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Spacelike Hypersurfaces in Spatially Parabolic Standard Static Spacetimes and Calabi–Bernstein-Type Problems

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Abstract. Complete spacelike hypersurfaces in spatially parabolic standard static spacetimes are studied. Under natural boundedness assumptions, we show how the parabolicity of the base is inherited by any spacelike hypersurface and vice versa. Moreover, we give new uniqueness and non-existence results for complete spacelike hypersurfaces in these ambient spacetimes as well as solve new Calabi–Bernstein-type problems.

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1. Introduction

The notion of symmetry is basic in Physics. In General Relativity, symmetry is usually based on the assumption of the existence of a one-parameter group of transformations generated by a Killing or, more generally, a conformal vector field. In fact, a usual simplification for the search of exact solutions to the Einstein equation is to assume a priori the existence of such an infinitesimal symmetry (see [12,15]). A complete general approach to symmetries can be found in [14]. Indeed, a symmetry in a spacetime $(\overline{M}^{n+1}, \overline{g})$ is due to the existence of a conformal vector field K. Let us recall that K is a conformal vector field provided that the Lie derivative of the metric tensor along K satisfies

$$\mathcal{L}_K \overline{g} = 2\rho \overline{g},$$

where ρ is a smooth function known as the conformal factor of K. Equivalently, a vector field K is conformal if and only if the stages ϕ_t of all its local flows are conformal maps. In the particular case where ρ identically vanishes, K is called a Killing vector field and the stages of all its local flows are isometries.

Moreover, we can classify a spacetime depending on the causal character of this infinitesimal symmetry. In particular, a Lorentzian manifold is called a stationary spacetime, if it admits a timelike Killing vector field K, since in this case we are able to choose around each point of the spacetime p such that $K_p \neq 0$ coordinates $(t, x^1, \ldots, x^n)$ such that $K = \partial_t$, being all the components of the metric independent of t. In fact, not only do the stationary observers along K see a non-changing metric but also they measure that $\mathcal{E} = \overline{g}(\partial_t, \gamma)$ is constant for any geodesic γ. Hence, photons and particles in free fall have constant energy $\mathcal{E}$ for these observers. Moreover, if this non-vanishing Killing vector field K is also irrotational (i.e., the orthogonal distribution $K^\perp$ is involutive), then a local warped product structure appears and the spacetime is called static (see, for instance, [1]). In fact, when this structure is global, this spacetime is known as a standard static spacetime.

More precisely, by a standard static spacetime, we denote the product manifold $\overline{M} = P \times \mathbb{R}$, where (P, σ) is an $n(\geq 2)$-dimensional (connected) Riemannian manifold, endowed with the Lorentzian metric

$$\overline{g} = \pi_P^*(\sigma) - h(\pi_P)^2 \pi_{\mathbb{R}}^*(\mathrm{d}t^2), \tag{1}$$

where π_P and $\pi_{\mathbb{R}}$ denote, respectively, the projections on P and $\mathbb{R}$, and h is a smooth positive function on P. This spacetime $(\overline{M} = P \times \mathbb{R}, \overline{g})$ is a warped product in the sense of [22, Ch. 7] with base (P, σ), fiber $(\mathbb{R}, -\mathrm{d}t^2)$ and warping function h. In [22], it was proved that any static spacetime is locally isometric to a standard static one (see also [1,28] for sufficient conditions for a static spacetime to be standard). In addition, in any static spacetime, there exists a distinguished foliation where the leaves are orthogonal to K. In particular, for the standard static case, the leaves of this foliation are given by the level hypersurfaces of the function $\pi_{\mathbb{R}}$, obtaining a foliation of the spacetime by totally geodesic spacelike hypersurfaces. The importance of standard static spacetimes also comes from the fact that they include some classical spacetimes, such as Lorentz–Minkowski spacetime, Einstein static universe as well as models that describe a universe where there is only a spherically symmetric non-rotating mass, as a star or a black hole, like exterior Schwarzschild spacetime [27].

On the other hand, the study of spacelike hypersurfaces has a long and fruitful history in General Relativity [21]. In fact, the existence of constant mean curvature spacelike hypersurfaces is necessary for the study of the structure of singularities in the space of solutions of Einstein's equations [4]. They also play a crucuial role in the proof of the positivity of the gravitational mass [30] as well as in the Cauchy problem for the Einstein's field equations [20]. Consequently, existence and uniqueness results for this type of hypersurfaces appear as fundamental topics in the literature. Indeed, one of the most important uniqueness results for spacelike hypersurfaces was the proof of the Calabi–Bernstein conjecture [9] for maximal hypersurfaces in the $(n+1)$-dimensional Lorentz–Minkowski spacetime given by Cheng and Yau

[10]. More recently, uniqueness results of this kind have been extended to a wide variety of spacetimes (see, for instance, [23–25]).

As far as spacelike hypersurfaces in standard static spacetimes are concerned, there have been obtained several results on the uniqueness of the splitting of these spacetimes in terms of their usual orthogonal decomposition for spatially closed standard static spacetimes in [1,29]. However, guaranteeing the uniqueness of splitting for spatially open standard static spacetimes reamins an open problem, since it is known that there exist spacetimes with different splittings of type (1), such as $\mathbb{L}^n$ or other nontrivial cases [31].

Considering all this, our aim in this article will be to study spacelike hypersurfaces in spatially open standard static spacetimes whose mean curvature function satisfies certain conditions. The importance of these spatially open models comes from the fact that despite the historical relevance of spatially closed models of the universe, recent observations suggest that our universe is actually spatially open [11]. Moreover, a spatially closed spacetime violates the so-called holographic principle [5], making spatially open models more suitable for a possible quantum theory of gravity [7].

To deal with a certain class of spatially open spacetimes, in [26], it was introduced the concept of spatially parabolic Generalized Robertson–Walker spacetime as one whose fiber is parabolic, i.e., it is a complete noncompact Riemannian manifold where the only superharmonic functions bounded from below are the constants. This notion was introduced as a natural counterpart and a first generalization of spatially closed GRW spacetimes (recall that the fiber of a spatially closed GRW spacetime is a compact Riemannian manifold [2, Prop. 3.2]). Taking this into account, we will introduce here the following notion: a standard static spacetime will be said to be spatially parabolic if its base is parabolic. Note that, in particular, our results improve and extend known results for spatially closed standard static spacetimes. The parabolicity of the base will provide wealth from the perspective of geometric analysis in the study of spacelike hypersurfaces. Furthermore, from a physical standpoint, the parabolicity of the base of a standard static spacetime presents a suggestive property. If we imagine the universe as a fluid where the galaxies are its molecules and we send a probe to the space whose motion can be approached by a Brownian one, parabolicity would imply that this probe could be observed in any region of the universe due to the recurrence of the Brownian motion in any parabolic Riemannian manifold [16].

Our paper is organized as follows. In Sect. 2, we define the geometric objects that we will study along the paper as well as the ambient spacetimes where we are going to work. Moreover, we also obtain the conformal change of metric that will enable us to get our main results. Section 3 is devoted to establish the following technical result (Proposition 7), which gives sufficient conditions for a spacelike hypersurface in a standard static spacetime to inherit the parabolicity from the universal Riemannian cover of the base.

Let $\psi : M \to \overline{M} = P \times_h \mathbb{R}$ be a complete spacelike hypersurface with bounded hyperbolic angle in a standard static spacetime whose base *has parabolic universal Riemannian covering. Then, M is parabolic with respect to the induced metric g.*

Thanks to this result and using a conformal change of metric, we are able to obtain in Sect. 4 our main parametric result for spacelike hypersurfaces in these ambiences (Theorem 11).

Consider a standard static spacetime $\overline{M} = P \times_h \mathbb{R}$ whose base has parabolic universal Riemannian covering. Let $\psi : M \longrightarrow \overline{M}$ be a complete spacelike hypersurface in $\overline{M}$ bounded away from future (resp. past) infinity with bounded hyperbolic angle and non-negative (resp. non-positive) mean curvature function such that the restriction of the warping function h to M satisfies $\inf h > 0$ and $\sup h < \infty$. Then, M is a (totally geodesic) spacelike slice.

As a consequence of this theorem, we obtain Corollary 12 for complete spacelike hypersurfaces with constant mean curvature.

Consider a standard static spacetime $\overline{M} = P \times_h \mathbb{R}$ whose base has parabolic universal Riemannian covering. Let $\psi : M \longrightarrow \overline{M}$ be a complete spacelike hypersurface with constant mean curvature $H \geq 0$ (resp. $H \leq 0$) in $\overline{M}$ bounded away from future (resp. past) infinity with bounded hyperbolic angle such that the restriction of the warping function h to M satisfies $\inf h > 0$ and $\sup h < \infty$. Then, M is a totally geodesic spacelike slice.

Finally, in Sect. 5, we show how to derive from the previous uniqueness results some Calabi–Bernstein-type results for the non-parametric case (Theorem 20).

Let $h : P \longrightarrow \mathbb{R}$ be a positive smooth function such that $\inf h > 0$ and $\sup h < \infty$ and consider any non-negative (resp. non-positive) function $H \in C^\infty(P \times \mathbb{R})$. The only entire solutions bounded from above (resp. bounded from below) of the problem

$$\operatorname{div}\left(\frac{h\ Du}{n\sqrt{1-h^2|Du|^2}}\right) + \frac{\sigma(Dh, Du)}{n\sqrt{1-h^2|Du|^2}} = H, \tag{E.1}$$

$$|Du| < \frac{\lambda}{h}, \quad 0 < \lambda < 1. \tag{E.2}$$

on a parabolic Riemannian manifold P are the constant functions.

2. Preliminaries

Let us consider a standard static spacetime $\overline{M} = P \times_h \mathbb{R}$ as defined in the previous section. In this ambience, the Killing vector field $\partial_t = \partial/\partial t$ is timelike; therefore, $\overline{M}$ is time-orientable. Moreover, in these spacetimes, there is a distinguished family of observers given by the integral curves of the unitary timelike vector field $\frac{1}{h}\partial_t$. Since the vector field ∂_t is Killing, its local flows are isometries in $\overline{M}$ that preserve the restpaces of the observers in $\frac{1}{h}\partial_t$. Physically, this means that the spatial universe measured by each observer in $\frac{1}{h}\partial_t$ always looks the same.

Particularly, in this article we will deal with a certain class of standard static spacetimes, namely, spatially parabolic ones. By definition, we say that a standard static spacetime $\overline{M} = P \times_h \mathbb{R}$ is spatially parabolic, if its base P is a parabolic Riemannian manifold. Due to technical reasons that will be discussed in Sect. 3, we will also study standard static spacetimes whose basis have parabolic universal Riemannian covering. In fact, in Sect. 3 we will give sufficient conditions for any spacelike hypersurface to inherit the parabolicity from the base (or from its universal Riemannian covering) and vice versa.

Given an n-dimensional manifold M, an immersion $\psi : M \to \overline{M}$ in a standard static spacetime $\overline{M} = P \times_h \mathbb{R}$ is said to be spacelike if the Lorentzian metric (1) induces, via ψ, a Riemannian metric g on M. In this case, M is called a spacelike hypersurface. We will denote by $\tau := \pi_{\mathbb{R}} \circ \psi$ the restriction of the projection $\pi_{\mathbb{R}}$ along the immersion ψ. Moreover, denoting by $\partial_t^T := \partial_t + \overline{g}(N, \partial_t)N$ the tangential component of ∂_t along ψ, it is easy to check that the gradient of τ on a spacelike hypersurface M is

$$\nabla \tau = -\frac{1}{h^2}\partial_t^T, \tag{2}$$

where h is evaluated on M.

Given $\psi : M \to \overline{M}$ an n-dimensional spacelike hypersurface immersed in a standard static spacetime $(\overline{M} = P \times_h \mathbb{R}, \overline{g})$ the time orientation of $\overline{M}$ allows us to take for every spacelike hypersurface M a unique unitary timelike vector field $N \in \mathfrak{X}^{\perp}(M)$ globally defined on M with the same time orientation as ∂_t, i.e., such that $\overline{g}(N, \partial_t) < 0$. From the wrong-way Schwarz inequality, we have $\overline{g}(N, \frac{1}{h}\partial_t) \leq -1$, with equality holding at a point $p \in M$ if and only if $N = \frac{1}{h}\partial_t$ at p. Hence, we can define the hyperbolic angle function θ on any point $p \in M$ as $\overline{g}(N, \frac{1}{h}\partial_t) = -\cosh\theta$. Therefore, from (2) we get

$$|\nabla \tau|^2 = \frac{\overline{g}(N, \partial_t)^2 - h^2}{h^4} = \frac{\sinh^2\theta}{h^2}. \tag{3}$$

We will denote by A the shape operator associated with N. Then, the mean curvature function associated with N is given by $H := -(1/n)\mathrm{trace}(A)$. As it is well known, the mean curvature is constant if and only if the spacelike hypersurface is, locally, a critical point of the n-dimensional area functional for compactly supported normal variations, under certain constraints of the volume [8]. When the mean curvature vanishes identically, the spacelike hypersurface is called a maximal hypersurface [21].

In any standard static spacetime $\overline{M} = P \times_h \mathbb{R}$, there is a remarkable family of spacelike hypersurfaces, namely, its spacelike slices $P \times t_0$, $t_0 \in I$. It can be easily seen that a spacelike hypersurface in $\overline{M}$ is a (piece of) spacelike slice if and only if the function τ is constant on M. Furthermore, the shape operator of any spacelike slice vanishes identically, i.e., they are totally geodesic. Indeed, there exists a foliation in any standard static spacetime by its totally geodesic spacelike slices.

Let $\psi : M \to \overline{M}$ be an n-dimensional spacelike hypersurface immersed in a standard static spacetime $\overline{M} = P \times_h \mathbb{R}$. Using that ∂_t is Killing and the Gauss and Weingarten formulas of the spacelike hypersurface M in $\overline{M}$, we get

$$g(\nabla_X \partial_t^T, X) + \overline{g}(N, \partial_t) g(AX, X) = 0, \tag{4}$$

for all $X \in \mathfrak{X}(M)$. From (2) and taking a local orthonormal reference frame $\{E_1, \ldots, E_n\}$ on (M, g) we can compute the Laplacian of τ, obtaining

$$\Delta\tau = -\sum_{i=1}^{n} g\left(\nabla_{E_i}\left(\frac{1}{h^2}\partial_t^T\right), E_i\right) = -\sum_{i=1}^{n} E_i\left(\frac{1}{h^2}\right)\, g(\partial_t^T, E_i)$$
$$-\frac{1}{h^2}\sum_{i=1}^{n} g(\nabla_{E_i}\partial_t^T, E_i). \tag{5}$$

Using (4) in (5), we obtain

$$\Delta\tau = -\frac{2}{h}\sum_{i=1}^{n} g(E_i, \nabla h)\; g(\nabla\tau, E_i) + \frac{\overline{g}(N, \partial_t)}{h^2}\sum_{i=1}^{n} g(AE_i, E_i). \tag{6}$$

Hence, the Laplacian of τ on (M, g) is given by

$$\Delta\tau = -\frac{2}{h}\; g(\nabla h, \nabla\tau) + nH\;\frac{\cosh\theta}{h}, \tag{7}$$

where H is the mean curvature function associated with N. Now, the key point in our reasoning is to endow our spacelike hypersurface M, when its dimension is $n \geq 3$, with the conformal metric given by

$$\widehat{g} = h^{\frac{4}{n-2}} g, \tag{8}$$

being g the metric induced on M by the spacetime. Knowing how differential operators behave under conformal changes of metric (see for instance [6]) we can get from (7) that the Laplacian of τ on $(M, \widehat{g})$ is

$$\widehat{\Delta}\tau = n\; h^{-\frac{n+2}{n-2}}\; H\cosh\theta. \tag{9}$$

In particular, the spacelike hypersurface is maximal ($H = 0$) if and only if the function τ is $\widehat{\Delta}$-harmonic on M.

Remark 1. Although in the 2-dimensional case this conformal metric does not work, we can make the following trick to use it. In fact, given a standard stationary spacetime $(P \times_h \mathbb{R}, \overline{g})$, we may consider a new one of higher dimension $(\mathbb{S}^2 \times P \times_h \mathbb{R}, g_{_{\mathbb{S}^2}} + \overline{g})$. Now, given a spacelike surface $\psi : S \to P \times_h \mathbb{R}$, take a new spacelike hypersurface in the extended spacetime, $\psi' : \mathbb{S}^2 \times S \to \mathbb{S}^2 \times P \times_h \mathbb{R}$ given by $\psi'(s, p) = (s, \psi(p))$, with $s \in \mathbb{S}^2$ and $p \in S$. Observe that the new mean curvature vector field of ψ' is the same than the one of ψ lifted to the extended spacetime. In this setting, we are able to apply the conformal change. Note that the manifold $\mathbb{S}^2$ may be replaced with $\mathbb{S}^1$. However, we choose $\mathbb{S}^2$ because it is simply connected.

Remark 2. On the other hand, when $n = 2$, there is no suitable conformal change for g. Indeed, let us consider $F(g) := h\Delta\tau + 2g(\nabla h, \nabla\tau)$ and the conformal metric $\widehat{g} := e^{2u} g$, with $u \in C^\infty(M)$ arbitrary. Now, making use of the expresions for the gradient and Laplacian operators under conformal changes of metric (see [6]), it is not difficult to see that

$$F(e^{2u}g) = e^{-2u}F(g), \quad \text{for all } u \in C^\infty(M).$$

3. Parabolicity of Spacelike Hypersurfaces

A complete (non-compact) $n(\geq 2)$-dimensional Riemannian manifold is said to be parabolic, if it admits no non-constant positive superharmonic functions on it (see [18]). The study of parabolicity can be approached from different points of view.

From a mathematical perspective, parabolicity of Riemannian surfaces is closely related to their Gaussian curvature. Indeed, a key result by Ahlfors and Blanc-Fiala-Huber [18] avers that a complete (non-compact) Riemannian surface with non-negative Gaussian curvature is parabolic. In higher dimension, parabolicity of Riemannian manifolds has no clear relation with the sectional curvature. Indeed, the Euclidean space $\mathbb{R}^n$ is parabolic if and only if $n \leq 2$. Nevertheless, there are sufficient conditions to ensure the parabolicity of a Riemannian manifold of arbitrary dimension based on the volume growth of its geodesic balls (see [3] and references therein).

A crucial fact that we will use in this article is that parabolicity is invariant under quasi-isometries, [16, Cor. 5.3]. We recall that given two Riemannian manifolds (M_1, g_1) and (M_2, g_2), we say that a global diffeomorphism φ from M_1 onto M_2 is a quasi-isometry, if there exists a constant $c \geq 1$ such that

$$c^{-1}\, g_1(v, v) \leq g_2(\mathrm{d}\varphi(v), \mathrm{d}\varphi(v)) \leq c\, g_1(v, v)$$

for all $v \in T_pM_1$, $p \in M_1$.

Moreover, we can also see that if a Riemannian covering $\widetilde{M}$ of a Riemannian manifold M is parabolic, then M is also parabolic. This is due to the fact that M if admitted a non-constant positive superharmonic function, then the composition of this function with the covering map would result in another function with the same properties on $\widetilde{M}$, contradicting the parabolicity of $\widetilde{M}$.

It is a relevant fact that parabolicity is equivalent to the recurrence of the Brownian motion on a Riemannian manifold [16]. This has an intriguing physical interpretation: any particle in the Riemannian manifold will pass through any of its open subsets at some arbitrarily large time.

Now, we will give some sufficient conditions for a spacelike hypersurface in a standard static spacetime to inherit the parabolicity from the base and vice versa. Let $\psi : M \to \overline{M} = P \times_h \mathbb{R}$ be a spacelike hypersurface in a standard static spacetime. The induced metric on M by 1 is given by

$$g = \varphi^*(\sigma) - h(\varphi)^2(\pi_{\mathbb{R}} \circ \psi)^*(\mathrm{d}t^2), \tag{10}$$

being $\varphi := \pi_P \circ \psi$, with $\pi_P : \overline{M} \longrightarrow P$ the projection onto P. For any tangent vector v on M, we obtain

$$g(v, v) = \sigma(\mathrm{d}\varphi(v), \mathrm{d}\varphi(v)) - h(\varphi)^2 g(\nabla\tau, v)^2 \leq \sigma(\mathrm{d}\varphi(v), \mathrm{d}\varphi(v)). \tag{11}$$

We also have from (classical) Schwarz inequality

$$g(v,v) \geq \sigma(\mathrm{d}\varphi(v), \mathrm{d}\varphi(v)) - h(\varphi)^2 g(\nabla\tau, \nabla\tau) g(v,v). \tag{12}$$

Moreover, using (3), we obtain from (12)

$$g(v,v) \geq \frac{1}{\cosh^2\theta}\, \sigma(\mathrm{d}\varphi(v), \mathrm{d}\varphi(v)). \tag{13}$$

Hence, from (11) and (13), we get

$$\frac{1}{\cosh^2\theta}\, \sigma(\mathrm{d}\varphi(v), \mathrm{d}\varphi(v)) \leq g(v,v) \leq \sigma(\mathrm{d}\varphi(v), \mathrm{d}\varphi(v)). \tag{14}$$

Now, using (14), we can get the following result

Lemma 3. *Let $\psi : M \to \overline{M} = P \times_h \mathbb{R}$ be a spacelike hypersurface in a standard static spacetime whose hyperbolic angle is bounded. Then, there is a constant $c \geq 1$ such that the projection of M on the base P, φ satisfies*

$$c^{-1}\, \sigma(\mathrm{d}\varphi(v), \mathrm{d}\varphi(v)) \leq g(v,v) \leq c\, \sigma(\mathrm{d}\varphi(v), \mathrm{d}\varphi(v)), \tag{15}$$

for all $v \in T_pM$, $p \in M$.

Compare with [26, Lemma 4.1] where the restriction of the warping function of the GRW spacetime was assumed to be bounded and having positive infimum.

Remark 4. The boundedness of the hyperbolic angle is an assumption with a clear physical meaning. At any point p of a spacelike hypersurface M in a standard static spacetime $\overline{M}$, we have two families of instantaneous observers [27], namely, the ones given by the unitary timelike vector field $\mathcal{T} = \frac{1}{h}\, \partial_t$ at $p \in M$ and the instantaneous normal observers, which are given by the velocities of the integral curves of the unitary timelike normal vector field N_p. From the orthogonal decomposition $N_p = \mathcal{E}(p)\mathcal{T}_p + N_p^P$ we obtain that $\cosh\theta(p)$ coincides with the energy $\mathcal{E}(p)$ that $\mathcal{T}_p$ measures for N_p. Moreover, the velocity measured by $\mathcal{T}_p$ for N_p is $v(p) := \frac{1}{\cosh\theta(p)} N_p^P$ [27, p. 67]. Hence, if the hyperbolic angle is bounded, the relative speed $|v| = |\tanh\theta|$ does not approach the speed of light $c = 1$ and the energy measured at each point $\mathcal{E}(p)$ remains bounded for any $p \in M$.

We will also need the following standard result on covering spaces (see for instance [17]).

Lemma 5. *Let $p : (\widetilde{X}, \widetilde{x}_0) \longrightarrow (X, x_0)$ be a covering space and consider a continuous map $f : (Y, y_0) \longrightarrow (X, x_0)$, where Y is a path-connected and locally path-connected topological space. Then, there exists a lift $\widetilde{f} : (Y, y_0) \longrightarrow (\widetilde{X}, \widetilde{x}_0)$ of f if and only if the induced homomorphism f_* between the corresponding fundamental groups satisfies $f_*(\pi_1(Y, y_0)) \subset p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$.*

Now, we will state sufficient conditions for the universal Riemannian covering space of the base manifold to inherit the parabolicity from a spacelike hypersurface.

Lemma 6. *Let $\overline{M} = P \times_h \mathbb{R}$ be a standard static spacetime. If it admits a simply connected parabolic spacelike hypersurface M with bounded hyperbolic angle, then the universal Riemannian covering space of the base manifold is parabolic; in particular, M is parabolic.*

Proof. Let us consider $\psi : M \to \overline{M} = P \times_h \mathbb{R}$ a parabolic spacelike hypersurface in a standard static spacetime. Using the completeness of M and (11), we know from [13, Lemma 7.3.3] that $\varphi : M \longrightarrow P$ is a covering map. Moreover, if M is simply connected, from Lemma 5 we have a lift of this map $\widetilde{\varphi} : M \longrightarrow \widetilde{P}$, where $\widetilde{P}$ is the universal Riemannian covering of P. Note that $\widetilde{\varphi}$ is in fact a global diffeomorphism (since any two simply connected coverings of a manifold are equivalent). Finally, we can use Lemma 3 to conclude that $\widetilde{\varphi}$ is a quasi-isometry, leading to the parabolicity of $(\widetilde{P}, \widetilde{\sigma})$ and, in particular, the parabolicity of (P, σ). □

Proposition 7. *Let $\psi : M \to \overline{M} = P \times_h \mathbb{R}$ be a complete spacelike hypersurface with bounded hyperbolic angle in a standard static spacetime whose base has parabolic universal Riemannian covering. Then, M is parabolic with respect to the induced metric g.*

Proof. As previously, the projection φ from M onto P is a covering map. Moreover, from Lemma 3, inequalities (15) hold true.

Let us denote by $(\widetilde{M}, \widetilde{g})$ the Riemannian universal covering of (M, g) and by $\widetilde{\pi}_M : \widetilde{M} \longrightarrow M$ the corresponding covering map, which is a local isometry. We can now consider the Riemannian universal covering $(\widetilde{P}, \widetilde{\sigma})$ of the base (P, σ) and use Lemma 5 to get a lift $\widetilde{\rho} : \widetilde{M} \longrightarrow \widetilde{P}$ of the map $\rho := \varphi \circ \widetilde{\pi}_M : \widetilde{M} \longrightarrow P$. We can easily see that $\widetilde{\rho}$ is a global diffeomorphism from $\widetilde{M}$ onto $\widetilde{P}$ and the conclusion of Lemma 3 is rewritten as follows

$$c^{-1}\, \widetilde{\sigma}(d\widetilde{\rho}(\widetilde{v}), d\widetilde{\rho}(\widetilde{v})) \leq \widetilde{g}(\widetilde{v}, \widetilde{v}) \leq c\, \widetilde{\sigma}(d\widetilde{\rho}(\widetilde{v}), d\widetilde{\rho}(\widetilde{v})), \tag{16}$$

for all $\widetilde{v} \in T_{\widetilde{p}}\widetilde{M}$, $\widetilde{p} \in \widetilde{M}$. Therefore, $\widetilde{g}$ is a quasi-isometry from $(\widetilde{M}, \widetilde{g})$ onto $(\widetilde{P}, \widetilde{\sigma})$. Hence, $(\widetilde{M}, \widetilde{g})$ is parabolic; in particular, (M, g) is also parabolic. □

Again, Proposition 7 assumes less hypotheses on the warping function than in the case of spacelike hyupersurfaces in a GRW spacetime [26, Prop. 4.3]. Since every covering of a simply connected manifold is trivial, we can obtain

Corollary 8. *Let $\psi : M \to \overline{M} = P \times_h \mathbb{R}$ be a complete spacelike hypersurface with bounded hyperbolic angle in a spatially parabolic standard static spacetime with simply connected base. Then, M is parabolic with respect to the induced metric g.*

4. Uniqueness Results

To get our uniqueness results, we will need the following lemma, which states sufficient conditions for a complete spacelike hypersurface in a standard static spacetime which is parabolic with the induced metric g to be parabolic with the conformal one $\widehat{g}$ given by (8).

Lemma 9. *Let $\psi : M \to \overline{M} = P \times_h \mathbb{R}$ be a complete spacelike hypersurface in a standard static spacetime which is parabolic with respect to the induced metric g. If the restriction of the warping function h to M satisfies* $\inf h > 0$ *and* $\sup h < \infty$*, then the same spacelike hypersurface M endowed with the conformal metric $\widehat{g}$ is also parabolic.*

As a consequence of Proposition 7 and Lemma 9, we have

Corollary 10. *Let $\overline{M} = P \times_h \mathbb{R}$ be a standard static spacetime whose base P has parabolic Riemannian universal covering. Consider a complete spacelike hypersurface $\psi : M \to \overline{M}$ with bounded hyperbolic angle such that the restriction of the warping function h to M satisfies* $\inf h > 0$ *and* $\sup h < \infty$*. Then, M with the conformal metric $\widehat{g}$ is parabolic.*

We are now in a position to prove the following uniqueness result.

Theorem 11. *Consider a standard static spacetime $\overline{M} = P \times_h \mathbb{R}$ whose base has parabolic universal Riemannian covering. Let $\psi : M \longrightarrow \overline{M}$ be a complete spacelike hypersurface in $\overline{M}$ bounded away from future (resp. past) infinity with bounded hyperbolic angle and non-negative (resp. non-positive) mean curvature function such that the restriction of the warping function h to M satisfies* $\inf h > 0$ *and* $\sup h < \infty$*. Then, M is a (totally geodesic) spacelike slice.*

Proof. If the restriction of the warping function to M satisfies $\inf h > 0$ and $\sup h < \infty$, Corollary 10 states that $(M, \widehat{g} = h^{\frac{4}{n-2}} g)$ is parabolic. Moreover, if the mean curvature function satisfies $H \geq 0$, (9) yields to $\widehat{\Delta}\tau \geq 0$.

Thus, if M is bounded away from future infinity, i.e., on M we have $\sup \tau = T < \infty$, then we get using (9), that the function given by $\tau - T$ is a negative subharmonic function on the parabolic manifold $(M, \widehat{g})$. Therefore, τ is constant on M and thus, M is a spacelike slice. We can analogously prove that if $H \leq 0$ and on M we have $\inf \tau > -\infty$, M is also a spacelike slice. □

In particular, for the case $H =$constant (which includes maximal hypersurfaces), we get

Corollary 12. *Consider a standard static spacetime $\overline{M} = P \times_h \mathbb{R}$ whose base has parabolic universal Riemannian covering. Let $\psi : M \longrightarrow \overline{M}$ be a complete spacelike hypersurface with constant mean curvature $H \geq 0$ (resp. $H \leq 0$) in $\overline{M}$ bounded away from future (resp. past) infinity with bounded hyperbolic angle such that the restriction of the warping function h to M satisfies* $\inf h > 0$ *and* $\sup h < \infty$*. Then, M is a totally geodesic spacelike slice.*

Remark 13. Since every Riemannian covering of a simply connected manifold is trivial, we can obtain from Theorem 11 and Corollary 12 analogous uniqueness results for spatially parabolic standard static spacetimes with simply connected base.

Remark 14. Corollary 12 provides an uniqueness result for complete maximal hypersurfaces as well as non-existence results for complete spacelike hypersurfaces whose constant mean curvature is different from zero. Furthermore, note that the assumption of the spacelike hypersurface to be bounded away from future (resp. past) infinity made in Theorem 11 cannot be omitted to obtain our uniqueness result. As a counterexample, we can easily see that in $\mathbb{L}^{n+1}$, which satisfies all the remaining assumptions in the theorem, there exist spacelike hyperplanes with bounded hyperbolic angle appart from the spacelike slices which are not bounded away from future (resp. past) infinity.

Remark 15. Even though our parabolic Riemannian manifolds are not compact, all the assumptions on the boundedness of the warping function h in Corollary 10 are automatically fulfilled, if M is assumed to be compact. Furthermore, a spacetime is called spatially closed if it admits a closed (compact without boundary) spacelike hypersurface. If M is a compact spacelike hypersurface in a standard static spacetime, its projection onto the base is a covering map; so, the base of a spatially closed standard static spacetime is compact [1]. Taking this into account, from Theorem 11, we can obtain the following uniqueness result for compact spacelike hypersurfaces in a standard static spacetimes with signed mean curvature function.

Corollary 16. *The only compact spacelike hypersurfaces in a spatially closed standard static spacetime with signed mean curvature function are the totally geodesic spacelike slices.*

Particularly, for spacelike hypersurfaces with constant mean curvature, we have

Corollary 17. *The only compact spacelike hypersurfaces with constant mean curvature in a spatially closed standard static spacetime are the totally geodesic spacelike slices.*

Remark 18. Corollary 17 extends the result obtained in [1, Thm. 3] for compact spacelike hypersurfaces in spatially closed standard static spacetimes, where only the maximal case was studied and the hyperbolic angle was assumed to be constant. Indeed, in Corollary 16 we are also extending [29, Prop. 2], where only the maximal case for spacelike hypersurfaces was studied.

5. Calabi–Bernstein-Type Results

Let (P, σ) be a (non-compact) Riemannian manifold and let $h : P \longrightarrow \mathbb{R}$ be a positive smooth function. For each $u \in C^\infty(P)$, we can consider its graph

$$\Sigma_u = \{(p, u(p)) : p \in P\}$$

in the standard static spacetime $\overline{M} = P \times_h \mathbb{R}$. The induced metric on P from the Lorentzian metric on $\overline{M}$ via the graph Σ_u is given by

$$g_u = \sigma - h^2 \mathrm{d}u^2, \tag{17}$$

which is positive definite (i.e., Σ_u is spacelike) if and only if u satisfies

$$|Du| < \frac{1}{h}, \tag{18}$$

where $|Du|$ denotes the norm of the gradient of u in (P, σ). In this case, the future pointing unit normal vector field on Σ_u is

$$N = \frac{h}{\sqrt{1 - h^2|Du|^2}} \left((Du, 0) + \frac{1}{h^2}\, \partial_t \right) \tag{19}$$

and the hyperbolic angle of this graph is

$$\cosh \theta = \frac{1}{\sqrt{1 - h^2|Du|^2}}. \tag{20}$$

On the other hand, standard computations show that the mean curvature function H of Σ_u in $\overline{M}$ with respect to N is given by

$$nH = \operatorname{div} \left(\frac{h\ Du}{\sqrt{1 - h^2|Du|^2}} \right) + \frac{\sigma(Dh, Du)}{\sqrt{1 - h^2|Du|^2}}, \tag{21}$$

being div the divergence operator in (P, σ).

The differential equation (21) for $H = 0$ with the constraint (18) is called the maximal hypersurface equation in $\overline{M}$. The solutions of this equation are maximal graphs in $\overline{M}$. Note that the constraint (18) is nothing but the ellipticity condition of this equation.

We will use our previous results to obtain new Calabi–Bernstein-type theorems in standard static spacetimes. Concretely, we will determine in some relevant cases all the entire solutions of

$$\operatorname{div} \left(\frac{h\ Du}{n\sqrt{1 - h^2|Du|^2}} \right) + \frac{\sigma(Dh, Du)}{n\sqrt{1 - h^2|Du|^2}} = H, \tag{E.1}$$

$$|Du| < \frac{\lambda}{h}, \quad 0 < \lambda < 1. \tag{E.2}$$

Note that the constraint (E.2) may be written as $\cosh \theta < \frac{1}{\sqrt{1-\lambda^2}}$, where θ is the hyperbolic angle of Σ_u. Thus, (E.2) implies that Σ_u has bounded hyperbolic angle. Indeed, this constraint means that the second order partial differential equation (E) is uniformly elliptic.

Since we want to use our previous results for complete spacelike hypersurfaces to give uniqueness results for entire spacelike graphs in these spacetimes, we need to recall that the induced metric on an entire spacelike graph in a standard static spacetime is not necessarily complete, since even in $\mathbb{L}^{n+1}$ there are examples of entire spacelike graphs that are not complete (see for instance [19]). This fact contrasts with the Riemannian case, where a closed hypersurface in a complete Riemannian manifold must also be complete. Hence, if we want to obtain a non-parametric uniqueness result from a parametric one, we need the following lemma.

Lemma 19. *Let $\overline{M} = P \times_h \mathbb{R}$ be a standard static spacetime whose base (P, σ) is a (non-compact) complete Riemannian manifold. Consider an entire spacelike graph Σ_u. If its hyperbolic angle is bounded, then Σ_u is complete with respect to the induced metric from $\overline{M}$.*

Proof. From (17) , the induced metric g_u on P satisfies

$$g_u(v,v) \geq \frac{1}{\cosh^2\theta}\sigma(v,v). \tag{22}$$

where v is tangent to P. Thus, if $\cosh\theta$ is bounded, any divergent curve in (P,σ) with infinite length also has infinite length in with respect to the induced metric g_u; which ends the proof. □

Taking this into account, Theorem 11 and Lemma 19 enable us to obtain the following theorem

Theorem 20. *Let* $h : P \longrightarrow \mathbb{R}$ *be a positive smooth function such that* $\inf h > 0$ *and* $\sup h < \infty$ *and consider any non-negative* (*resp. non-positive*) *function* $H \in C^\infty(P\times\mathbb{R})$. *The only entire solutions bounded from above* (*resp. bounded from below*) *of the problem*

$$\operatorname{div}\left(\frac{h\ Du}{n\sqrt{1-h^2|Du|^2}}\right) + \frac{\sigma(Dh,Du)}{n\sqrt{1-h^2|Du|^2}} = H, \tag{E.1}$$

$$|Du| < \frac{\lambda}{h}, \quad 0<\lambda<1. \tag{E.2}$$

on a parabolic Riemannian manifold P *are the constant functions.*

Remark 21*.* Note that Theorem 20 yields to a non-existence result, if there exists $p \in P\times\mathbb{R}$ such that $H(p)$ is different from zero. In particular, under the assumptions of the theorem, there are no entire spacelike graphs with constant mean curvature different from zero. Moreover, we can obtain as a consequence of this theorem the following Calabi–Bernstein-type result for complete maximal hypersurfaces.

Corollary 22. *Let* $h : P \longrightarrow \mathbb{R}$ *be a positive smooth function that satisfies* $\inf h > 0$ *and* $\sup h < \infty$. *Then, the only entire solutions bounded from above* (*resp. bounded from below*) *to*

$$\operatorname{div}\left(\frac{h\ Du}{\sqrt{1-h^2|Du|^2}}\right) = -\frac{\sigma(Dh,Du)}{\sqrt{1-h^2|Du|^2}}, \tag{E'.1}$$

$$|Du| < \frac{\lambda}{h}, \quad 0<\lambda<1, \tag{E'.2}$$

on a parabolic Riemannian manifold P *are the constant functions.*

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