in $W_0^{1,\delta}(\Omega)$. Moreover, testing (3.2) by $T_k(u_\varepsilon)$, it follows that $T_k(u_\varepsilon)$ is bounded in $H_0^1(\Omega)$ for every $k > 0$. □

5. Proof of Main Results

5.1. Proof of Theorem 2.1

By Lemma 4.3, the sequence of approximated solutions u_ε is bounded in $H_0^1(\Omega) \cap L^{m^{**}(2-\theta)}(\Omega)$. Therefore, there exists a function u belongs to $H_0^1(\Omega) \cap L^{m^{**}(2-\theta)}(\Omega)$ such that, up to subsequences, u_ε converges to u weakly in

$H_0^1(\Omega)$, and almost everywhere in Ω. Now, we are going to prove the almost everywhere convergence of ∇u_ε to ∇u.

Lemma 5.1. *The sequence* $\nabla u_\varepsilon(x)$ *converges a.e. to* $\nabla u(x)$.

Proof. The proof is in the spirit of [6, Lemma 2.3] and also [7, Lemma 2.6], we fix $h, k > 0$. Plugging $T_h(u_\varepsilon - T_k(u))$ as a test function in (3.2), and using the estimate (4.1), we get

$$\int_\Omega M(x, T_{\frac{1}{\varepsilon}}(u_\varepsilon))(\nabla u_\varepsilon - T_k(u)) \cdot \nabla T_h(u_\varepsilon - T_k(u)) \\ \leq - \int_\Omega M(x, T_{\frac{1}{\varepsilon}}(u_\varepsilon)) \nabla T_k(u) \cdot \nabla T_h(u_\varepsilon - T_k(u)) + 2h \int_\Omega (\lambda u_\varepsilon^r + f).$$

Using the fact that the sequence u_ε is bounded in $L^{(2-\theta)m^{**}}(\Omega)$ (recall that $r < m^{**}(2-\theta)$), we get

$$\int_\Omega M(x, T_{\frac{1}{\varepsilon}}(u_\varepsilon))(\nabla u_\varepsilon - T_k(u)) \cdot \nabla T_h(u_\varepsilon - T_k(u)) \\ \leq - \int_\Omega M(x, T_{\frac{1}{\varepsilon}}(u_\varepsilon)) \nabla T_k(u) \cdot \nabla T_h(u_\varepsilon - T_k(u)) + 2Ch,$$

where C is a positive constant depend only of λ, $\|f\|_{L^1(\Omega)}$ and $\|u_\varepsilon\|_{L^{(2-\theta)m^{**}}(\Omega)}$. Using hypothesis (1.2), we obtain

$$\int_{\{|u_\varepsilon - T_k(u)| \leq h\}} \frac{\alpha |\nabla T_h(u_\varepsilon - T_k(u))|^2}{(\rho + u_\varepsilon)^\gamma} \\ \leq - \int_\Omega M(x, T_{\frac{1}{\varepsilon}}(u_\varepsilon)) \nabla T_k(u) \cdot \nabla T_h(u_\varepsilon - T_k(u)) + 2Ch.$$

Since $u_\varepsilon \leq h + k$ on the set $\{|u_\varepsilon - T_k(u)| \leq h\}$, we get

$$\int_\Omega |\nabla T_h(u_\varepsilon - T_k(u))|^2 \\ \leq - \frac{(\rho + h + k)^\gamma}{\alpha} \int_\Omega M(x, T_{\frac{1}{\varepsilon}}(u_\varepsilon)) \nabla T_k(u) \cdot \nabla T_h(u_\varepsilon - T_k(u)) \\ + 2Ch \frac{(\rho + h + k)^\gamma}{\alpha}.$$

Thus it follows

$$\limsup_{\varepsilon \to 0} \int_\Omega |\nabla T_h(u_\varepsilon - T_k(u))|^2 \leq 2Ch \frac{(\rho + h + k)^\gamma}{\alpha}.$$

Now, we fix s such that $1 < s < 2$. Then, we have

$$\int_\Omega |\nabla(u_\varepsilon - u)|^s = \int_{\{|u_\varepsilon - u| \leq h,\ |u| \leq k\}} |\nabla(u_\varepsilon - u)|^s \\ + \int_{\{|u_\varepsilon - u| \leq h,\ |u| > k\}} |\nabla(u_\varepsilon - u)|^s \\ + \int_{\{|u_\varepsilon - u| > h\}} |\nabla(u_\varepsilon - u)|^s \tag{5.1}$$

Since the sequence $u_\varepsilon - u$ is bounded in $W_0^{1,s}(\Omega)$ (since $s < 2$), then using Hölder's inequality with exponent $\frac{2}{s}$ on the two last terms of right-hand side of (5.1), we obtain

$$\int_\Omega |\nabla(u_\varepsilon - u)|^s \le \int_\Omega |\nabla T_h(u_\varepsilon - T_k(u))|^2 + 2^s R^s \text{meas}\{|u| > k\}^{1-\frac{s}{2}} + 2^s R^s \text{meas}\{|u_\varepsilon - u| > h\}^{1-\frac{s}{2}},$$

where R is a positive constant such that $\|u_\varepsilon\|_{H_0^1(\Omega)} \le R$. Thus, for every $h > 0$,

$$\limsup_{\varepsilon\to 0} \int_\Omega |\nabla(u_\varepsilon - u)|^s \le 2Ch^{\frac{(\rho+h+k)^\gamma}{\alpha}} + C_1 \text{meas}\{|u| > k\}^{1-\frac{s}{2}}.$$

That is, letting $h \to 0$ and then $k \to +\infty$,

$$\int_\Omega |\nabla(u_\varepsilon - u)|^s \to 0, \quad \text{for all } s < 2.$$

In consequence, we conclude (up to a subsequence) that $\nabla u_\varepsilon(x)$ converges almost everywhere to $\nabla u(x)$. □

Now, we are going to prove the strict positivity of the weak limit u of the sequence of approximated solutions u_ε.

Lemma 5.2. *Let u the weak limit of the sequence of approximated solutions u_ε. Then,*

$$u > 0 \quad in\, \Omega.$$

Proof. Following the ideas in [11, Lemma 2.3]. We define, for $t \ge 0$,

$$H_\varepsilon(t) = \int_0^t \frac{(\rho+\tau)^\gamma}{(\tau+\varepsilon)^\theta}\, \mathrm{d}\tau, \qquad H_0(t) = \int_0^t \frac{(\rho+\tau)^\gamma}{\tau^\theta}\, \mathrm{d}\tau, \tag{5.2}$$

and

$$\Phi_\varepsilon(t) = e^{-\nu\frac{H_\varepsilon(t)}{\alpha}}, \qquad \Phi_0(t) = e^{-\nu\frac{H_0(t)}{\alpha}}. \tag{5.3}$$

Note that the function H_0 is well defined since $\theta < 1$. Let v be fixed in $H_0^1(\Omega) \cap L^\infty(\Omega)$, with $v \ge 0$, and taking $v\,\Phi_\varepsilon(u_\varepsilon)$ as test function in (3.2) (which is admissible since it belongs to $H_0^1(\Omega) \cap L^\infty(\Omega)$), and using (1.2), (1.3), (1.7), and the fact that

$$\Phi'_\varepsilon(t) = \frac{-\nu}{\alpha}\frac{(\rho+t)^\gamma}{(t+\varepsilon)^\theta}\Phi_\varepsilon(t),$$

we obtain

$$\begin{aligned}
&\int_\Omega M(x, T_{\frac{1}{\varepsilon}}(u_\varepsilon))\nabla u_\varepsilon \cdot \nabla v\, \Phi_\varepsilon(u_\varepsilon) - \int_\Omega \left[\frac{\lambda u_\varepsilon^r}{1+\varepsilon u_\varepsilon^r} + f_\varepsilon\right]\Phi_\varepsilon(u_\varepsilon)v \\
&\ge \nu \int_\Omega \frac{(\rho+u_\varepsilon)^\gamma}{(u_\varepsilon+\varepsilon)^\theta(\rho+T_{\frac{1}{\varepsilon}}(u_\varepsilon))^\gamma}|\nabla u_\varepsilon|^2\Phi_\varepsilon(u_\varepsilon)v - \nu\int_\Omega \frac{u_\varepsilon|\nabla u_\varepsilon|^2}{(u_\varepsilon+\varepsilon)^{\theta+1}}\Phi_\varepsilon(u_\varepsilon)v \\
&\ge \nu\varepsilon \int_\Omega \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon+\varepsilon)^{\theta+1}}\, \Phi_\varepsilon(u_\varepsilon)v \\
&\ge 0.
\end{aligned}$$

Since $u_\varepsilon \geq 0$ and $f_\varepsilon \geq T_1(f)$ (being $\varepsilon < 1$), we have

$$\int_\Omega M(x, T_{\frac{1}{\varepsilon}}(u_\varepsilon))\nabla u_\varepsilon \cdot \nabla v\, \Phi_\varepsilon(u_\varepsilon) \geq \int_\Omega T_1(f)\Phi_\varepsilon(u_\varepsilon)v, \tag{5.4}$$

for all v in $H_0^1(\Omega) \cap L^\infty(\Omega)$, with $v \geq 0$.

Taking into account (1.5) and the fact that $u_\varepsilon \geq 0$, we can assure that for some $h \geq 1$, we have that $f \not\equiv 0$ in $\{0 \leq u \leq h\}$. We assume without loss of generality that $h = 1$. Now, let us define for $\sigma > 0$, the function

$$\psi_\sigma(t) = \begin{cases} 1 & \text{if } 0 \leq t < 1, \\ -\frac{1}{\sigma}(t - 1 - \sigma) & \text{if } 1 \leq t < \sigma + 1, \\ 0 & \text{if } \sigma + 1 \leq t, \end{cases} \tag{5.5}$$

and fix a function φ in $H_0^1(\Omega) \cap L^\infty(\Omega)$, with $\varphi \geq 0$. Taking $v = \psi_\sigma(u_\varepsilon)\varphi$ in (5.4) and using (1.2), we obtain

$$\begin{aligned} &\int_\Omega M(x, T_{\frac{1}{\varepsilon}}(u_\varepsilon))\nabla u_\varepsilon \cdot \nabla\varphi\, \psi_\sigma(u_\varepsilon)\Phi_\varepsilon(u_\varepsilon) \\ &\quad \geq \int_\Omega T_1(f)\Phi_\varepsilon(u_\varepsilon)\psi_\sigma(u_\varepsilon)\varphi + \frac{\alpha}{\sigma}\int_{\{1 \leq u_\varepsilon < \sigma+1\}} \Phi_\varepsilon(u_\varepsilon)\frac{|\nabla u_\varepsilon|^2}{(\rho + u_\varepsilon)^\gamma}\,\varphi, \end{aligned}$$

and thus, dropping the nonnegative term,

$$\int_\Omega M(x, T_{\frac{1}{\varepsilon}}(u_\varepsilon))\nabla u_\varepsilon \cdot \nabla\varphi\, \psi_\sigma(u_\varepsilon)\Phi_\varepsilon(u_\varepsilon) \geq \int_\Omega T_1(f)\Phi_\varepsilon(u_\varepsilon)\psi_\sigma(u_\varepsilon)\varphi.$$

Then, letting σ tend to 0, and using the fact that $T_{\frac{1}{\varepsilon}}(T_1(u_\varepsilon)) = T_1(u_\varepsilon)$ (since $\varepsilon < 1$), we get

$$\int_\Omega M(x, T_1(u_\varepsilon))\nabla T_1(u_\varepsilon) \cdot \nabla\varphi\, \Phi_\varepsilon(T_1(u_\varepsilon)) \geq \int_{\{0 \leq u_\varepsilon \leq 1\}} T_1(f)\Phi_\varepsilon(T_1(u_\varepsilon))\varphi. \tag{5.6}$$

Since the sequence $M(x, T_1(u_\varepsilon))\nabla T_1(u_\varepsilon)$, up to subsequences, converges almost everywhere to $M(x, T_1(u))\nabla T_1(u)$ in Ω , and it is bounded in $(L^2(\Omega))^N$ (by (4.10) and the boundedness of the matrix M), then using the Vitali's theorem we can conclude that $M(x, T_1(u_\varepsilon))\nabla T_1(u_\varepsilon)$ converges weakly in $(L^2(\Omega))^N$ to $M(x, T_1(u))\nabla T_1(u)$. Letting ε tend to the zero in (5.6), we obtain

$$\int_\Omega M(x, T_1(u))\nabla T_1(u) \cdot \nabla\varphi\, \Phi_0(T_1(u)) \geq \int_{\{0 \leq u \leq 1\}} T_1(f)\Phi_0(T_1(u))\varphi, \tag{5.7}$$

for all φ in $H_0^1(\Omega) \cap L^\infty(\Omega)$, with $\varphi \geq 0$, and then, by density, for every nonnegative φ in $H_0^1(\Omega)$. Now, we define the function

$$P(t) = \int_0^t \Phi_0(\tau)\,\mathrm{d}\tau.$$

If we set $w = P(T_1(u))$, we have that w belongs to $H_0^1(\Omega)$; furthermore, since

$$\Phi_0(T_1(u)) \geq \Phi_0(1) = e^{\frac{-\nu}{\alpha} H_0(1)} > 0,$$

we deduce from (5.7) that

$$\int_\Omega \tilde{M}(x, \nabla w) \cdot \nabla \varphi \geq \int_\Omega g(x)\, \varphi,$$

where we have set

$$\tilde{M}(x,\xi) = M(x, T_1(u(x))\xi, \text{ and } g(x) = T_1(f)\, e^{\frac{-\nu}{\alpha} H_0(1)} \chi_{\{0 \leq u(x) \leq 1\}}. \tag{5.8}$$

The comparison principle in $H_0^1(\Omega)$ says that $w(x) \geq z(x)$, where z is the bounded weak solution of

$$\begin{cases} z \in H_0^1(\Omega), \\ -\text{div}\left(\tilde{M}(x, \nabla z)\right) = g(x). \end{cases}$$

Using (1.2), it is easy to verify that the vector-valued function $\tilde{M}$ satisfies for almost every $x \in \Omega$, for every ξ, $\xi' \in \mathbb{R}^N$, with $\xi \neq \xi'$

$$\begin{aligned} &\tilde{M}(x,\xi)\xi \geq \frac{\alpha}{(\rho+1)^\gamma} |\xi|^2, \\ &|\tilde{M}(x,\xi)| \leq \beta |\xi|, \\ &\left[\tilde{M}(x,\xi) - \tilde{M}(x,\xi')\right] \cdot [\xi - \xi'] > \frac{\alpha}{(\rho+1)^\gamma} |\xi - \xi'|^2. \end{aligned}$$

Since g is nonnegative and not identically zero, the weak Harnack inequality [29, Theorem 1.2] yields $z > 0$ in Ω and so $w > 0$. Since $T_1(u) \geq w$ (due to the fact that $\Phi_0(t) \leq 1$), we conclude that $T_1(u) > 0$ in Ω, which then implies that $u > 0$ in Ω, since $u \geq T_1(u)$. □

In the sequel, we need the following corollary.

Corollary 5.3. *Let u the weak limit of the sequence of approximated solutions u_ε. Then, $\frac{|\nabla u|^2}{u^\theta}$ is in $L^1(\Omega)$.*

Proof. Thanks to (4.1), and (1.7), we have □

$$\mu \int_\Omega \frac{u_\varepsilon |\nabla u_\varepsilon|^2}{(u_\varepsilon + \varepsilon)^{\theta+1}} \leq \lambda \int_\Omega u_\varepsilon^r + \int_\Omega f. \tag{5.9}$$

Using Fatou's lemma as well as the weak convergence of u_ε to u in $H_0^1(\Omega)$, and the strict positivity of u, we obtain

$$\mu \int_\Omega \frac{|\nabla u|^2}{u^\theta} \leq \lambda \int_\Omega u^r + \int_\Omega f \leq C. \tag{5.10}$$

Hence, the Corollary is proved. To complete the proof of the Theorem 2.1, it remains to prove that u is a weak solution of the problem (1.1). This is the aim of the following lemma.

Lemma 5.4. *Let u be the weak limit of the sequence u_ε. Then u satisfies*

$$\int_\Omega M(x,u)\nabla u\cdot\nabla\varphi+\int_\Omega \frac{b(x)|\nabla u|^2}{u^\theta}\phi=\int_\Omega(\lambda u^r+f)\,\phi, \tag{5.11}$$

for every ϕ in $H_0^1(\Omega)\cap L^\infty(\Omega)$.

Proof. The proof of this lemma is based on the particular choice of test functions and the use of Fatou's lemma. We proceed as in [14, Theorem 2.6]. For every $k>0$, let us define

$$R_k(s)=\begin{cases}1 & \text{if } s\le k,\\ k+1-s & \text{if } k<s\le k+1,\\ 0 & \text{if } s>k+1.\end{cases} \tag{5.12}$$

Let $\phi\in H_0^1(\Omega)\cap L^\infty(\Omega)$, with $\phi\ge 0$, and consider the function

$$v_\varepsilon=e^{\frac{-\nu H_\varepsilon(u_\varepsilon)}{\alpha}}\,e^{\frac{\nu H_{1/j}(T_j(u))}{\alpha}}R_k(u_\varepsilon)\phi. \tag{5.13}$$

The function v_ε belongs also to $H_0^1(\Omega)\cap L^\infty(\Omega)$, so it is a legitimate test function for (3.2), and upon using it, we obtain

$$\begin{aligned}&\int_\Omega M(x,T_{\frac{1}{\varepsilon}}(u_\varepsilon))\nabla u_\varepsilon\cdot\nabla\phi\; e^{\frac{-\nu H_\varepsilon(u_\varepsilon)}{\alpha}}\,e^{\frac{\nu H_{1/j}(T_j(u))}{\alpha}}R_k(u_\varepsilon)\\ &\quad+\frac{\nu}{\alpha}\int_\Omega \frac{M(x,T_{\frac{1}{\varepsilon}}(u_\varepsilon))\nabla u_\varepsilon\cdot\nabla T_j(u)}{(T_j(u)+1/j)^\theta(\rho+T_j(u))^{-\gamma}}\,e^{\frac{-\nu H_\varepsilon(u_\varepsilon)}{\alpha}}\,e^{\frac{\nu H_{1/j}(T_j(u))}{\alpha}}R_k(u_\varepsilon)\phi\\ &=\int_\Omega\left[\frac{\lambda u_\varepsilon^r}{1+\varepsilon u_\varepsilon^r}+f_\varepsilon\right]e^{\frac{-\nu H_\varepsilon(u_\varepsilon)}{\alpha}}\,e^{\frac{\nu H_{1/j}(T_j(u))}{\alpha}}R_k(u_\varepsilon)\phi\\ &\quad+\int_\Omega\left[\frac{\nu}{\alpha}\frac{M(x,T_{\frac{1}{\varepsilon}}(u_\varepsilon))\nabla u_\varepsilon\cdot\nabla u_\varepsilon}{(u_\varepsilon+\varepsilon)^\theta(\rho+u_\varepsilon)^{-\gamma}}\right.\\ &\quad\left.-b(x)\frac{u_\varepsilon|\nabla u_\varepsilon|^2}{(u_\varepsilon+\varepsilon)^{\theta+1}}\right]e^{\frac{-\nu H_\varepsilon(u_\varepsilon)}{\alpha}}\,e^{\frac{\nu H_{1/j}(T_j(u))}{\alpha}}R_k(u_\varepsilon)\phi\\ &\quad+\int_{\{k<u_\varepsilon<k+1\}}M(x,T_{\frac{1}{\varepsilon}}(u_\varepsilon))\nabla u_\varepsilon\cdot\nabla u_\varepsilon\; e^{\frac{-\nu H_\varepsilon(u_\varepsilon)}{\alpha}}\,e^{\frac{\nu H_{1/j}(T_j(u))}{\alpha}}\phi.\end{aligned} \tag{5.14}$$

Note that by (1.2), (1.3), and (1.7), the function in the second integral of the right-hand side is nonnegative. Dropping the last term (which is nonnegative), and using Fatou's lemma as well the weak convergence of u_ε to u in $H_0^1(\Omega)$ in the right-hand side, and the weak convergence of $M(x,T_{\frac{1}{\varepsilon}}(u_\varepsilon))\nabla u_\varepsilon$ to $M(x,u)\nabla u$ in $(L^2(\Omega))^N$ (recall that the matrix M is bounded) in the left-hand side, we can pass to limit as ε tends to 0 in (5.14) to get

$$\begin{aligned}&\int_\Omega M(x,u)\nabla u\cdot\nabla\phi\, e^{\frac{-\nu H_0(u)}{\alpha}}\,e^{\frac{\nu H_{1/j}(T_j(u))}{\alpha}}R_k(u)\\ &\quad+\frac{\nu}{\alpha}\int_\Omega\frac{M(x,u)\nabla u\cdot\nabla T_j(u)}{(T_j(u)+1/j)^\theta(\rho+T_j(u))^{-\gamma}}\,e^{\frac{-\nu H_0(u)}{\alpha}}\,e^{\frac{\nu H_{1/j}(T_j(u))}{\alpha}}R_k(u)\phi\end{aligned}$$

$$\geq \int_\Omega (\lambda u^r + f)\, e^{\frac{-\nu H_0(u)}{\alpha}}\, e^{\frac{\nu H_{1/j}(T_j(u))}{\alpha}} R_k(u)\phi$$
$$+ \int_\Omega \left[\frac{\nu}{\alpha}\, \frac{M(x,u)\nabla u\cdot\nabla u}{u^\theta(\rho+u)^{-\gamma}} - b(x)\frac{|\nabla u|^2}{u^\theta}\right] e^{\frac{-\nu H_0(u)}{\alpha}}\, e^{\frac{\nu H_{1/j}(T_j(u))}{\alpha}} R_k(u)\phi. \tag{5.15}$$

Using (1.7), (5.10), the fact that $e^{\frac{-\nu H_0(u)}{\alpha}}\, e^{\frac{\nu H_{1/j}(T_j(u))}{\alpha}} \leq 1$ (since $H_{1/j}(T_j(u)) \leq H_{1/j}(u) \leq H_0(u)$) and $R_k(u) = 0$ if $u > k+1$, so by Lebesgue's convergence theorem, we can pass to the limit in (5.15) as j tends to infinity to obtain

$$\int_\Omega M(x,u)\nabla u\cdot\nabla\phi\, R_k(u) + \frac{\nu}{\alpha}\int_\Omega \frac{M(x,u)\nabla u\cdot\nabla u}{u^\theta(\rho+u)^{-\gamma}}\, R_k(u)\phi$$
$$\geq \int_\Omega (\lambda u^r + f) R_k(u)\phi + \int_\Omega \left[\frac{\nu}{\alpha}\, \frac{M(x,u)\nabla u\cdot\nabla u}{u^\theta(\rho+u)^{-\gamma}} - b(x)\frac{|\nabla u|^2}{u^\theta}\right] R_k(u)\phi. \tag{5.16}$$

Then, since $\frac{M(x,u)\nabla u\cdot\nabla u}{u^\theta(\rho+u)^{-\gamma}}\, R_k(u)$ belongs to $L^1(\Omega)$ (by (1.2), (5.10), and the fact that $R_k(u) = 0$, when $u > k+1$), we have

$$\int_\Omega M(x,u)\nabla u\cdot\nabla\phi\, R_k(u) + \int_\Omega b(x)\frac{|\nabla u|^2}{u^\theta}\, R_k(u)\phi \geq \int_\Omega (\lambda u^r + f) R_k(u)\phi. \tag{5.17}$$

Letting k tend to infinity (observing that $R_k(u)$ tends to 1), we obtain

$$\int_\Omega M(x,u)\nabla u\cdot\nabla\phi + \int_\Omega b(x)\frac{|\nabla u|^2}{u^\theta}\, \phi \geq \int_\Omega (\lambda u^r + f)\phi. \tag{5.18}$$

To prove the opposite inequality, we choose $\phi \in H_0^1(\Omega)\cap L^\infty(\Omega)$ with $\phi \geq 0$, as test function in (3.2), to obtain

$$\int_\Omega M(x, T_{\frac{1}{\varepsilon}}(u_\varepsilon))\nabla u_\varepsilon\cdot\nabla\phi + \int_\Omega b(x)\frac{u_\varepsilon|\nabla u_\varepsilon|^2}{(u_\varepsilon+\varepsilon)^\theta}\phi \leq \int_\Omega (\lambda u_\varepsilon^r + f_\varepsilon)\,\phi, \tag{5.19}$$

Passing to the limit in (5.19), using the weak convergence of sequence $M(x, T_{\frac{1}{\varepsilon}}(u_\varepsilon))\nabla u_\varepsilon$ to $M(x,u)\nabla u$ in $(L^2(\Omega))^N$, Fatou's lemma, and the strong convergence of u_ε in $L^r(\Omega)$ (due to the fact that u_ε is bounded in $L^{m^{**}(2-\theta)}(\Omega)$ and $r < m^{**}(2-\theta)$), it follows that

$$\int_\Omega M(x,u)\nabla u\cdot\nabla\phi + \int_\Omega b(x)\frac{|\nabla u|^2}{u^\theta}\phi \leq \int_\Omega (\lambda u^r + f)\phi. \tag{5.20}$$

Combining (5.18) and (5.20), we deduce that

$$\int_\Omega M(x,u)\nabla u\cdot\nabla\phi + \int_\Omega b(x)\frac{|\nabla u|^2}{u^\theta}\phi = \int_\Omega (\lambda u^r + f)\phi,$$

for every ϕ in $H_0^1(\Omega)\cap L^\infty(\Omega)$, with $\phi \geq 0$. Thus, we have that (2.2) holds for every nonnegative test function. The case of a general test function ϕ is then obtained by choosing ϕ^+ and ϕ^-, and then adding up the two equalities. $\square$

5.2. Proof of Theorem 2.2

In virtue of the Lemma 4.4, the sequence of approximated solutions u_ε is bounded in $H_0^1(\Omega)\cap L^\infty(\Omega)$. Therefore, there exists a function u belongs to $H_0^1(\Omega)\cap L^\infty(\Omega)$ such that, up to subsequences, u_ε converges weakly in $H_0^1(\Omega)$ to u, which satisfies $u>0$ in Ω, and $\frac{|\nabla u|^2}{u^\theta}$ is in $L^1(\Omega)$ (by the Lemma 5.2 and the Corollary 5.3. Thanks to Lemma 5.1, we have that ∇u_ε converges almost everywhere to ∇u in Ω. To prove that u is a weak solution of problem (1.1), it suffices to proceed as in the proof of Lemma 5.4, by testing (3.2) with the function $e^{\frac{-\nu H_\varepsilon(u_\varepsilon)}{\alpha}}\, e^{\frac{BH_\varepsilon(u)}{\alpha}}\phi$, instead of the test function given in (5.13), since in this case, the function u is bounded.

Remark 5.5. Taking into account the boundedness of u_ε in $L^\infty(\Omega)$, then the degenerate coercivity of the operator $Au=-\mathrm{div}\,(M(x,u)\nabla u)$ disappears. Therefore, we can apply the result of [12] to prove the almost everywhere convergence of ∇u_ε to ∇u, since both lower-order term and right one are bounded in $L^1(\Omega)$.

5.3. Proof of Theorem 2.3

According to the Lemma 4.5, the sequences u_ε and $T_k(u_\varepsilon)$ (for every $k>0$) are bounded, respectively, in $W_0^{1,q}(\Omega)\cap L^{m^{**}(2-\theta)}(\Omega)$, and $H_0^1(\Omega)$. Therefore, there exists a function u belonging to $W_0^{1,q}(\Omega)\cap L^{m^{**}(2-\theta)}(\Omega)$ such that, up to subsequences, u_ε and $T_k(u_\varepsilon)$ converge weakly, respectively, in $W_0^{1,q}(\Omega)$ and $H_0^1(\Omega)$, and almost everywhere in Ω, respectively, to u and $T_k(u)$. Moreover, by repeating the argument in the proof of Lemma 5.2, it follows that $u>0$ in Ω. The Corollary 5.3 ensures that $\frac{|\nabla u|^2}{u^\theta}$ belongs to $L^1(\Omega)$. The argument in the proof of Lemma 5.1 is still valid and gives the almost everywhere convergence of the sequence ∇u_ε to ∇u in Ω. To finish the proof of the Theorem 2.3, it remains to prove that u is a distributional solution of the problem (1.1). This is the goal of the next lemma.

Lemma 5.6. *Let u be the weak limit of the sequence u_ε. Then u satisfies*

$$\int_\Omega M(x,u)\nabla u\cdot\nabla\phi+\int_\Omega b(x)\frac{|\nabla u|^2}{u^\theta}\phi=\int_\Omega(\lambda u^r+f)\,\phi, \tag{5.21}$$

for every $\phi\in\mathcal{C}_0^1(\Omega)$.

Proof. To prove Lemma 5.6, we repeat the proof of Lemma 5.4, obtaining two inequalities; the second one can be obtained exactly as before, while for the first one we have to slightly modify the test function, since we no longer have the estimate of u_ε in $H_0^1(\Omega)$. So, we take in (3.2) the test function $e^{\frac{-\nu H_\varepsilon(u_\varepsilon)}{\alpha}}\, e^{\frac{\nu H_{1/j}(T_j(u))}{\alpha}} R_k(u_\varepsilon)\phi$, with $\phi\in\mathcal{C}_0^1(\Omega),\ \phi\ge 0$, we obtain

$$\begin{aligned}&\int_\Omega M(x,T_{\frac{1}{\varepsilon}}(u_\varepsilon))\nabla u_\varepsilon\cdot\nabla\phi\; e^{\frac{-\nu H_\varepsilon(u_\varepsilon)}{\alpha}}\, e^{\frac{\nu H_{1/j}(T_j(u))}{\alpha}} R_k(u_\varepsilon)\\&\quad+\frac{\nu}{\alpha}\int_\Omega\frac{M(x,T_{\frac{1}{\varepsilon}}(u_\varepsilon))\nabla u_\varepsilon\cdot\nabla T_j(u)}{(T_j(u)+1/j)^\theta(\rho+T_j(u))^{-\gamma}}\, e^{\frac{-\nu H_\varepsilon(u_\varepsilon)}{\alpha}}\, e^{\frac{\nu H_{1/j}(T_j(u))}{\alpha}} R_k(u_\varepsilon)\phi\\&=\int_\Omega\left[\frac{\lambda u_\varepsilon^r}{1+\varepsilon u_\varepsilon^r}+f_\varepsilon\right] e^{\frac{-\nu H_\varepsilon(u_\varepsilon)}{\alpha}}\, e^{\frac{\nu H_{1/j}(T_j(u))}{\alpha}} R_k(u_\varepsilon)\phi\end{aligned}$$

$$
\begin{aligned}
&+\int_\Omega \left[\frac{\nu}{\alpha}\,\frac{M(x,T_{\frac{1}{\varepsilon}}(u_\varepsilon))\nabla u_\varepsilon\cdot\nabla u_\varepsilon}{(u_\varepsilon+\varepsilon)^\theta(\rho+u_\varepsilon)^{-\gamma}}\right.\\
&\left.-\,b(x)\frac{u_\varepsilon|\nabla u_\varepsilon|^2}{(u_\varepsilon+\varepsilon)^{\theta+1}}\right] e^{\frac{-\nu H_\varepsilon(u_\varepsilon)}{\alpha}}\, e^{\frac{\nu H_{1/j}(T_j(u))}{\alpha}} R_k(u_\varepsilon)\phi\\
&+\int_{\{k<u_\varepsilon<k+1\}} M(x,T_{\frac{1}{\varepsilon}}(u_\varepsilon))\nabla u_\varepsilon\cdot\nabla u_\varepsilon\; e^{\frac{-\nu H_\varepsilon(u_\varepsilon)}{\alpha}}\, e^{\frac{\nu H_{1/j}(T_j(u))}{\alpha}}\phi.
\end{aligned}
\tag{5.22}
$$

Dropping the last term (which is nonnegative), and using Fatou's lemma as well as the weak convergence of u_ε to u in $W_0^{1,q}(\Omega)$, and of $M(x,T_{\frac{1}{\varepsilon}}(u_\varepsilon))\nabla T_{k+1}(u_\varepsilon)$ to $M(x,u)\nabla T_{k+1}(u)$ in $(L^2(\Omega))^N$ for the first term, we obtain

$$
\begin{aligned}
&\int_\Omega M(x,u)\nabla u\cdot\nabla\phi\; e^{\frac{-\nu H_0(u)}{\alpha}}\, e^{\frac{\nu H_{1/j}(T_j(u))}{\alpha}} R_k(u)\\
&\quad+\frac{\nu}{\alpha}\int_\Omega \frac{M(x,u)\nabla u\cdot\nabla T_j(u)}{(T_j(u)+1/j)^\theta(\rho+T_j(u))^{-\gamma}}\, e^{\frac{-\nu H_0(u)}{\alpha}}\, e^{\frac{\nu H_{1/j}(T_j(u))}{\alpha}} R_k(u)\phi\\
&\geq \int_\Omega (\lambda u^r+f)\, e^{\frac{-\nu H_0(u)}{\alpha}}\, e^{\frac{\nu H_{1/j}(T_j(u))}{\alpha}} R_k(u)\phi\\
&\quad+\int_\Omega \left[\frac{\nu}{\alpha}\,\frac{M(x,u)\nabla u\cdot\nabla u}{u^\theta(\rho+u)^{-\gamma}}-b(x)\frac{|\nabla u|^2}{u^\theta}\right] e^{\frac{-\nu H_0(u)}{\alpha}}\, e^{\frac{\nu H_{1/j}(T_j(u))}{\alpha}} R_k(u)\phi.
\end{aligned}
\tag{5.23}
$$

We conclude the proof, as in Lemma 5.4, letting first j tend to infinity, and then k tend to infinity. □

5.4. Proof of Theorem 2.4

Lemma 4.6 asserts that the sequence u_ε is bounded in $W_0^{1,\delta}(\Omega)$, and the sequence $T_k(u_\varepsilon)$ is bounded in $H_0^1(\Omega)$ for every $k>0$. Therefore, there exists a function u belonging to $W_0^{1,\delta}(\Omega)$ such that, up to subsequences, u_ε converges weakly in $W_0^{1,\delta}(\Omega)$, and almost everywhere in Ω to u, and $T_k(u_\varepsilon)$ weakly converges in $H_0^1(\Omega)$, and almost every in Ω to $T_k(u)$ for every $k>0$. Furthermore, by the same technique used in the proof of Lemma 5.1, we have ∇u_ε converges almost everywhere in Ω to ∇u. The technique used in the proof of Lemma 5.2 can be still applied, yielding that $u>0$ in Ω. By the Corollary 5.3, we have $\frac{|\nabla u|^2}{u^\theta}\in L^1(\Omega)$. Since $T_k(u_\varepsilon)$ weakly converges in $H_0^1(\Omega)$, almost everywhere in Ω to $T_k(u)$, and u_ε strongly converges to u in $L^r(\Omega)$ (due to the fact that the sequence u_ε is bounded in $W_0^{1,\delta}(\Omega)$ and $r<2-\theta<\delta$) then, we can pass to the limit in (3.2) exactly as in the proof of Theorem 2.3 to conclude that u is a distributional solution of the problem (1.1).

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