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Fixed Point Theorems for Multi-valued Nonexpansive Mappings in Banach Spaces

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Abstract. In this paper, we present new fixed point theorems for multivalued nonexpansive mappings. Since Banach space can have any geometric structure, we consider mappings such that their perturbation by the identity operator is expansive. Then we derive some fixed point results including existence theorems for the sum and product of some classes of nonlinear operators. Three illustrating examples for functional and differential inclusions are supplied.

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1. Introduction and Preliminaries

Nonexpansive mappings (that is mappings with Lipschitz constant equal to 1) appear in many nonlinear equations for which the Banach principle for contraction mappings fails. For single-valued nonexpansive mappings, the fixed point theory for nonexpansive is well developed in the literature (see, e.g., [8, 15] and references therein). Since the pioneer work of Browder, Göhde, and Kirk, it is well known that the geometry of the space plays a key role in the existence theory for such classes of mappings (see, e.g., [15, Theorem 1.3]); in this respect, uniformly convex Banach spaces were first considered as early as in the 1970s. Since then much progress has been made in the research for existence results as regards the functional space and/or the structure of the domain of mappings under consideration (for instance when the so-called Opial's condition holds) (see, e.g., [13] and references therein).

Concerning multivalued nonexpansive mappings, the theory also developed in a parallel way starting from the extension of the Leray–Schauder topological degree to some classes of multifunctions by Petryshyn, Fitzpatrick, and Nussbaum in the 1970s [20–22]. When the space X is uniformly convex and the multifunction is nonexpansive and compact valued, the classical result due to Lim [19] naturally extends Browder–Göhde–Kirk Theorem. The structure of the domain together with different kinds of boundary conditions is imposed by many authors to get existence of fixed points for nonexpansive multifunctions. We also mention paper [24] where an inwardness condition both with a condition on the asymptotic center of bounded sequences are assumed (see also [17]). A condition on the characteristic of noncompact convexity of the space is introduced in [11], where some existence results are derived. When the domain is weakly compact, a fixed point theorem was obtained in [3] when the nonexpansive mapping satisfies some further growth condition.

Our aim in this paper is to extend the existence results obtained in [10,12] to multivalued nonexpansive mappings. We present a constructive proof for nonexpansive mappings. In this section, we have also collected the necessary material we need in our proofs.
Let X, Y be two normed spaces and $T : D \subset X \to 2^Y$ (also denoted $T : X \rightsquigarrow Y$ or $T : X \multimap Y$ in the literature) a multivalued mapping. The set $\mathcal{CL}(X)$ will refer to the family of closed subsets of X, $\mathcal{CLC}(X)$ the family of convex closed subsets of X, $\mathcal{B}(X)$ the family of bounded subsets of X, $\mathcal{BCL}(X)$ the family of bounded closed subsets of X, $\mathcal{K}(X)$ the family of compact subsets of X, and $\mathcal{KC}(X)$ the family of compact convex subsets of X.

A mapping T is called lower semi-continuous (l.s.c for short) provided that, whenever $x \in X$ and V is an open set such that $T(x) \cap V \neq \emptyset$, there exists an open neighborhood U of x such that for all $y \in U$, we have $T(y) \cap V \neq \emptyset$.

T is called upper semi-continuous (u.s.c for short) provided that, whenever $x \in X$ and V is an open set containing $T(x)$, there exists an open set U such that for all $y \in U$, we have $T(y) \subset V$.

More details on multivalued analysis can be found, e.g., in [14].

On $\mathcal{B}(X)$, one may define a metric, the Hausdorff metric $H(.,.)$ by

$$H(A, B) = \max\{H_d(A, B), H_d(B, A)\}, \ A, B \in \mathcal{B}(X),$$

where $H_d(A, B) = \sup_{x \in A}(\inf_{y \in B} \|x - y\|)$. Some basic properties of the Hausdorff metric can be easily deduced. We mention some of them that will be used in this work. The following lemma is readily proved.

Lemma 1.1 [16, Thm. 1.15]. *Let $A, B \in X$. Then,*

$$\mathrm{d}(x, A) \leq \mathrm{d}(x, B) + H(A, B), \ \forall\, x \in X.$$

Lemma 1.2. *Let A, B be two bounded subsets of a normed space X and $x_0 \in B$. Then,*

$$0 \leq H_d(A, x_0 - B) \leq \|A\|,$$

where $\|A\| = \sup\{\|x\|, \ x \in A\}$.

Proof.

$$\begin{aligned} H_d(A, x_0 - B) &= \sup_{a\in A} \inf_{b\in B} \|a - (x_0 - b)\| \\ &\leq \sup_{a\in A} \inf_{b\in B} \{\|a\| + \|x_0 - b\|\} \\ &= \sup_{a\in A} \{\|a\| + \inf_{b\in B} \|x_0 - b\|\} \\ &= \sup_{a\in A} \{\|a\|\} = \|A\|. \end{aligned}$$

□

Lemma 1.3. *Let A be a bounded convex subset of a normed space X and $a_0, x_0 \in A$. Then, for all $0 \leq \alpha, \beta \leq 1$*

$$\inf_{a\in A} \{\|(1-\alpha)a + \alpha a_0 - \beta x_0\|\} \leq (1-\beta)\|x_0\|.$$

Proof.

$$\begin{aligned} \|(1-\alpha)a + \alpha a_0 - \beta x_0\| &= \|((1-\alpha)a + \alpha a_0) - x_0 + (1-\beta)x_0\| \\ &\leq \|b - x_0\| + (1-\beta)\|x_0\|, \end{aligned}$$

where $b = (1-\alpha)a + \alpha a_0 \in A$ for A is convex. Since $\inf_{a\in A} \|b - x_0\| = 0$, the result follows. □

The following two lemmas are concerned with the sum and product of sets when the space has the structure of a Banach algebra, that is when for all subsets $X_1, X_2 \subset X$, we denote

$$X_1 + X_2 = \{a + b : \ a \in X_1, \ b \in X_2\}.$$
$$X_1 X_2 = \{ab : \ a \in X_1, \ b \in X_2\}.$$

The proof can be found in [16].

Lemma 1.4. *Let X be a normed space and A_1, A_2, B_1, and B_2 bounded subsets of X. Then,*

$$H(A_1 + A_2, B_1 + B_2) \leq H(A_1, B_1) + H(A_2, B_2).$$

Lemma 1.5. *Let X be a normed space and $A, B, C \in \mathcal{BCL}(X)$. Then,*

$$H(AC, BC) \leq H(0, C)H(A, B).$$

Definition 1.1. We shall call $T : D \subset X \rightsquigarrow X$ a contractive multivalued mapping if there exists some $k \in (0, 1)$ such that

$$H(T(x), T(y)) \leq k\|x - y\|, \ \text{for all} \ x, y \in D.$$

T is called nonexpansive when $k = 1$. Then recall the classical Covitz and Nadler fixed point theorem [4].

Theorem 1.6. *Let (X, d) be a complete metric space and $T : X \rightsquigarrow \mathcal{CL}(X)$ a contractive mapping. Then T has at least one fixed point, i.e., $x \in T(x) \cap X$.*

Finally, we will make use of:

Definition 1.2. Let $T : D \subset X \rightsquigarrow X$ be a multivalued mapping. Then T is ψ-expansive if there exists a nondecreasing or, continuous function $\psi : [0,\infty) \to [0,\infty)$ with $\psi(0) = 0$ and $\psi(r) > 0$, $\forall\, r > 0$ such that for all $x, y \in D$

$$H(T(x), T(y)) \geq \psi(\|x - y\|).$$

We are ready to state and prove our main existence result.

2. Existence and Uniqueness Result

Theorem 2.1. *Let X be a Banach space, $D \ni 0$ a nonempty closed convex subset, and $T : D \rightsquigarrow \mathcal{CLC}(X)$ a multivalued map with $T(D)$ bounded. Assume that T is a nonexpansive and $(\mathrm{I_d} - \lambda_n T)$ is ψ-expansive, where $(\lambda_n)_{n\in\mathbb{N}} \subset (0,1]$ is a real sequence converging to 1. Then T has a unique fixed point in D.*

Proof. For each integer $n \geq 1$, define the approximation operator $T_n : D \rightsquigarrow \mathcal{CLC}(D)$ by $T_n(x) = \lambda_n T(x)$, for all $x \in D$. Since T_n is a contractive multivalued mapping, then by Theorem 1.6 for every $n \in \{1, 2, \ldots\}$, T_n has a fixed point $x_n \in T_n(x_n) \cap D$ for each $n \geq 1$.

Claim 1. *$(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.* Since $(\mathrm{I_d} - \lambda_n)T$ is ψ-expansive, by the triangle inequality, we have, for all $m \geq n \geq 1$:

$$\begin{aligned}
\psi(\|x_n - x_m\|) &\leq H((\mathrm{I_d} - \lambda_n T)(x_n), (\mathrm{I_d} - \lambda_n T)(x_m)) \\
&\leq H((\mathrm{I_d} - \lambda_n T)(x_n), (1 - \lambda_n)T(x_n)) \\
&\quad + H((1 - \lambda_n)T(x_n), (1 - \lambda_n)T(x_m)) \\
&\quad + H((1 - \lambda_n)T(x_m), (\mathrm{I_d} - \lambda_n T)(x_m)).
\end{aligned}$$

The three terms in the right-hand side are now estimated separately:

(a) *The first term.* Since $T(D)$ is bounded, let

$$M = \|T(D)\| = \sup\{\|y\|,\ y \in \cup_{x\in D} T(x)\}.$$

By Lemma 1.2, we have

$$\begin{aligned}
H_d((1 - \lambda_n)T(x_n), (\mathrm{I_d} - \lambda_n T)(x_n)) &= H_d((1 - \lambda_n)T(x_n), x_n - \lambda_n T(x_n)) \\
&\leq \|(1 - \lambda_n)T(x_n)\| \\
&\leq (1 - \lambda_n)M.
\end{aligned}$$

Taking $\alpha = \beta = \lambda_n$, $x_0 = \frac{x_n}{\lambda_n}$, $a_0 = v_n$, and $A = T(x_n)$ in Lemma 1.3, we get

$$\begin{aligned}
&H_d((\mathrm{I_d} - \lambda_n T)(x_n), (1 - \lambda_n)T(x_n)) \\
&\quad = \sup_{v_n \in T(x_n)} \left\{ \inf_{w_n \in T(x_n)} \|x_n - \lambda_n v_n - (1 - \lambda_n)w_n\| \right\} \\
&\quad = \sup_{v_n \in T(x_n)} \left\{ \inf_{w_n \in T(x_n)} \|(1 - \lambda_n)w_n + \lambda_n v_n - \lambda_n x_n/\lambda_n\| \right\} \\
&\quad \leq \sup_{v_n \in T(x_n)} (1 - \lambda_n)\|x_n/\lambda_n\| \\
&\quad \leq (1 - \lambda_n)M,
\end{aligned}$$

for $\frac{x_n}{\lambda_n} \in T(x_n) \subset T(D)$. Hence,

$$H((\mathrm{I_d} - \lambda_n T)(x_n), (1-\lambda_n)T(x_n)) \leq (1-\lambda_n)M. \tag{2.1}$$

(*b*) *The second term.* Since T is nonexpansive, we have the estimates:

$$\begin{aligned} H((1-\lambda_n)T(x_n), (1-\lambda_n)T(x_m)) &= (1-\lambda_n)H(T(x_n), T(x_m)) \\ &\leq (1-\lambda_n)\|x_n - x_m\| \\ &\leq (1-\lambda_n)(\|x_n\| + \|x_m\|) \\ &\leq (1-\lambda_n)\left(\lambda_n\|x_n/\lambda_n\| + \lambda_m\|x_m/\lambda_m\|\right). \end{aligned}$$

Then,

$$H((1-\lambda_n)T(x_n), (1-\lambda_n)T(x_m)) \leq (1-\lambda_n)(\lambda_n + \lambda_m)M. \tag{2.2}$$

(*c*) *The third term.* Taking $\alpha = \lambda_n, \beta = \lambda_m$, $a_0 = w_m$, $x_0 = x_m/\lambda_m$, and $A = T(x_m)$ in Lemma 1.3, we obtain

$$\begin{aligned} &H_d((\mathrm{I_d} - \lambda_n T)(x_m), (1-\lambda_n)T(x_m)) \\ &\quad = \sup_{w_m \in T(x_m)} \left\{ \inf_{v_m \in T(x_m)} \|(1-\lambda_n)v_m - x_m + \lambda_n w_m\| \right\} \\ &\quad = \sup_{w_m \in T(x_m)} \left\{ \inf_{v_m \in T(x_m)} \|(1-\lambda_n)v_m + \lambda_n w_m - \lambda_m x_m/\lambda_m\| \right\} \\ &\quad \leq (1-\lambda_m)\|x_m/\lambda_m\| \\ &\quad \leq (1-\lambda_m)M. \end{aligned}$$

Again, Lemma 1.3 with $\alpha = 1-\lambda_n, \beta = \lambda_m$, $a_0 = v_m$, $x_0 = x_m/\lambda_m$, and $A = T(x_m)$ yields

$$\begin{aligned} &H_d((1-\lambda_n)T(x_m), (\mathrm{I_d} - \lambda_n T)(x_m)) \\ &\quad = \sup_{v_m \in T(x_m)} \left\{ \inf_{w_m \in T(x_m)} \|(1-\lambda_n)v_m - x_m + \lambda_n w_m\| \right\} \\ &\quad \leq (1-\lambda_m)\|x_m/\lambda_m\| \\ &\quad \leq (1-\lambda_m)M. \end{aligned}$$

Collecting the above estimates, we conclude that

$$H((1-\lambda_m)T(x_m), (\mathrm{I_d} - \lambda_n T)(x_m)) \leq (1-\lambda_m)M. \tag{2.3}$$

To summarize, (2.1), (2.2), and (2.3) imply that

$$\begin{aligned} \psi(\|x_n - x_m\|) &\leq ((1-\lambda_n) + (\lambda_m + \lambda_n)(1-\lambda_n) + (1-\lambda_m))M \\ &= u_{(n,m)}M, \end{aligned} \tag{2.4}$$

where the double sequence $u_{n,m}$ converges to 0, as $n, m \to \infty$.

Arguing by contradiction, assume that there exists $\varepsilon_0 > 0$, two subsequences $(x_{n_k})_k, (x_{m_k})_k$, and $k_0 = 1, 2, \ldots$ such that

$$t_k = \|x_{n_k} - x_{m_k}\| \geq \varepsilon_0, \quad \forall\, k \geq k_0.$$

Then we distinguish between two cases:

(i) ψ is nondecreasing. We have

$$u_{(n_k,m_k)}M \geq \psi(\|x_{n_k} - x_{m_k}\|) \geq \psi(\varepsilon_0) > 0.$$

Letting $k \to \infty$ in (2.4) implies that $\psi(\varepsilon_0) = 0$. Given the properties of ψ, it follows that $\varepsilon_0 = 0$, a contradiction.

(ii) ψ is continuous. Since $T(D)$ is bounded and $x_n \in \lambda_n T(x_n)$, we have

$$0 < \varepsilon_0 \leq t_k \leq (\lambda_{m_k} + \lambda_{n_k})M \leq 2M.$$

Taking again subsequences, if need be, the real sequence $(t_k)_k$ converges to some limit $t_0 \geq \varepsilon_0$, as $k \to +\infty$. The function ψ being continuous together with the property $\psi(r) = 0 \iff r = 0$, we find that $\psi(t_0) > 0$, while, by (2.4), $\psi(t_0) = \lim_{k\to+\infty} \psi(t_k) = 0$, a contradiction.

Therefore, we have proved that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in D. The subset D being closed in the Banach space X, the sequence $(x_n)_{n\in\mathbb{N}} \subset D$ converges to some limit $x \in D$.

Claim 2. *x is the unique fixed point of T.* Using Lemma 1.1 and the nonexpansiveness of T, we get

$$\begin{aligned}
\mathrm{d}(x, T(x)) &\leq \mathrm{d}(x, x_n) + \mathrm{d}(x_n, T(x)) \\
&\leq \mathrm{d}(x, x_n) + \mathrm{d}(x_n, T(x_n)) + H(T(x_n), T(x)) \\
&\leq \|x - x_n\| + \inf_{y_n \in T(x_n)} \|x_n - y_n\| + \|x_n - x\| \\
&\leq 2\|x - x_n\| + \left(1 - \frac{1}{\lambda_n}\right) \|x_n\|.
\end{aligned}$$

Passing to the limit as $n \to \infty$, we get $\mathrm{d}(x, T(x)) = 0$, that is $x \in T(x)$.

To check the uniqueness of the fixed point, consider $x, y \in D$ two possible fixed points of T. Since $(\mathrm{I_d} - \lambda_n)T$ is expansive, we have

$$\begin{aligned}
\psi(\|x - y\|) &\leq H((\mathrm{I_d} - \lambda_n)T(x), (\mathrm{I_d} - \lambda_n T)(y)) \\
&\leq H((\mathrm{I_d} - \lambda_n T)(x), (1 - \lambda_n)T(x)) \\
&\quad + H((1 - \lambda_n)T(x), (1 - \lambda_n)T(y)) \\
&\quad + H((1 - \lambda_n)T(y), (\mathrm{I_d} - \lambda_n T)(y)).
\end{aligned}$$

We have to deal with each term of the right-hand side.

For all $x_1, x_2 \in T(x)$, we have

$$\|x - \lambda_n x_1 - (1 - \lambda_n)x_2\| = \|x - b(x_1, x_2)\|,$$

where $b(x_1, x_2) = \lambda_n x_1 + (1 - \lambda_n)x_2 \in T(x)$, for T is convex valued. Since

$$\inf_{x_2 \in T(x)} \|x - b(x_1, x_2)\| = 0 \text{ and } \inf_{x_1 \in T(x)} \|x - b(x_1, x_2)\| = 0,$$

we deduce that

$$\begin{aligned}
H((\mathrm{I_d} - \lambda_n T)(x), (1 - \lambda_n)T(x)) &= H_d((\mathrm{I_d} - \lambda_n T)(x), (1 - \lambda_n)T(x)) \\
&= \sup_{x_2 \in T(x)} \inf_{x_1 \in T(x)} \|x - \lambda_n x_1 - (1 - \lambda_n)x_2\| \\
&= 0.
\end{aligned}$$

In the same way, we can check that

$$H((1-\lambda_n)T(y), (\mathrm{I_d}-\lambda_n T)(y)) = 0.$$

Regarding the second term, we have by the nonexpansiveness of T,

$$\begin{aligned}\psi(\|x-y\|) &\le H((1-\lambda_n)T(x), (1-\lambda_n)T(y)) \\ &\le (1-\lambda_n)\|x-y\| \\ &\le (1-\lambda_n)2M.\end{aligned}$$

Letting $n \to \infty$ guarantees that $\psi(\|x-y\|) = 0$, hence $\|x-y\| = 0$. The proof of Theorem 2.1 is completed. □

Remark 2.1. (a) If D is bounded, then so is $T(D)$, as already seen in the single-valued case (see, e.g., [9, Lemma 1.1]) and the situation is the same for k-contractions multivalued mappings $T : X \rightsquigarrow Y$. Indeed, B is a bounded subset of X, $x, x_0 \in B$, then, by Lemma 1.1:

$$\|y\| \le \|Tx\| \le H(Tx, Tx_0) + \|Tx_0\| \le k\|x-x_0\| + \|x_0\|, \ \forall\, y \in Tx.$$

(b) Convexity of D is required for T_n to self-map the subset D with $T_n(x) = \lambda_n T(x) + (1-\lambda_n)\{0\} \subset D$.

(c) The closure of values of T comes from application of Covitz–Nadler Theorem 1.6, while convexity of values is required in the proof of Theorem 2.1.

3. Consequences

Immediately, we have:

Theorem 3.1. *Let X be a Banach space, $D \ni 0$ a nonempty closed convex subset, and $T : D \rightsquigarrow \mathcal{CLC}(X)$ a multivalued map with $T(D)$ bounded. Assume that T is a nonexpansive mapping and $(\mathrm{I_d} - T)$ is ψ-expansive. Then T has a unique fixed point in D.*

The proof of this theorem is the same as in Theorem 2.1, where $(\lambda_n)_{n\in\mathbb{N}} \subset (0, 1]$. We only check the triangle inequality in Claim 1:

$$\begin{aligned}\psi(\|x_n - x_m\|) &\le H((\mathrm{I_d}-T)x_n, (\mathrm{I_d}-T)x_m) \\ &\le H((\mathrm{I_d}-T)x_n, (1-\lambda_n)T(x_n)) \\ &\quad + H((1-\lambda_n)T(x_n), (1-\lambda_m)T(x_m)) \\ &\quad + H((1-\lambda_m)T(x_m), (\mathrm{I_d}-T)x_m).\end{aligned}$$

When T is a single-valued mapping, we first recover from Theorem 2.1, a result proved by Garcia–Falset in 2010 (see [12, Proposition 3.4]).

Corollary 3.2. *Let X be a Banach space, $C \ni 0$ a bounded convex closed subset of X, and $T : C \longrightarrow C$ a nonexpansive mapping such that $(\mathrm{I_d} - T)$ is ψ-expansive. Then, T has a unique fixed point in C.*

Next, we prove a Krasnosel'skii type fixed point theorem. We will make use of a result proved by Rybinski [23] for contractive mappings, namely [23, Thm. 1]. The latter result turns out to be still valid for nonexpansive mappings. The proof, just an adaptation based on two technical lemmas, follows the same lines and thus is not reproduced.

Lemma 3.3. *Let X be a paracompact topological space and Y be a closed subset of Banach space E. Let $T : X \times Y \rightsquigarrow \mathcal{CLC}(Y)$ be such that for every $x \in X$, $T(x, .)$ is nonexpansive in the second argument y, and for every $y \in Y$, the multivalued mapping $T(., y)$ is l.s.c. on Y. Then there exists a continuous mapping $h : X \times Y \to Y$ such that $h(x, y) \in P_T(x) = \{z \in Y : \ z \in T(x, z)\}$, for every $(x, y) \in X \times Y$.*

As a consequence, the following result extends Theorem [7, Thm. 3.2].

Corollary 3.4. *Let D be a bounded closed convex subset of X and let $T_1 : X \rightsquigarrow \mathcal{CLC}(X)$ and $T_2 : D \rightsquigarrow \mathcal{BC}(D)$ be two multivalued mappings satisfying*

(a) *T_1 is a nonexpansive mapping and $(\mathrm{I_d} - T_1)$ is ψ-expansive,*
(b) *T_2 is l.s.c. and compact,*
(c) *for all $x, y \in D$, $T_1x + T_2y \subset D$.*
Then the sum $T_1 + T_2$ has a fixed point in D.

Proof. Define the multivalued operator $T : X \times D \rightsquigarrow \mathcal{CLC}(X)$ by $T(x, y) = T_2(x) + T_1(y)$. First, we show that $T(x, y)$ is a nonexpansive multivalued mapping in y, for each fixed $x \in X$. Let $y_1, y_2 \in X$ be arbitrary, then by Lemma 1.4,

$$\begin{aligned} H(T(x, y_1), T(x, y_2)) &= H(T_2x + T_1y_1, T_2x + T_1y_2) \\ &\le H(T_1y_1, T_1y_2) \\ &\le \|y_1 - y_2\|, \end{aligned}$$

i.e., the multivalued operator $T_x(.) = T(x, .)$ is nonexpansive. In addition, $(\mathrm{I_d} - T_x)$ is ψ-expansive, which follows from Lemma 1.4. Indeed,

$$\begin{aligned} H((\mathrm{I_d} - T_x)(y_1), (\mathrm{I_d} - T_x)(y_2)) &= H(y_1 - (T_2x + T_1y_1), y_2 - (T_2x + T_2y_2)) \\ &\ge H(y_1 - (T_2x + T_1y_1) + T_2x, \\ &\qquad y_2 - (T_2x + T_1y) + T_2x) \\ &= H(y_1 - T_1y_1, y_2 - T_1y_2) \\ &\ge \psi(\|y_1 - y_2\|). \end{aligned}$$

By Theorem 2.1, $T_x(.)$ has a unique fixed point $y_x \in T_x(y_x)$. Hence, $P_T(x) = \{z \in D : z \in T(x, z)\} = \{y_x\}$. Moreover, $T(., .)$ satisfies all conditions of the nonexpansive version of [23, Thm. 1]. As a consequence, there exists a continuous mapping $h : X \times D \to X$ such that $h(x, y) \in P_T(x)$, that is

$$\forall\, x \in X, \ \text{ there exists a unique } \ y_x \in D : \ h(x, y_x) = \{y_x\}.$$

In particular, this defines a single-valued mapping

$$\varrho : \ x \mapsto h(x, x),$$

from X to X. We have that ϱ is compact on D. Since T_2 is compact, $T_2(D)$ is totally bounded. Let $\varepsilon > 0$. Then there exists $Y = \{y_1, \ldots, y_n\} \subset X$ such that

$$T_2(D) \subset \cup_{i=1}^{n} T_2(y_i) + B(0, \psi(\varepsilon)).$$

In particular, for each $y \in D$, we have that $T_2(y) \subset \cup_{i=1}^{n} T_2(y_i) + B(0, \psi(\varepsilon))$ and there exists $y_k \in Y$ $(1 \le k \le n)$ such that

$$H(T_2(y), T_2(y_k)) < \psi(\varepsilon).$$

By definition of $P_T(x)$, $\varrho(y) \in T_2(y) + T_1(\varrho(y))$ and $\varrho(y_k) \in T_2(y_k) + T_1(\varrho(y_k))$. Thus, $(\mathrm{I_d} - T_1)\varrho(y) \subset T_2(y)$ and $(\mathrm{I_d} - T_1)\varrho(y_k) \subset T_2(y_k)$. Since T_1 is $\psi-$ expansive, we deduce that

$$\begin{aligned} \psi(\|\varrho(y) - \varrho(y_k)\|) &\le H((\mathrm{I_d} - T_1)\varrho(y), (\mathrm{I_d} - T_1)\varrho(y_k)) && (3.1)\\ &\le H(T_2(y), T_2(y_k)) && (3.2)\\ &< \psi(\varepsilon). && (3.3) \end{aligned}$$

(a) If ψ is continuous, then so is ψ^{-1}. By (3.1) there exists η_0 such that $\psi(\|\varrho(y) - \varrho(y_k)\|) - \psi(\varepsilon) \le \eta_0$. Thus for all $\delta > 0$, we have

$$|\psi^{-1}(\psi(\|\varrho(y) - \varrho(y_k)\|)) - \psi^{-1}(\psi(\varepsilon))| \le \delta,$$

that is, $|\|\varrho(y) - \varrho(y_k)\| - \varepsilon| \le \delta$. The number δ being arbitrary, we get $\|\varrho(y) - \varrho(y_k)\| < \varepsilon$.

(b) If ψ is nondecreasing, then $\|\varrho(y) - \varrho(y_k)\| < \varepsilon$.

We have proved that, for each $y \in D$, there exist $y_k \in Y$ such that $\varrho(y) \subset B(\varrho(y_k), \varepsilon)$, that is, $\varrho(D) \subset \cup_{i=1}^{n} B(\varrho(y_i), \varepsilon)$, showing that ϱ is compact.

As a consequence, by Schauder's fixed point theorem, there is some x^* such that

$$\begin{aligned} x^* = \varrho(x^*) = h(x^*, x^*) \in T_{x^*}(\varrho(x^*)) &= T(x^*, (\varrho(x^*))\\ &= T(x^*, x^*) = T_2(x^*) + T_1(x^*), \end{aligned}$$

a fixed point of the sum operator $T_1 + T_2$. □

The third result concerns the existence of a fixed point for the product of two nonexpansive mappings in a Banach algebra X. The proof follows the same lines as in Corollary 3.4 applied to the product mapping $T(x, y) = T_1 x . T_2 y$ by making use of Lemma 1.5. We omit the proof.

Corollary 3.5. *Let D be a nonempty and closed subset of a Banach algebra X and let $T_1 : D \rightsquigarrow \mathcal{BC}(X), T_2 : D \rightsquigarrow \mathcal{BC}(X)$ be two multivalued operators such that*

(a) *T_1 is a nonexpansive mapping,*
(b) *T_2 is l.s.c. compact and $\|T_2(D)\| = M \le 1$,*
(c) *for all $y \in D$, $\mathrm{I_d}(.) - T_1(.)T_2(y)$ is $\psi-$expansive,*
(d) *$T_1(x)T_2(y)$ is a convex subset of D, for each $x, y \in D$.*

Then the product $T = T_1 . T_2$ has a fixed point in D.

4. Applications: Functional Integral Inclusions

Given a closed bounded interval of the real line $J = [0,1]$ in $\mathbb{R}$, $L^1(J,\mathbb{R})$ and $L^\infty(J,\mathbb{R})$ will denote the Lebesgue spaces of integrable and essentially bounded functions, respectively. These are Banach spaces with norms in L^1 and L^∞ denoted by $\|.\|_1$ and $\|.\|_\infty$, respectively. The space of real continuous functions on $J, X = C(J,\mathbb{R})$ is endowed with the sup-norm $\|x\| = \sup_{t\in J}|x(t)|$.

4.1. Example 1

Consider the following functional integral inclusion:

$$x(t) \in F(t,x(t)) + \int_0^t k(t,s)G(s,x(s))\mathrm{d}s, \quad t \in J, \tag{4.1}$$

where $k : J \times J \to \mathbb{R}$ is L^∞ and $F, G : J \times \mathbb{R} \rightsquigarrow \mathcal{CLC}(\mathbb{R})$. By solution of problem 4.1, we mean a function $x \in C(J,\mathbb{R})$ such that

$$x(t) = v_1(t) + \int_0^1 k(t,s)v_2(s)\mathrm{d}s, \quad t \in J,$$

where $v_1(t) \in F(t,x(t))$, $v_2(t) \in G(t,x(t))$, $v_1 \in C^1(J,\mathbb{R})$, and $v_2 \in L^1(J,\mathbb{R})$.

First, recall:

Definition 4.1. A multivalued map $G : J \times \mathbb{R} \rightsquigarrow \mathcal{KC}(\mathbb{R})$ is said to be L^1-Carathéodory if

(a) the mapping $t \mapsto G(t,x)$ is measurable for each $x \in \mathbb{R}$,
(b) the mapping $x \mapsto G(t,x)$ is upper semi-continuous for almost all $t \in J$,
(c) for each real number $\rho > 0$, there exists a function $g_\rho \in L^1([0,1],\mathbb{R}^+)$ such that

$$\|G(t,u)\| = \sup\{|v| : \ v \in G(t,u)\} \le g_\rho(t), \ a.e. \ t \in J,$$

and for all $u \in \mathbb{R}$ with $|u| \le \rho$, i.e., G is locally integrably bounded. (G is integrably bounded whenever g does not depend on ρ).

For any function x, consider the selection set

$$S_{G,x} = \{v \in L^1([0,1],\mathbb{R}) : v(t) \in G(t,x(t)), \quad a.e. \ t \in J\}.$$

When G is L^1-Carathéodory, we know from a result due to Lasota and Opial [18] that for each $x \in C(J,\mathbb{R})$, the set $S_{G,x}$ is nonempty. Our first existence result in this section is:

Theorem 4.1. *Assume that*

(H_1) $F : J \times \mathbb{R} \rightsquigarrow \mathcal{KC}(\mathbb{R})$ *is a nonexpansive mapping and* $(\mathrm{I_d} - F)$ *is* ψ-*expansive in its second argument,*
(H_2) *there exists* $0 \le L < 1$ *such that*

$$H(F(t,x),F(t,0)) \le L|x|, \quad \textit{for each } x \in \mathbb{R} \ \textit{ and } t \in J,$$

(H_3) $G : J \times \mathbb{R} \rightsquigarrow \mathcal{KC}(\mathbb{R})$ *is* L^1-*Carathéodory,*

(H_4) *there exists a real number* $r > 0$ *such that*

$$r \geq \frac{\|F(t,0)\| + \|k\|_\infty \|g_r\|_1}{1-L}.$$

Then problem (4.1) *has a solution in* $C(J, \mathbb{R})$.

Remark 4.1. Of course, when G is integrably bounded, Assumption (H_4) is not needed.

Proof. Let $D = \overline{B}(0, r)$ be the closed ball in X centered at the origin, with radius r in X, where the real number r satisfies (H_4). Define the operators

$$T_1 x(t) := F(t, x(t)), \quad t \in J \tag{4.2}$$

and

$$T_2 x := \left\{ u \in X : u(t) = \int_0^t k(t,s) v(s) \mathrm{d}s, \quad t \in J, \ v \in S_{G,x} \right\}. \tag{4.3}$$

Then problem (4.1) is equivalent to the operational inclusion

$$x(t) \in T_1 x(t) + T_2 x(t), \quad t \in J. \tag{4.4}$$

For $i = 1, 2$, $T_i(x)$ are shortened here to $T_i x$. We will show that the multivalued operators T_1 and T_2 satisfy all conditions of Corollary 3.4. Clearly, T_1 and T_2 are well defined since $S_{F,x} \neq \emptyset$ and $S_{G,x} \neq \emptyset$ for each $x \in X$. Moreover by (H_2), $T_1 : X \rightsquigarrow \mathcal{CLC}(\mathbb{R})$ is a nonexpansive multivalued mapping and $(\mathrm{I_d} - T_1)$ is ψ-expansive.

Claim 1. *T_2 takes bounded convex values.* Given $x \in D$ and $y \in T_2 x$, there is a $v \in S_{G,x}$ such that

$$y(t) = \int_0^1 k(t,s) v(s) \mathrm{d}s, \ t \in J.$$

By (H_3), we have for all $t \in J$,

$$\begin{aligned} |y(t)| &\leq \int_0^1 |k(t,s)||v(s)| \mathrm{d}s \leq \int_0^1 |k(t,s)| \|G(t, x(t))\| \mathrm{d}s \\ &\leq \int_0^1 |k(t,s)||g_r(s)| \mathrm{d}s \leq \|k\|_\infty \|g_r\|_1. \end{aligned}$$

Then,

$$\|y\|_\infty \leq \|k\|_\infty \|g_r\|_1.$$

Hence, T_2 has bounded values. Let $u_1, u_2 \in T_2 x$, then there exist $v_1, v_2 \in S_{G,x}$ such that

$$u_1(t) = \int_0^t k(t,s) v_1(s) \mathrm{d}s \ t \in J \ \text{ and } \ u_2(t) = \int_0^t k(t,s) v_2(s) \mathrm{d}s \quad t \in J.$$

For $\lambda \in [0, 1]$, we write

$$\begin{aligned} \lambda u_1(t) + (1-\lambda) u_2(t) &= \int_0^t k(t,s)(\lambda v_1(s) + (1-\lambda) v_2(s)) \mathrm{d}s \\ &= \int_0^t k(t,s) v(s) \mathrm{d}s, \end{aligned}$$

where $v(t) = \lambda v_1(t) + (1-\lambda)v_2(t) \in G(t, x(t))$ for G takes convex values. Thus, T_2x is convex for each $x \in X$.

Claim 2. *T_2 is lower semi-continuous on D.* Let (x_n) be a sequence that converges to some limit x_0 in X and let $V = B(0,\delta)$, $\delta > 0$ be the open ball of X centered at the origin with radius δ such that $T_2x_0 \cap V \neq \emptyset$. Then there exists $y_0 \in T_0x_0$ with $\|y_0\|_\infty < \delta$ and there exists $v_0 \in S_{G,x_0}$ such that

$$y_0(t) = \int_0^t k(t,s)v_0(s)\mathrm{d}s, \quad t \in J.$$

We can show that $|y_0(t)| \leq \|k\|_\infty \|g_r\|_1 < \delta$. Since, by (H_3), G is L^1-Carathéodory, then $G(t,x_n) \subset G(t,x_0) + \varepsilon B(0,1)$, for all $\varepsilon > 0$ and large enough n. To show that $T_2x_n \cap V \neq \emptyset$, let $y_n \in T_2x_n$. Then there exists $v_n \in S_{G,x_n} \subset (S_{G,x_0} + \varepsilon B(0,1))$ such that

$$y_n(t) = \int_0^t k(t,s)v_n(s)\mathrm{d}s, \quad t \in J.$$

Moreover, there exist $v \in S_{G,x_0}$, $b \in B(0,1)$ such that $v_n = v + \varepsilon b$. Hence, $|v_n| \leq |v| + \varepsilon$. As in Claim 1, we have the estimate:

$$|y_n(t)| = \left|\int_0^t k(t,s)(v(s) + \varepsilon b)\mathrm{d}s\right| < \delta + \|k\|_\infty \varepsilon.$$

Letting $\varepsilon \to 0$, there exists $y_n^0 \in T_2x_n$ such that $\|y_n^0\|_\infty < \delta$. Thus, $T_2x_n \cap V \neq \emptyset$. This shows that T_2 is lower semi-continuous on X.

Claim 3. *For all $x, y \in D$, $T_1x + T_2y \subset D$.* For this, let $u_1 \in T_1x$ and $u_2 \in T_2y$. Then there exists $v_2 \in S_{G,y}$ such that $u_2(t) = \int_0^t k(t,s)v_2(s)\mathrm{d}s$, $t \in J$. Using (H_2) and (H_3), we get

$$\begin{aligned}|u_1(t) + u_2(t)| &= |u_1(t) + \int_0^t k(t,s)v_2(s)\mathrm{d}s| \\ &\leq |u_1(t)| + \int_0^t |k(t,s)v_2(s)|\mathrm{d}s \\ &\leq \|F(t,x)\| + \|k\|_\infty \|g_r\|_1 \\ &\leq L\|x\| + \|F(t,0)\| + \|k\|_\infty \|g_r\|_1.\end{aligned}$$

Hence,

$$\|u_1 + u_2\| \leq Lr + \|F(t,0)\| + \|k\|_\infty \|g_r\|_1.$$

By Assumption (H_4), we deduce that $\|u_1 + u_2\|_\infty \leq r$, proving our claim. Finally, the compactness of T_2 can be shown using Ascoli–Arzela Lemma both with Assumption (H_3) (see the proof of [7, Theorem 4.1]). We conclude that the operators T_1, T_2 satisfy all conditions of Corollary 3.4. As a consequence, the functional integral inclusion (4.1) admits a continuous solution on J.

□

4.2. Example 2

Our second result concerns the following differential inclusion:

$$x(t) \in \int_0^t k_1(t,s)F(s,x(s))\mathrm{d}s + \int_0^t k_2(t,s)G(s,x(s))\mathrm{d}s, \quad t \in J, \tag{4.5}$$

where the kernels $k_1, k_2 : J \times J \to \mathbb{R}$ are L^∞. We have:

Theorem 4.2. *Assume that*

(H_1) $F, G : J \times \mathbb{R} \rightsquigarrow \mathcal{KC}(\mathbb{R})$ *are* L^1*-Carathéodory multivalued mappings,*

(H_2) *there exists a function* $l \in L^1(J, \mathbb{R}^+)$ *such that*

$$H(F(t,x), F(t,y)) \leq l(t)|x-y|, \quad a.e.\ t \in [0,1] \text{ and } x, y \in \mathbb{R},$$

(H_3) *for all* $x, y \in X$ *there exists* $u_0 \in S_{F,x}$ *and* $v_0 \in S_{F,y}$ *such that*

$$\|u_0 - v_0\| \leq \frac{l(t)}{2}\|x-y\|, \quad a.e.\ t \in [0,1],$$

(H_4) *there exists a real number* $r > 0$ *such that*

$$r \geq \frac{\|k_1\|_\infty \overline{F} + \|k_2\|_\infty \|g_r\|_1}{1 - \|l\|_1 \|k_1\|_\infty / 2} \quad \text{and} \quad \|l\|_1 \|k_1\|_\infty \leq 1,$$

where $\overline{F} = \int_0^1 \|F(s,0)\|\mathrm{d}s.$

Then problem (4.5) *has a solution in* $C(J, \mathbb{R})$.

Proof. Let $D = \overline{B}(0,r)$, where r is as introduced in Hypothesis (H_4). Define the operators

$$T_1 x := \left\{ u \in X : u(t) = \int_0^t k_1(t,s)v(s)\mathrm{d}s, \quad t \in J,\ v \in S_{F,x} \right\} \tag{4.6}$$

and

$$T_2 x := \left\{ u \in X : u(t) = \int_0^t k_2(t,s)v(s)\mathrm{d}s, \quad t \in J,\ v \in S_{G,x} \right\}. \tag{4.7}$$

Then problem (4.5) is equivalent to the operator inclusion

$$x \in T_1 x + T_2 x, \quad x \in D. \tag{4.8}$$

The fact that T_2 takes bounded convex values, is lower semi-continuous on D, and compact can be proved as in Theorem 4.1.

Claim 1. T_1 *is a nonexpansive mapping and* $(\mathrm{I_d} - T_1)$ *is* ψ*-expansive.* Let $x, y \in X$ and $u_1 \in T_1 x$. Then there is a $v_1 \in S_{F,x}$ such that

$$u_1(t) = \int_0^t k_1(t,s)v_1(s)\mathrm{d}s, \quad t \in J.$$

By Hypothesis (H_2), there exists $w \in F(\cdot, y(\cdot))$ such that

$$|v_1(t) - w| \leq l(t)|x(t) - y(t)|, \quad \text{a.e. } t \in J.$$

Then the multivalued operator U defined by $U(t) = S_{F,y} \cap K(t)$, where

$$K(t) = \{w \in \mathbb{R} : |v_1(t) - w| \leq l(t)|x(t) - y(t)|, \ \text{a.e. } t \in J\}$$

has nonempty values and is measurable. Let v_2 be a measurable selection for U, which exists by Kuratowski–Ryll–Nardzewski Selection Theorem (see [2,4]). Then $v_2 \in F(\cdot, y(\cdot))$ and

$$|v_1(t) - v_2(t)| \leq l(t)|x(t) - y(t)|, \quad \text{a.e. } t \in J.$$

Define

$$u_2(t) = \int_0^t k_1(t,s)v_2(s)\mathrm{d}s, \quad t \in J.$$

Then $u_2 \in T_1 y$ and

$$\begin{aligned}
|u_1(t) - u_2(t)| &\leq \left| \int_0^t k_1(t,s)v_1(s)\mathrm{d}s - \int_0^t k_1(t,s)v_2(s)\mathrm{d}s \right| \\
&\leq \int_0^t k_1(t,s)|v_1(s) - v_2(s)|\mathrm{d}s \\
&\leq \int_0^t k_1(t,s)l(s)\|x - y\|\mathrm{d}s \\
&\leq \|k_1\|_\infty \|l\|_1 \|x - y\|.
\end{aligned}$$

As a consequence for all $x, y \in X$ and all $u_1 \in T_1 x$, we have

$$\mathrm{d}(u_1, T_1 y) \leq \|u_1 - u_2\| \leq \|k_1\|_\infty \|l\|_1 \|x - y\|.$$

Hence,

$$H(T_1 x, T_1 y) \leq \|k_1\|_\infty \|l\|_1 \|x - y\|, \quad \text{for all } \ x, y \in X.$$

Since $\|k_1\|_\infty |l|_1 \leq 1$, T_1 is a nonexpansive mapping. By Hypothesis (H_3), we can find u_0, v_0 such that

$$\|u_0^* - v_0^*\| \leq \frac{\|k_1\|_\infty \|l\|_1}{2} \|x - y\|, \tag{4.9}$$

where $u_0^*(t) = \int_0^t k_1(t,s)u_0(s)\mathrm{d}s$ and $v_0^*(t) = \int_0^t k_1(t,s)v_0(s)\mathrm{d}s$ for $t \in J$. Let $\psi : [0,\infty) \to [0,\infty)$ be the continuous nondecreasing function defined by $\psi(r) = \left(1 - \frac{\|k_1\|_\infty \|l\|_1}{2}\right) r$. For all $x, y \in X$, we have the estimates, where $u \in T_1 x$

$$\begin{aligned}
\psi(\|x - y\|) &= \|x - y\| - \frac{\|k_1\|_\infty \|l\|_1}{2} \|x - y\| \\
&\leq \|(x - u) - (y - v_0^*)\| + \|u - u_0^*\| + \|u_0^* - v_0^*\| \\
&\quad - \frac{\|k_1\|_\infty \|l\|_1}{2} \|x - y\| \\
&\leq \|(x - u) - (y - v_0^*)\| + \|u - u_0^*\| + \frac{\|k_1\|_\infty \|l\|_1}{2} \|x - y\| \\
&\quad - \frac{\|k_1\|_\infty \|l\|_1}{2} \|x - y\| \\
&\leq \|(x - u) - (y - v_0^*)\| + \|u - u_0^*\|.
\end{aligned}$$

Therefore,

$$\begin{aligned}\psi(\|x-y\|) &\leq \inf_{u\in T_1x}\{\|(x-u)-(y-v_0^*)\|+\|u-u_0^*\|\}\\ &\leq \inf_{u\in T_1x}\|(x-u)-(y-v_0^*)\|.\end{aligned}$$

As a consequence,

$$\psi(\|x-y\|)\leq H_d((\mathrm{I_d}-T_1)y,(\mathrm{I_d}-T_1)x).$$

Interchanging the roles of x, y, we obtain that the multivalued mapping $(\mathrm{I_d}-T_1)$ is ψ-expansive.

Claim 2. *For all $x, y \in D$, $T_1x+T_2y\in D$.* For this purpose, let $u_1\in T_1x$ and $u_2\in T_2y$. Then there exist $v_1\in S_{F,x}$ and $v_2\in S_{G,y}$ such that $u_1(t)=\int_0^t k_1(t,s)v_1(s)\mathrm{d}s$ and $u_2(t)=\int_0^t k_2(t,s)v_2(s)\mathrm{d}s$, $t\in J$. Let $u_0\in S_{F,x}$ and $v_0\in S_{F,0}$ satisfy condition (H_3). Then for all $t\in[0,1]$, $v\in F(t,x)$, and all $w\in S_{F,x}$,

$$\begin{aligned}|v(t)| &\leq |v(t)-u_0(t)|+|u_0(t)-v_0(t)|+|v_0(t)|\\ &\leq |v(t)-w(t)|+|w(t)-u_0(t)|+\frac{l(t)}{2}\|x\|+\|F(t,0)\|.\end{aligned}$$

Taking the infimum over $w\in S_{F,x}$ yields

$$|v(t)|\leq \frac{l(t)}{2}\|x\|+\|F(t,0)\|.$$

Then,

$$\begin{aligned}|u_1(t)+u_2(t)| &= \left|\int_0^t k_1(t,s)v_1(s)\mathrm{d}s+\int_0^t k_2(t,s)v_2(s)\mathrm{d}s\right|\\ &\leq \int_0^t |k_1(t,s)v_1(s)|+\int_0^t |k_2(t,s)v_2(s)|\\ &\leq \int_0^t |k_1(t,s)|\left(\frac{l(t)}{2}\|x\|+\|F(t,0)\|\right)\mathrm{d}s+\|k_2\|_\infty\|g_r\|_1\\ &\leq \frac{\|k_1\|_\infty\|l\|_1}{2}\|x\|+\|k_1\|_\infty\int_0^t\|F(s,0)\|\mathrm{d}s+\|k_2\|_\infty\|g_r\|_1.\end{aligned}$$

Finally,

$$\|u_1+u_2\|\leq \frac{\|k_1\|_\infty\|l\|_1}{2}r+\|k_1\|_\infty\overline{F}+\|k_2\|_\infty\|g_r\|_1,$$

which implies, by Assumption (H_5), that $\|u_1+u_2\|_\infty\leq r$.

We conclude that the operators T_1, T_2 satisfy all conditions of Corollary 3.4, proving that the functional integral inclusion (4.5) has a solution in X.

□

4.3. Example 3

Our third application makes use of Corollary 3.5. Given a closed bounded interval $J=[0,a],\ a>0$ of the real line, consider the differential inclusion

$$\begin{cases} \left(\frac{x(t)}{f(t,x(t))}\right)' \in k(t)G(t,x(t)),\ t\in J \\ x(0)=x_0\in\mathbb{R}, \end{cases} \tag{4.10}$$

where $f: J\times\mathbb{R}\to\mathbb{R}\backslash\{0\}$ is continuous, $w\in\mathbb{R}$, $k: J\to\mathbb{R}$ is a bounded function, and $G: J\times\mathbb{R}\rightsquigarrow\mathcal{KC}(\mathbb{R})$. By solution of problem (4.10), it is meant a function $x\in AC(J,\mathbb{R})$ such that

(a) the function $t\mapsto\frac{x(t)}{f(t,x(t))}$ is differentiable, and

(b) $\left(\frac{x(t)}{f(t,x(t))}\right)'=v(t),\ t\in J$ for some $v\in L^1(J,\mathbb{R})$ with $v(t)\in G(t,x(t))$, a.e. $t\in J$.

Here, $AC(J,\mathbb{R})$ is the space of all absolutely continuous real-valued functions in J. Problem (4.10) has been discussed in the literature in the particular case where $f(t,x)=1$ and $k(t)=1$, that is:

$$\begin{cases} x'\in G(t,x(t)),\ t\in J, \\ x(0)=x_0\in\mathbb{R} \end{cases} \tag{4.11}$$

(see Aubin and Celina [1, Chapter 2], Deimling [5]). Also, the problem

$$\begin{cases} \left(\frac{x(t)}{f(t,x(t))}\right)' \in G(t,x(t)), \quad t\in J \\ x(0)=x_0\in\mathbb{R} \end{cases} \tag{4.12}$$

was discussed by Dhage [6] under mixed generalized Lipschitz and Carathéodory conditions. Problem 4.10 is investigated here under nonexpansive and Carathéodory conditions. Recall the norm $\|x\|=\sup_{t\in J}|x(t)|$ in $C(J,\mathbb{R})$ and consider the hypotheses:

(H_1) The function f is bounded on $J\times\mathbb{R}\to\mathbb{R}\backslash\{0\}$ with bound $K>0$, and $x\mapsto f(t,x)$ is a nonexpansive mapping.

(H_2) The multivalued operator $G: J\times\mathbb{R}\rightsquigarrow\mathcal{KC}(\mathbb{R})$ is L^1-Carathéodory.

Theorem 4.3. *Further to Hypotheses $(H_1),(H_2)$, assume that*

$$\left|\frac{x_0}{f(0,x_0)}\right|+\|k\|_\infty\|g\|_1<1. \tag{4.13}$$

Then problem 4.11 has a solution in $C(J,\mathbb{R})$.

Proof. Let $D=\overline{B}(0,r)$ be the closed ball in X centered at the origin and with radius $r=K\left(\left|\frac{x_0}{f(0,x_0)}\right|+\|k\|_\infty\|h\|_1\right)$. Define the operators T_1,T_2 for $t\in J$ by

$$T_1x(t)=f(t,x(t)) \tag{4.14}$$

and

$$T_2x(t)=\left\{u\in X:\ u(t)=\frac{x_0}{f(t,x_0)}+\int_0^t k(s)v(s)\mathrm{d}s,\quad v\in S_{G,x}\right\}. \tag{4.15}$$

Then the differential inclusion (4.11) is equivalent to the operational inclusion:

$$x(t) \in T_1x(t)T_2x(t), \quad t \in J. \tag{4.16}$$

We will show that the multivalued operators T_1 and T_2 satisfy all conditions of Corollary 3.5. Clearly, T_1 is well defined and T_2 is also well defined for $S_{G,x} \neq \emptyset$, for each $x \in X$. Moreover by Hypothesis (H_1), $T_1 : X \rightsquigarrow \mathcal{CLC}(\mathbb{R})$ is a nonexpansive multivalued mapping. The fact that T_2 takes bounded convex values, is lower semi-continuous on D, and compact can be proved as Theorem 4.1.

Claim 1. $\|T_2(D)\| \leq 1$. Given $x \in D$, there exist $w \in T_2x$ and $u \in S_{G,x}$ such that

$$w(t) = \frac{x_0}{f(0,x_0)} + \int_0^t k(s)u(s)\mathrm{d}s.$$

We have

$$\begin{aligned} |w(t)| &= \left| \frac{x_0}{f(0,x_0)} + \int_0^t k(s)u(s)\mathrm{d}s \right| \\ &\leq \left| \frac{x_0}{f(0,x_0)} \right| + \int_0^t |k(s)||u(s)|\mathrm{d}s \\ &\leq \left| \frac{x_0}{f(0,x_0)} \right| + \int_0^t |k(s)||g(s)|\mathrm{d}s. \end{aligned}$$

Then,

$$\|w\| \leq \left| \frac{x_0}{f(0,x_0)} \right| + \|k\|_\infty \|g\|_1.$$

Our claim then follows from condition (4.13).

Claim 2. *T_1xT_2y is a convex subset of D for each $x, y \in D$.* Let $x, y \in D$ be arbitrary and $w, z \in D$. Then there exist $u, v \in S_{G,y}$ such that

$$w = f(t,x(t)) \left(\frac{x_0}{f(0,x_0)} + \int_0^t k(s)u(s)\mathrm{d}s \right)$$

and

$$z = f(t,x(t)) \left(\frac{x_0}{f(0,x_0)} + \int_0^t k(s)v(s)\mathrm{d}s \right).$$

For $\lambda \in [0,1]$, we have

$$\begin{aligned}\lambda w + (1-\lambda)z &= \lambda f(t,x(t))\left(\frac{x_0}{f(0,x_0)} + \int_0^t k(s)u(s)\mathrm{d}s\right)\\ &\quad + (1-\lambda)f(t,x(t))\left(\frac{x_0}{f(0,x_0)} + \int_0^t k(s)v(s)\mathrm{d}s\right)\\ &= f(t,x(t))\left(\lambda\left(\frac{x_0}{f(0,x_0)} + \int_0^t k(s)u(s)\mathrm{d}s\right)\right.\\ &\quad \left. + (1-\lambda)\left(\frac{x_0}{f(0,x_0)} + \int_0^t k(s)v(s)\mathrm{d}s\right)\right)\\ &= f(t,x(t))\left(\frac{x_0}{f(0,x_0)} + \int_0^t k(s)(\lambda u(s) + (1-\lambda)v(s))\mathrm{d}s\right).\end{aligned}$$

Since $G(t,y(t))$ is convex, then $\tilde{w}(t) = \lambda u(t) + (1-\lambda)v(t) \in G(t,y(t))$, for all $t \in J$. Then $\tilde{w} \in S_{G,y}$ and $\lambda w + (1-\lambda)z \in T_1xT_2y$, showing that T_1xT_2y is a convex subset of X. Moreover,

$$\begin{aligned}|\lambda w(t) + (1-\lambda)z(t)| &\le |f(t,x(t))|\left|\frac{x_0}{f(0,x_0)}\right| + \int_0^t |k(s)||w'(s)|\mathrm{d}s\\ &\le |f(t,x(t))|\left|\frac{x_0}{f(0,x_0)}\right| + \|k\|\int_0^t g(s)\mathrm{d}s.\end{aligned}$$

Hence,

$$\|\lambda w + (1-\lambda)z\| \le \|f\|\left|\frac{x_0}{f(0,x_0)}\right| + \|g\|\|h\|_1 \le K\left(\left|\frac{x_0}{f(0,x_0)}\right| + \|k\|\|g\|_1\right).$$

As a consequence, T_1xT_2y is a convex subset of D, for all $x, y \in D$.

Claim 3. $(\mathrm{I_d} - T_1(.)T_2(y))$ *is* ψ*-expansive, for all* $y \in D$. Let $x, y \in D$, $u \in S_{G,y}$, and $\psi : [0,\infty) \to [0,\infty)$ given by

$$\psi(r) = \left(1 - \left|\frac{x_0}{f(0,x_0)}\right| - \|k\|_\infty\|g\|_1\right) r.$$

Clearly, ψ is a continuous nondecreasing function. In addition, the following estimates hold:

$$\begin{aligned}\psi(\|x-z\|) &= \left(1 - \left|\frac{x_0}{f(0,x_0)}\right| - \|k\|_\infty\|g\|_1\right)\|x-z\|\\ &= \left\|x - z + \left((f(t,x(t)) - f(t,z(t))\left(\frac{x_0}{f(0,x_0)} + \int_0^t k(s)u(s)\mathrm{d}s\right)\right)\right.\\ &\quad \left. - \left((f(t,x(t)) - f(t,z(t))\left(\frac{x_0}{f(0,x_0)} + \int_0^t k(s)u(s)\mathrm{d}s\right)\right)\right\|\\ &\quad - \left(\left|\frac{x_0}{f(0,x_0)}\right| + \|k\|_\infty\|g\|_1\right)\|x-z\|\\ &\le \left\|x - f(t,x(t))\left(\frac{x_0}{f(0,x_0)} + \int_0^t k(s)u(s)\mathrm{d}s\right)\right.\end{aligned}$$

$$
\begin{aligned}
& -z + f(t,z(t))\left(\frac{x_0}{f(0,x_0)} + \int_0^t k(s)u(s)\mathrm{d}s\right)\Bigg\| \\
& \|f(t,x(t)) - f(t,z(t))\| \left\| \frac{x_0}{f(0,x_0)} + \int_0^t k(s)u(s)\mathrm{d}s \right\| \\
& - \left(\left| \frac{x_0}{f(0,x_0)} \right| + \|k\| \|g\|_1 \right) \|x - z\| \\
\leq & \left\| x - f(t,x(t))\left(\frac{x_0}{f(0,x_0)} + \int_0^t k(s)u(s)\mathrm{d}s\right) \right. \\
& -z + f(t,z(t))\left(\frac{x_0}{f(0,x_0)} + \int_0^t k(s)u(s)\mathrm{d}s\right)\Bigg\| \\
& + \|x - z\| \left(\left| \frac{x_0}{f(0,x_0)} \right| + \|k\| \|g\|_1 \right) \\
& - \left(\left| \frac{x_0}{f(0,x_0)} \right| + \|k\|_\infty \|g\|_1 \right) \|x - z\|.
\end{aligned}
$$

Interchanging the roles of x and z yields the estimate:

$$\psi(\|x - z\|) \leq H(\mathrm{I_d}(x) - T_1(x)T_2(y), \mathrm{I_d}(z) - T_1(z)T_2(y)), \quad \text{for all } x, y \in D.$$

Finally, T_1 and T_2 satisfy all conditions of Corollary 3.5, showing that the operator inclusion T_1T_2 has a fixed point, solution to problem (4.11).

□

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