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Non-orthogonal Fusion Frames of an Analytic Operator and Application to a One-Dimensional Wave Control System

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Abstract. In this paper, we are mainly concerned with a one-dimensional wave control system. We assert the existence of non-orthogonal fusion frames by extending this problem to a theoretical one introduced by Sz. Nagy (Acta Sci Math Szeged 14, 1951). The key idea of this work is based on the estimate inspired from Sz. Nagy (1951) using the spectral analysis method. More precisely, we prove that if the eigenvalues of the unperturbed operator are isolated and with finite multiplicity, we can construct non-orthogonal fusion frames.

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1. Introduction

In the present paper, we consider the following one-dimensional string equation

$$w_{tt}(x,t) - w_{xx}(x,t) = 0, \quad 0 < x < 1, \ t > 0, \tag{1.1}$$

where $w(x,t)$ denotes the transversal displacement of the string depart from its equilibrium position at x and time t. The initial conditions are given by

$$w(x,0) = w_0(x), \quad w_t(x,0) = w_1(x),$$

whereas the Neumann boundary conditions are

$$\begin{cases} w_x(0,t) = u_1(t) \\ w_x(1,t) = u_2(t), \end{cases}$$

where $u_j(t), j = 1,2$ designate the control input. To stabilize Eq. (1.1), many authors such as Kobayashi [15] and Xu [17] used various control strategies. For instance, we adopt the following general linear controllers from [17, 19]

$$\begin{cases} u_1(t) = k_1 w_t(0,t) + \gamma w(0,t) \\ u_2(t) = -k_2 w_t(1,t) - \delta w(1,t), \end{cases} \tag{1.2}$$

where

$$\gamma \geq 0, \quad \delta \geq 0, \quad k_1 \geq 0, \quad k_2 \geq 0 \quad \text{and} \quad k_1 + k_2 \neq 0, \quad \gamma + \delta \neq 0.$$

The abstract formulation of the problem is obtained by considering the following Hilbert space

$$\mathcal{X} := H^1(0,1) \times L^2(0,1),$$

where $H^1(0,1)$ is the usual Sobolev space order 1 and is equipped with the inner product

$$(u,v)_{H^1} := \int_0^1 u'(x)\overline{v'(x)}\mathrm{d}x + \gamma u(0)\overline{v(0)} + \delta u(1)\overline{v(1)}.$$

The inner product of two elements $F = (f_1, f_2)$, $G = (g_1, g_2) \in \mathcal{X}$ is defined by

$$\langle F, G\rangle_{\mathcal{X}} := \int_0^1 f_1'(x)\overline{g_1'(x)}\mathrm{d}x + \gamma f_1(0)\overline{g_1(0)} + \delta f_1(1)\overline{g_1(1)} + \int_0^1 f_2(x)\overline{g_2(x)}\mathrm{d}x.$$

Here and hereafter we use the notation $u'(x) = \frac{\mathrm{d}u}{\mathrm{d}x} = u_x(x)$.

Define the operator A in $\mathcal{X}$ by

$$\begin{aligned} \mathcal{D}(A) := \big\{&(u,v) \in H^2(0,1) \times H^1(0,1) \text{ such that } u'(0) = \gamma u(0) + k_1 v(0), \\ &u'(1) = -\delta u(1) - k_2 v(1)\big\}, \end{aligned} \tag{1.3}$$

$$A(u,v) := (v, u''), \quad (u,v) \in \mathcal{D}(A). \tag{1.4}$$

With the help of these notations, we can rewrite Eq. (1.1) into an evolutionary equation in $\mathcal{X}$:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}W(t) = AW(t), & t > 0, \\ W(0) \quad = W_0, \end{cases}$$

where $W(t) := (w(x,t), w_t(x,t))$ and $W_0 := (w_0(x), w_1(x))$.

It is interesting to note that in control and transport theory, it is very difficult to show that a system satisfies the spectrum determined growth condition. So, using the spectrum of the system operator to verify this property becomes an attractive alternative. The authors in literatures achieved this aim by studying the asymptotic behavior of the spectrum such as in [12,13] or by proving the Riesz basis property of the eigenvectors of the system operator (see [11,17,18]).

However, the Riesz basis property is not always verified. For instance, in Eqs. (1.2) and (1.3) if we assume that $k_1 = 1$ and $k_2 \geq 0$ then the eigenvectors of the system operator (1.4) fail to form a basis since $\sup_n \|P_n\| = \infty$, where P_n is the eigenprojection corresponding to the isolated eigenvalue λ_n of A. So, can we extend the Riesz basis property to the notion of non-orthogonal fusion frame?

The motivations for studying such a generalization are various and meaningful. In fact, the concept of non-orthogonal fusion frame was initially motivated by Cahill et al. [4] as a slight modification of the fusion frame (or frame of subspaces) which has been established by Casazza and Kutyniok [5] as a natural generalization of frame theory. The relevance of this notion, as remarked in [5], is that it gives criteria for constructing a frame for $\mathcal{H}$

by joining sequences of frames for subspaces of $\mathcal{H}$. This notion has been intensely studied during the last years and several new applications has been discovered. The difference between the non-orthogonal fusion frames and the fusion frames is the use of the non-orthogonal projections instead of orthogonal projections.

Among this direction and in order to provide a positive answer, we are interested in this paper by an analytic operator investigated in some valuable papers such as [3,7–10,16]. More precisely, we deal with the following operator

$$T(\varepsilon) := T_0 + \varepsilon T_1 + \varepsilon^2 T_2 + \cdots + \varepsilon^k T_k + \cdots , \tag{1.5}$$

where $\varepsilon \in \mathbb{C}$, T_0 is a closed densely defined linear operator on a separable Hilbert space $\mathcal{H}$ with domain $\mathcal{D}(T_0)$ while $T_1, T_2, \dots$ are linear operators on $\mathcal{H}$ having the same domain $\mathcal{D} \supset \mathcal{D}(T_0)$ and satisfying

$$\|T_k \varphi\| \le q^{k-1}(a\|\varphi\| + b\|T_0\varphi\|)$$

for all $\varphi \in \mathcal{D}(T_0)$ and for all $k \ge 1$, where a, b and q are positive constants.

Motivated by a classical work due to Sz. Nagy [16], we study the existence of non-orthogonal fusion frames related to the perturbed operator (1.5). Indeed, in [16] the author proved that if we designate by P_n the eigenprojection of T_0 related to the eigenvalue λ_n then for $|\varepsilon|$ enough small there exists a sequence of eigenprojections $\{P_n(\varepsilon)\}_{n\in\mathbb{N}^*}$ of $T(\varepsilon)$ that can be developed as entire series of ε as follow:

$$P_n(\varepsilon) = P_n + \varepsilon P_{n,1} + \varepsilon^2 P_{n,2} + \cdots . \tag{1.6}$$

Based on the estimates given in [16], we establish, under sufficient conditions, the existence of a sequence of complex numbers $(\varepsilon_n)_{n\in\mathbb{N}^*}$ and a sequence of eigenprojections $\{P_n(\varepsilon_n)\}_{n\in\mathbb{N}^*}$ of $(T(\varepsilon_n))_{n\in\mathbb{N}^*}$ having the form (1.6) such that the system $\{P_n(\varepsilon_n), v_n\}_{n\in\mathbb{N}^*}$ is a non-orthogonal fusion frame for $\mathcal{H}$, where $(v_n)_{n\in\mathbb{N}^*}$ is a family of weight.

Note here that Eq. (1.6) plays a crucial role in the existence of the non-orthogonal fusion frame related to the perturbed operator (1.5). In fact, it allows us to get a considerable improvement to the results developed in [3] and [10] since the eigenvalue λ_n of T_0 is not necessarily with multiplicity one.

However, the non-orthogonal fusion frame $\{P_n(\varepsilon_n), v_n\}_{n\in\mathbb{N}^*}$ depends on $(\varepsilon_n)_{n\in\mathbb{N}^*}$. Further, it is related to a sequence of operators $(T(\varepsilon_n))_{n\in\mathbb{N}^*}$ and not to the operator (1.5).

In this context and in order to get such improvements, we study the existence of a fixed ε for which the families $\{P_n(\varepsilon), v_n\}_1^N \cup \{P_n(\varepsilon_n), v_n\}_{N+1}^\infty$ and $\{P_n(\varepsilon), v_n\}_1^N \cup \{P_n, v_n\}_{N+1}^\infty$ $(N > 1)$ form non-orthogonal fusion frames for $\mathcal{H}$. More precisely, we show that for $|\varepsilon|$ enough small the first N projections coincide with a sequence of eigenprojections $(P_n(\varepsilon))_{1\le n\le N}$ of $T(\varepsilon)$ that can be developed as an entire series of ε.

Again, these families more or less rely on ε. Indeed, it is clear here that either the first N projections are associated to the perturbed operator $T(\varepsilon)$ or all the projections are related to a sequence of operators $(T(\varepsilon_n))_{n\in\mathbb{N}^*}$. So, can we assure the existence of a fixed complex number ε so that the family $\{P_n(\varepsilon), v_n\}_{n\in\mathbb{N}^*}$ is a non-orthogonal fusion frame for $\mathcal{H}$?

In order to get a positive answer, we provide sufficient conditions assuring the existence of a non-orthogonal fusion frame for a fixed complex number ε. In fact, based on the spectral analysis developed in [16], we show that for $|\varepsilon|$ enough small the family $\{P_n(\varepsilon), v_n\}_{n\in\mathbb{N}^*}$ associated to the perturbed operator (1.5) forms a non-orthogonal fusion frame for $\mathcal{H}$. Here the sequence of eigenprojection $(P_n(\varepsilon))_n$ is associated to $T(\varepsilon)$ and can be developed as an entire series of ε.

The content of the present paper is as follows: In the next section, we state some definitions and preliminary results concerning the concept non-orthogonal fusion frames and function of finite order. We advise that section 3 contributes to the main body of this paper. In this section, we prove the existence of non-orthogonal fusion frames where the eigenvalues of T_0 are not necessarily simple. In the last section, an application to a one-dimensional controlled wave system is presented.

2. Preliminaries

The objective of this section is to present some definitions and basic properties concerning the notion of fusion frames and non-orthogonal fusion frames that will be needed in the sequel. To this interest, let $\mathcal{H}$ denotes a separable Hilbert space and I a countable index set.

We begin this part by introducing the concept of frames of subspaces which have been renamed, recently, as fusion frames. This new concept can be considered as a generalization of frames (see [5]).

Definition 2.1. An operator $P \in \mathcal{L}(\mathcal{H})$ is called a projection if $P^2 = P$. If in addition we have $P^* = P$ then P is called an orthogonal projection. ♢

Definition 2.2 [5]. Let $\{W_i\}_{i\in I}$ be a family of closed subspaces in $\mathcal{H}$ and let $\{w_i\}_{i\in I}$ be a family of weights, i.e., $w_i > 0$ for all $i \in I$. Then, we say that $\{W_i\}_{i\in I}$ is a frame of subspaces with respect to $\{w_i\}_{i\in I}$ for $\mathcal{H}$ (or $\{(W_i, w_i)\}_{i\in I}$ is a fusion frame), if there exist constants $0 < \mathbb{A} \leq \mathbb{B} < \infty$ such that

$$\mathbb{A}\|f\|^2 \leq \sum_{i\in I} w_i^2 \|\pi_{W_i}(f)\|^2 \leq \mathbb{B}\|f\|^2, \quad \text{for all} \quad f \in \mathcal{H},$$

where π_{W_i} is the orthogonal projection onto the subspace W_i. The numbers $\mathbb{A}, \mathbb{B}$ are called the fusion frame bounds. ♢

Now, we give a formal definition of non-orthogonal fusion frames which are a modification of fusion frames with a sequence of non-orthogonal projections operators.

Definition 2.3 [4]. Let $\{W_i\}_{i\in I}$ be a sequence of closed subspaces in $\mathcal{H}$ and let $\{w_i\}_{i\in I}$ be a family of weights, i.e., $w_i > 0$ for all $i \in I$. For each $i \in I$ let P_{W_i} be a projection onto W_i. Then, we say that $\{P_{W_i}, w_i\}_{i\in I}$ is a non-orthogonal fusion frame for $\mathcal{H}$, if there exist constants $0 < \mathcal{A} \leq \mathcal{B} < \infty$ such that

$$\mathcal{A}\|f\|^2 \leq \sum_{i\in I} w_i^2 \|P_{W_i}(f)\|^2 \leq \mathcal{B}\|f\|^2, \quad \text{for all} \quad f \in \mathcal{H}.$$

The numbers $\mathcal{A}, \mathcal{B}$ are called the non-orthogonal fusion frame bounds. $\diamondsuit$

The following Theorem is an extension of [2, Proposition 2.4] to the case of non-orthogonal fusion frame.

Theorem 2.1. *Let $\{P_{W_i}, w_i\}_{i\in I}$ be a non-orthogonal fusion frame for $\mathcal{H}$ with frame bounds $\mathcal{A}$ and $\mathcal{B}$, $\{Z_i\}_{i\in I}$ a family of closed subspaces in $\mathcal{H}$ and $\{v_i\}_{i\in I}$ a family of weights such that $0 < w_i \leq v_i \leq \sqrt{2}w_i$. Suppose that there exists an $0 < R < \mathcal{A}$ such that*

$$\sum_{i\in I} v_i^2 \|P_{W_i}(f) - P_{Z_i}(f)\|^2 \leq R\|f\|^2, \quad \textit{for all} \quad f \in \mathcal{H},$$

where P_{W_i} (respectively, P_{Z_i}) denotes the non-orthogonal projection onto W_i (respectively, Z_i). Then $\{P_{Z_i}, v_i\}_{i\in I}$ is a non-orthogonal fusion frame with frame bounds $\mathcal{A}\left(1-\sqrt{\frac{R}{\mathcal{A}}}\right)^2$ and $\mathcal{B}\left(\sqrt{2}+\sqrt{\frac{R}{\mathcal{B}}}\right)^2$. $\diamondsuit$

Proof. Let $f \in \mathcal{H}$. Using Minkowski's inequality, we have

$$\left(\sum_{i\in I} v_i^2 \|P_{W_i}(f)\|^2\right)^{\frac{1}{2}} \leq \left(\sum_{i\in I} v_i^2 \|P_{W_i}(f) - P_{Z_i}(f)\|^2\right)^{\frac{1}{2}} + \left(\sum_{i\in I} v_i^2 \|P_{Z_i}(f)\|^2\right)^{\frac{1}{2}}.$$

Hence, we obtain

$$\begin{aligned}\left(\sum_{i\in I} v_i^2 \|P_{Z_i}(f)\|^2\right)^{\frac{1}{2}} &\geq \left(\sum_{i\in I} v_i^2 \|P_{W_i}(f)\|^2\right)^{\frac{1}{2}} - \left(\sum_{i\in I} v_i^2 \|P_{W_i}(f) - P_{Z_i}(f)\|^2\right)^{\frac{1}{2}} \\ &\geq \left(\sum_{i\in I} w_i^2 \|P_{W_i}(f)\|^2\right)^{\frac{1}{2}} - \left(\sum_{i\in I} v_i^2 \|P_{W_i}(f) - P_{Z_i}(f)\|^2\right)^{\frac{1}{2}} \\ &\geq \left(\sqrt{\mathcal{A}} - \sqrt{R}\right)\|f\|.\end{aligned}$$

Therefore,

$$\sum_{i\in I} v_i^2 \|P_{Z_i}(f)\|^2 \geq \mathcal{A}\left(1-\sqrt{\frac{R}{\mathcal{A}}}\right)^2 \|f\|^2.$$

Similarly, we have

$$\sum_{i\in I} v_i^2 \|P_{Z_i}(f)\|^2 \leq \mathcal{B}\left(\sqrt{2}+\sqrt{\frac{R}{\mathcal{B}}}\right)^2 \|f\|^2.$$

Indeed, it follows from Minkowski's inequality that

$$\left(\sum_{i\in I} v_i^2 \|P_{Z_i}(f)\|^2\right)^{\frac{1}{2}} \leq \left(\sum_{i\in I} v_i^2 \|P_{W_i}(f) - P_{Z_i}(f)\|^2\right)^{\frac{1}{2}} + \left(\sum_{i\in I} v_i^2 \|P_{W_i}(f)\|^2\right)^{\frac{1}{2}}.$$

Since $w_i \leq v_i \leq \sqrt{2}w_i$, we get

$$\begin{aligned}\left(\sum_{i\in I} v_i^2 \|P_{Z_i}(f)\|^2\right)^{\frac{1}{2}} &\leq \left(\sum_{i\in I} v_i^2 \|P_{W_i}(f) - P_{Z_i}(f)\|^2\right)^{\frac{1}{2}} + \left(2\sum_{i\in I} w_i^2 \|P_{W_i}(f)\|^2\right)^{\frac{1}{2}} \\ &\leq \left(\sqrt{R} + \sqrt{2\mathcal{B}}\right) \|f\|.\end{aligned}$$

Consequently, $\{P_{Z_i}, v_i\}_{i\in I}$ is a non-orthogonal fusion frame with frame bounds $\mathcal{A}\left(1 - \sqrt{\frac{R}{\mathcal{A}}}\right)^2$ and $\mathcal{B}\left(\sqrt{2} + \sqrt{\frac{R}{\mathcal{B}}}\right)^2$. □

We close this part by recalling some results from [20] concerning function of finite order.

Definition 2.4 [20, p. 61]. An entire function $f(z)$ is said to be of exponential type if the inequality

$$|f(z)| \leq Ae^{B|z|}, \quad \forall z \in \mathbb{C} \tag{2.1}$$

holds for some positive constants A and B.

The smallest of constants B such that (2.1) holds is said to be exponential type of f. ♢

Definition 2.5 [20, p. 63]. An entire function $f(z)$ is said to be of finite order if there exists a positive number k such that

$$M(r) = \max\{|f(z)| \text{ such that } |z| = r\} \leq e^{r^k}$$

as soon as r is "sufficiently large", i.e., $r > r(k)$. The greatest lower bound of all positive numbers k for which this is true is called the order of the function. ♢

Remark 2.1 [20, p. 64]. An entire function of exponential type is of finite order at most 1. ♢

Theorem 2.2 [20, Theorem 5, p. 64]. *If $f(z)$ is an entire function of finite order ρ, then*

$$n(r) = O(r^{\rho+\xi})$$

for every positive number ξ, where $n(r)$ denotes the number of zeros of $f(z)$ contained in the disk $\{z \in \mathbb{C} \quad \text{such that} \quad |z| \leq r\}$. ♢

Theorem 2.3 [20, Theorem 6, p. 64]. *If $f(z)$ is an entire function of finite order ρ and if $z_1, z_2, z_3, \ldots$ are its zeros, other than $z = 0$, then the series*

$$\sum_{n=1}^{\infty} \frac{1}{|z_n|^\alpha}$$

is convergent whenever $\alpha > \rho$. ♢

3. Main Results

In this section, we provide some sufficient conditions that ensure the existence of non-orthogonal fusion frames related to the perturbed operator $T(\varepsilon)$ [see Eq. (1.5)] when the eigenvalues of T_0 are not necessarily with multiplicity one. Let then $\mathcal{H}$ be a separable Hilbert space and T_0 a linear operator on $\mathcal{H}$ verifying

$(H1)$ T_0 is closed with domain $\mathcal{D}(T_0)$ dense in $\mathcal{H}$.
$(H2)$ The eigenvalues $(\lambda_n)_{n\in\mathbb{N}^*}$ are isolated and with finite multiplicity.

Let $T_1, T_2, T_3, \ldots$ be linear operators on $\mathcal{H}$ having the same domain $\mathcal{D}$ and satisfying the following hypothesis:

$(H3)$ $\mathcal{D} \supset \mathcal{D}(T_0)$ and there exist positive constants a, b and q such that for all $k \geq 1$

$$\|T_k\varphi\| \leq q^{k-1}(a\|\varphi\| + b\|T_0\varphi\|), \quad \text{for all} \quad \varphi \in \mathcal{D}(T_0).$$

Let ε be a non zero complex number and consider the eigenvalue problem

$$\begin{cases} T_0\varphi + \varepsilon T_1\varphi + \varepsilon^2 T_2\varphi + \cdots + \varepsilon^k T_k\varphi + \cdots = \lambda\varphi \\ \varphi \in \mathcal{D}(T_0) \end{cases}$$

Before going further, we state the following result established in [16] assuring the convergence and the closure of the series $\sum_{k\geq 0}\varepsilon^k T_k$.

Theorem 3.1 [16, Theorem 3]. *Assume that assumptions $(H1)$ and $(H3)$ hold. Then for $|\varepsilon| < \frac{1}{q}$, the series*

$$T_0\varphi + \varepsilon T_1\varphi + \varepsilon^2 T_2\varphi + \cdots + \varepsilon^k T_k\varphi + \cdots$$

converges for all $\varphi \in \mathcal{D}(T_0)$. If $T(\varepsilon)\varphi$ denotes its limit, then $T(\varepsilon)$ is a linear operator with domain $\mathcal{D}(T_0)$ and for $|\varepsilon| < \frac{1}{q+b}$, the operator $T(\varepsilon)$ is closed.$\diamondsuit$

Let $n \in \mathbb{N}^*$ and λ_n the eigenvalue number n of the operator T_0. Since $(T_0 - zI)^{-1}$ is an analytic function of z and $\|(T_0 - zI)^{-1}\|$ is a continuous function of z then we designate by:

$$M_n := \max_{z_n\in\mathcal{C}_n} \|(T_0 - z_nI)^{-1}\|,$$
$$N_n := \max_{z_n\in\mathcal{C}_n} \|T_0(T_0 - z_nI)^{-1}\| = \max_{z_n\in\mathcal{C}_n} \|I + z_n(T_0 - z_nI)^{-1}\|,$$

and

$$\alpha_n := aM_n + bN_n,$$

where $\mathcal{C}_n = \mathcal{C}(\lambda_n, r_n)$ the circle with center λ_n and with radii $r_n = \frac{d_n}{2}$ and $d_n = d(\lambda_n, \sigma(T_0)\backslash\{\lambda_n\})$ is the distance between λ_n and $\sigma(T_0)\backslash\{\lambda_n\}$.

Let P_n be the eigenprojection for the eigenvalue λ_n defined as:

$$P_n := -\frac{1}{2\pi i}\int_{\mathcal{C}_n} (T_0 - z_nI)^{-1}\mathrm{d}z_n.$$

Now, we are ready to state the objective of this section.

Theorem 3.2. *Assume that hypotheses* $(H1)$–$(H3)$ *hold and the family* $\{P_n, w_n\}_{n\in\mathbb{N}^*}$ *is a non-orthogonal fusion frame for* $\mathcal{H}$ *with lower and upper non-orthogonal fusion frame bounds* $\mathcal{A}$ *and* $\mathcal{B}$*, respectively. Then, there exist a sequence of complex numbers* $(\varepsilon_n)_n$ *and a sequence of projections* $\{P_n(\varepsilon_n)\}_{n\in\mathbb{N}^*}$ *having the form*

$$P_n(\varepsilon_n) = P_n + \varepsilon_n P_{n,1} + \varepsilon_n^2 P_{n,2} + \cdots + \varepsilon_n^i P_{n,i} + \cdots ,$$

such that for $|\varepsilon_n| < \frac{\sqrt{6\mathcal{A}}}{\pi n v_n r_n M_n \alpha_n + \sqrt{6\mathcal{A}}(q+\alpha_n)}$ *the family* $\{P_n(\varepsilon_n), v_n\}_{n\in\mathbb{N}^*}$ *forms a non-orthogonal fusion frame for* $\mathcal{H}$*. Here* $\{w_n\}_{n\in\mathbb{N}^*}$ *and* $\{v_n\}_{n\in\mathbb{N}^*}$ *are two families of weights verifying* $w_n \le v_n \le \sqrt{2}w_n$. ♢

Remark 3.1. Notice that in [3, Theorem 3.2] the author proved the existence of a Riesz basis associated to the perturbed operator $T(\varepsilon_n)$ using a spectral analysis method based on the fact that the eigenvalues $(\lambda_n)_{n\in\mathbb{N}*}$ of T_0 are with multiplicity one. However, due to [6, Theorem 2.13], a frame is a Riesz basis if and only if it is ω-linearly independent. Further, the concept of non-orthogonal fusion frame can be viewed as a generalization of the one of the frame. So, Theorem 3.2 can be considered as an extension of [3, Theorem 3.2] to the non-orthogonal fusion frame. On the other hand, we assure the existence of non-orthogonal fusion frame by assuming that $(\lambda_n)_{n\in\mathbb{N}*}$ are not necessarily simple. ♢

Proof of Theorem 3.2. Let $n \in \mathbb{N}^*$ and $z_n \in \mathcal{C}_n$. We have

$$\begin{aligned} T(\varepsilon) - z_n I &= T_0 - z_n I + \varepsilon T_1 + \varepsilon^2 T_2 + \ldots \\ &= \left(I + \varepsilon T_1 (T_0 - z_n I)^{-1} + \varepsilon^2 T_2 (T_0 - z_n I)^{-1} - \ldots\right)(T_0 - z_n I) \\ &= (I+S)(T_0 - z_n I), \end{aligned} \tag{3.1}$$

where

$$S := \sum_{k=1}^{\infty} \varepsilon^k T_k (T_0 - z_n I)^{-1}.$$

Let $\varphi \in \mathcal{H}$ such that $\varphi \neq 0$. It follows from hypothesis $(H3)$ that

$$\begin{aligned} \|S\varphi\| &\le \sum_{k=1}^{\infty} |\varepsilon|^k \|T_k (T_0 - z_n I)^{-1}\varphi\| \\ &\le \|\varphi\| \sum_{k=1}^{\infty} |\varepsilon|^k q^{k-1} (a\|(T_0 - z_n I)^{-1}\| + b\|z_n (T_0 - z_n I)^{-1} + I\|) \\ &\le \|\varphi\| \sum_{k=1}^{\infty} |\varepsilon|^k q^{k-1} \alpha_n. \end{aligned}$$

Hence,

$$\|S\| \le \sum_{k=1}^{\infty} |\varepsilon|^k q^{k-1} \alpha_n.$$

Then, for $|\varepsilon| < \frac{1}{q}$ we have

$$\|S\| \le \frac{\alpha_n |\varepsilon|}{1 - |\varepsilon| q}.$$

So, for $|\varepsilon| < \frac{1}{q+\alpha_n}$ we get $\|S\| < 1$ and $I - S$ is invertible with bounded inverse. Thus, Eq. (4.17) implies that $T(\varepsilon) - z_n$ is invertible with bounded inverse. Consequently, for $|\varepsilon| < \frac{1}{q+\alpha_n}$ we obtain $\mathcal{C}_n \subset \rho(T(\varepsilon))$. Hence, let $P_n(\varepsilon)$ be the eigenprojection on $W_{n,\varepsilon} = R(P_n(\varepsilon))$ defined by

$$P_n(\varepsilon) := \frac{-1}{2\pi i}\int_{\mathcal{C}_n} (T(\varepsilon) - z_n I)^{-1} \mathrm{d}z_n,$$

where $R(P_n(\varepsilon))$ designates the range of $P_n(\varepsilon)$. It follows from [16, p. 134] that for $|\varepsilon| < \frac{1}{q+\alpha_n}$ we have

$$P_n(\varepsilon) = P_n + \varepsilon P_{n,1} + \varepsilon^2 P_{n,2} + \cdots + \varepsilon^i P_{n,i} + \cdots \tag{3.2}$$

and

$$\|P_{n,i}\| \leq r_n M_n \alpha_n (q+\alpha_n)^{i-1}, \quad \text{for all } i \geq 1. \tag{3.3}$$

For each eigenvalue λ_n of T_0, we fix an $\varepsilon_n \in \mathbb{C}$ such that

$$|\varepsilon_n| \in \left]0, \frac{\sqrt{6\mathcal{A}}}{\pi n v_n r_n M_n \alpha_n + \sqrt{6\mathcal{A}}(q+\alpha_n)}\right[.$$

It is easy to see that $|\varepsilon_n| < \frac{1}{q+\alpha_n}$, then Eqs. (3.2) and (3.3) imply that $P_n(\varepsilon_n)$ can be developed as entire series of ε_n as follow

$$P_n(\varepsilon_n) = P_n + \varepsilon_n P_{n,1} + \varepsilon_n^2 P_{n,2} + \cdots + \varepsilon_n^i P_{n,i} + \cdots,$$

with

$$\|P_{n,i}\| \leq r_n M_n \alpha_n (q+\alpha_n)^{i-1}, \quad \text{for all } i \geq 1.$$

Hence, we obtain

$$\begin{aligned}\|P_n(\varepsilon_n) - P_n\| &= \left\|\sum_{i=1}^{\infty} \varepsilon_n^i P_{n,i}\right\| \\ &\leq \sum_{i=1}^{\infty} |\varepsilon_n^i| \|P_{n,i}\| \\ &\leq \sum_{i=1}^{\infty} |\varepsilon_n|^i r_n M_n \alpha_n (q+\alpha_n)^{i-1} \\ &\leq r_n M_n \alpha_n |\varepsilon_n| \sum_{i=1}^{\infty} (|\varepsilon_n|(q+\alpha_n))^{i-1}.\end{aligned}$$

Since $|\varepsilon_n| < \frac{\sqrt{6\mathcal{A}}}{\pi n v_n r_n M_n \alpha_n + \sqrt{6\mathcal{A}}(q+\alpha_n)}$, we get

$$\begin{aligned}\|P_n(\varepsilon_n) - P_n\| &\leq \frac{|\varepsilon_n| r_n M_n \alpha_n}{1 - |\varepsilon_n|(q+\alpha_n)} \\ &< \frac{\sqrt{6\mathcal{A}}}{n\pi v_n}.\end{aligned} \tag{3.4}$$

If we denote by

$$R := \sum_{n=1}^{\infty} v_n^2 \|P_n(\varepsilon_n) - P_n\|^2,$$

Eq. (3.4) yields $R < \sum_{n=1}^{\infty} \frac{6\mathcal{A}}{n^2\pi^2} = \mathcal{A}$. Hence, we have

$$\sum_{n=1}^{\infty} v_n^2 \|P_n(\varepsilon_n)f - P_n f\|^2 \leq R\|f\|^2, \quad \forall f \in \mathcal{H},$$

with $R < \mathcal{A}$. Then, due to Theorem 2.1 the family $\{P_n(\varepsilon_n), v_n\}_{n\in\mathbb{N}^*}$ forms a non-orthogonal fusion frame for $\mathcal{H}$. □

Now, it remains to show the existence of a fixed complex number ε such that the families $\{P_n(\varepsilon), v_n\}_1^N \cup \{P_n(\varepsilon_n), v_n\}_{N+1}^{\infty}$ and $\{P_n(\varepsilon), v_n\}_1^N \cup \{P_n, v_n\}_{N+1}^{\infty}$ are non-orthogonal fusion frames for $\mathcal{H}$.

Theorem 3.3. *Suppose that hypotheses* $(H1)$*–*$(H3)$ *hold and the family* $\{P_n, w_n\}_{n\in\mathbb{N}^*}$ *is a non-orthogonal fusion frame for* $\mathcal{H}$ *with lower and upper non-orthogonal fusion frame bounds* $\mathcal{A}$ *and* $\mathcal{B}$*, respectively. Then, there exist a sequence of complex numbers* $(\varepsilon_n)_{n\in\mathbb{N}^*}$ *and two sequences of projections* $\{P_n(\varepsilon)\}_{n\in\mathbb{N}^*}$ *and* $\{P_n(\varepsilon_n)\}_{n\in\mathbb{N}^*}$ *having the form*

$$P_n(\varepsilon) = P_n + \varepsilon P_{n,1} + \varepsilon^2 P_{n,2} + \cdots + \varepsilon^i P_{n,i} + \cdots$$
$$P_n(\varepsilon_n) = P_n + \varepsilon_n P_{n,1} + \varepsilon_n^2 P_{n,2} + \ldots + \varepsilon_n^i P_{n,i} + \ldots$$

such that for $|\varepsilon| < \frac{\sqrt{6\mathcal{A}}}{\sup_{n\in[1,N]}\left(\pi n v_n r_n M_n \alpha_n + \sqrt{6\mathcal{A}}(q+\alpha_n)\right)}$*, where* $N \in \mathbb{N}^*$*, the systems*

(i) $\{P_n(\varepsilon), v_n\}_1^N \cup \{P_n(\varepsilon_n), v_n\}_{N+1}^{\infty}$
(ii) $\{P_n(\varepsilon), v_n\}_1^N \cup \{P_n, v_n\}_{N+1}^{\infty}$

form non-orthogonal fusion frames for $\mathcal{H}$*. Here* $\{w_n\}_{n\in\mathbb{N}^*}$ *and* $\{v_n\}_{n\in\mathbb{N}^*}$ *are two families of weights verifying* $w_n \leq v_n \leq \sqrt{2}w_n$*.* ♢

Remark 3.2. It is interesting to note that in Theorem 3.2 the obtained non-orthogonal fusion frame depend on a sequence of complex numbers $(\varepsilon_n)_{n\in\mathbb{N}^*}$. Further, they are related to a family of operators $(T(\varepsilon_n))_{n\in\mathbb{N}^*}$, whereas in Theorem 3.3 the first N projections in the two non-orthogonal fusion frames are associated to the perturbed operator $T(\varepsilon)$ for a fixed complex number ε. ♢

Proof of Theorem 3.3. Let $n \in [1, N],\ N \in \mathbb{N}^*$. Clearly, we have

$$\begin{aligned} |\varepsilon| &< \frac{\sqrt{6\mathcal{A}}}{\sup_{n\in[1,N]}\left(\pi n v_n r_n M_n \alpha_n + \sqrt{6\mathcal{A}}(q+\alpha_n)\right)} \\ &< \frac{1}{q+\alpha_n}. \end{aligned}$$

Hence, in view of [16, p. 134] the eigenprojection $P_n(\varepsilon)$ can be developed as entire series of ε, i.e.,

$$P_n(\varepsilon_n) = P_n + \varepsilon_n P_{n,1} + \varepsilon_n^2 P_{n,2} + \cdots + \varepsilon_n^i P_{n,i} + \cdots,$$

with

$$\|P_{n,i}\| \leq r_n M_n \alpha_n (q+\alpha_n)^{i-1}, \quad \text{for all } i \geq 1.$$

So, we have

$$\begin{aligned}\|P_n(\varepsilon) - P_n\| &= \left\| \sum_{i=1}^{\infty} \varepsilon^i P_{n,i} \right\| \\ &\leq \sum_{i=1}^{\infty} |\varepsilon^i| \|P_{n,i}\| \\ &\leq \sum_{i=1}^{\infty} |\varepsilon|^i r_n M_n \alpha_n (q+\alpha_n)^{i-1} \\ &\leq r_n M_n \alpha_n |\varepsilon| \sum_{i=1}^{\infty} (|\varepsilon|(q+\alpha_n))^{i-1}.\end{aligned}$$

Consequently, we get

$$\begin{aligned}\|P_n(\varepsilon) - P_n\| &\leq \frac{|\varepsilon| r_n M_n \alpha_n}{1-|\varepsilon|(q+\alpha_n)} \\ &< \frac{\sqrt{6\mathcal{A}}}{n\pi v_n}.\end{aligned} \tag{3.5}$$

Let $n \geq N+1$. For each eigenvalue λ_n of T_0, we fix an $\varepsilon_n \in \mathbb{C}$ such that

$$|\varepsilon_n| \in \left] 0, \frac{\sqrt{6\mathcal{A}}}{\pi n v_n r_n M_n \alpha_n + \sqrt{6\mathcal{A}}(q+\alpha_n)} \right[.$$

As $|\varepsilon_n| < \frac{1}{q+\alpha_n}$, thus following some ideas of the above we get

$$\|P_n(\varepsilon_n) - P_n\| < \frac{\sqrt{6\mathcal{A}}}{n\pi v_n}. \tag{3.6}$$

Now, let $\mathcal{P}_n \in \{P_n(\varepsilon)\}_1^N \cup \{P_n(\varepsilon_n)\}_{N+1}^{\infty}$. It follows from Eqs. (3.5) and (3.6) that

$$\|\mathcal{P}_n(\varepsilon_n) - P_n\| < \frac{\sqrt{6\mathcal{A}}}{n\pi v_n}. \tag{3.7}$$

Setting

$$R := \sum_{n=1}^{\infty} v_n^2 \|\mathcal{P}_n - P_n\|^2,$$

then Eq. (3.7) implies that $R < \sum_{n=1}^{\infty} \frac{6\mathcal{A}}{n^2\pi^2} = \mathcal{A}$. Therefore, we have

$$\sum_{n=1}^{\infty} v_n^2 \|\mathcal{P}_n f - P_n f\|^2 \leq R\|f\|^2, \quad \forall f \in \mathcal{H},$$

with $R < \mathcal{A}$. Consequently, in view of Theorem 2.1 the family $\{P_n(\varepsilon), v_n\}_1^N \cup \{P_n(\varepsilon_n), v_n\}_{N+1}^{\infty}$ forms a non-orthogonal fusion frame for $\mathcal{H}$. This achieves the proof of the first item.

The proof of the second item is similar to the one of the first item. □

We note here that the non-orthogonal fusion frames obtained above depend totally on the sequence $(\varepsilon_n)_n$ or partially on the fixed complex number ε. So, our objective now is to prove the existence of a family of non-orthogonal fusion frame for $T(\varepsilon)$.

Theorem 3.4. *Suppose that hypotheses* $(H1)$-$(H3)$ *hold and the family* $\{P_n, w_n\}_{n\in\mathbb{N}^*}$ *is a non-orthogonal fusion frame for* $\mathcal{H}$ *with lower and upper non-orthogonal fusion frame bounds* $\mathcal{A}$ *and* $\mathcal{B}$*, respectively. Further, assume that for all* $n \in \mathbb{N}^*$ *there exists a sequence* $(r_n)_{n\in\mathbb{N}^*}$ *in* $\mathbb{R}_+^*$ *satisfying*

(i) $\{z \in \mathbb{C}$ *such that* $|z-\lambda_n| \le r_n\} \cap \sigma(T_0) = \{\lambda_n\}$*;*
(ii) $\sup_{n\ge 1}\alpha_n < \infty$*;*
(iii) $\sum_{n=1}^{\infty}(v_n r_n M_n \alpha_n)^2 < \infty$.

Then, for $|\varepsilon| < \frac{\sqrt{\mathcal{A}}}{\sqrt{\sum_{n=1}^{\infty}(v_n r_n M_n\alpha_n)^2}+\sqrt{\mathcal{A}}\left(q+\ \sup_{n\ge1}\alpha_n\right)}$ *there exists a sequence of projections* $\{P_n(\varepsilon)\}_{n\in\mathbb{N}^*}$ *having the form*

$$P_n(\varepsilon) = P_n + \varepsilon P_{n,1} + \varepsilon^2 P_{n,2} + \cdots$$

such that the family $\{P_n(\varepsilon), v_n\}_{n\in\mathbb{N}^*}$ *forms a non-orthogonal fusion frame for* $\mathcal{H}$*. Here* $\{w_n\}_{n\in\mathbb{N}^*}$ *and* $\{v_n\}_{n\in\mathbb{N}^*}$ *are two families of weights verifying* $w_n \le v_n \le \sqrt{2}w_n$*.* ♢

Remark 3.3. Theorem 3.4 improves Theorems 3.2 and 3.3. Indeed, in Theorem 3.2 we have proved that for each eigenprojection P_n of T_0, there exist a sequence of complex numbers $(\varepsilon_n)_{n\in\mathbb{N}^*}$ and a sequence of eigenprojections $(P_n(\varepsilon_n))_{n\in\mathbb{N}^*}$ of $(T(\varepsilon_n))_{n\in\mathbb{N}^*}$ such that the system $\{P_n(\varepsilon_n), v_n\}_{n\in\mathbb{N}^*}$ forms a non-orthogonal fusion frame for $\mathcal{H}$. We point out here that the non-orthogonal fusion frame $\{P_n(\varepsilon_n), v_n\}_{n\in\mathbb{N}^*}$ is related to the eigenprojections of a sequence of operators $(T(\varepsilon_n))_{n\in\mathbb{N}^*}$ and depends on the sequence $(\varepsilon_n)_{n\in\mathbb{N}^*}$. Further, in Theorem 3.3 we assert the existence of a fixed complex number ε only for the N first projections; whereas in Theorem 3.4, we give a fixed complex number ε for which the system $\{P_n(\varepsilon), v_n\}_{n\in\mathbb{N}^*}$ forms a non-orthogonal fusion frame for $\mathcal{H}$. Furthermore, the family of subspaces $\{W_{n,\varepsilon}\}_{n\in\mathbb{N}^*}$ coincide with the range of the eigenprojections $(P_n(\varepsilon))_{n\in\mathbb{N}^*}$ of $T(\varepsilon)$. ♢

Proof of Theorem 3.4. Let $n \in \mathbb{N}^*$. It is easy to see that

$$\begin{aligned} |\varepsilon| &< \frac{\sqrt{\mathcal{A}}}{\sqrt{\sum_{n=1}^{\infty}(v_n r_n M_n \alpha_n)^2} + \sqrt{\mathcal{A}}\left(q + \sup_{n\ge1}\alpha_n\right)} \\ &< \frac{1}{q+\alpha_n}. \end{aligned}$$

Then, it follows from [16, p. 134] that the eigenprojection $P_n(\varepsilon)$ can be developed as entire series of ε as follow

$$P_n(\varepsilon_n) = P_n + \varepsilon_n P_{n,1} + \varepsilon_n^2 P_{n,2} + \cdots + \varepsilon_n^i P_{n,i} + \cdots, \tag{3.8}$$

with

$$\|P_{n,i}\| \le r_n M_n \alpha_n (q+\alpha_n)^{i-1}, \quad \text{for all } i \ge 1. \tag{3.9}$$

Hence, Eqs. (3.8) and (3.9) yield

$$\begin{aligned}\|P_n(\varepsilon)-P_n\| &\le \sum_{i=1}^{\infty}|\varepsilon^i|\|P_{n,i}\| \\ &\le \sum_{i=1}^{\infty}|\varepsilon|^i r_n M_n\alpha_n(q+\alpha_n)^{i-1} \\ &\le r_n M_n\alpha_n|\varepsilon|\sum_{i=1}^{\infty}(|\varepsilon|(q+\alpha_n))^{i-1} \\ &\le r_n M_n\alpha_n\frac{|\varepsilon|}{1-|\varepsilon|(q+\alpha_n)}. \end{aligned} \tag{3.10}$$

So, Eq. (3.10) entails the estimate

$$\|P_n(\varepsilon)-P_n\| < r_n M_n\alpha_n\frac{\sqrt{\mathcal{A}}}{\sqrt{\sum_{n=1}^{\infty}(v_n r_n M_n\alpha_n)^2}}. \tag{3.11}$$

Setting

$$R := \sum_{n=1}^{\infty} v_n^2\|P_n(\varepsilon_n)-P_n\|^2,$$

Eq. (3.11) implies that $R < \mathcal{A}$. Hence, we have

$$\sum_{n=1}^{\infty} v_n^2\|P_n(\varepsilon_n)f-P_nf\|^2 \le R\|f\|^2, \quad \forall f\in\mathcal{H},$$

with $R < \mathcal{A}$. Consequently, the result follows immediately from Theorem 2.1. □

4. Application to a One-Dimensional Wave Control System

To illustrate the importance of the above mentioned results, we consider a controlled wave system given by

$$\begin{cases} w_{tt}(x,t) = w_{xx}(x,t), \quad 0<x<1,\ t>0, \\ w_x(0,t) = w_t(0,t)+\gamma w(0,t), \\ w_x(1,t) = -k_2 w_t(1,t)-\delta w(1,t), \\ w(x,0) = w_0(x), \quad w_t(x,0)=w_1(x), \end{cases} \tag{4.1}$$

where $\gamma,\ \delta>0$ and $k_2\ge 0$. The abstract formulation of Eq. (4.1) is equivalent to consider the Hilbert space $\mathcal{X}$ defined by

$$\mathcal{X} := H^1(0,1)\times L^2(0,1)$$

and the operator A given by

$$A(u,v) := (v,u''), \quad (u,v)\in\mathcal{D}(A),$$

where

$$\mathcal{D}(A) := \big\{(u,v) \in H^2(0,1) \times H^1(0,1) \text{ such that } u'(0) = \gamma u(0) + v(0),\\ u'(1) = -\delta u(1) - k_2 v(1)\big\}.$$

Then, our initial problem can be written as

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} W(t) = AW(t), & t > 0, \\ W(0) \quad = W_0, \end{cases}$$

where $W(t) := (w(x,t), w_t(x,t))$ and $W_0 := (w_0(x), w_1(x))$.

It is easy to see that the operator A is a densely defined closed operator in $\mathcal{X}$ and it's adjoint A^*, is given by

$$A^*(u,v) = -(v, u''), \quad (u,v) \in \mathcal{D}(A^*),$$

where

$$\mathcal{D}(A^*) := \big\{(u,v) \in H^2(0,1) \times H^1(0,1) \text{ such that } u'(0) = -v(0) + \gamma u(0),\\ u'(1) = k_2 v(1) - \delta u(1)\big\}.$$

Before going further, we recall the following result from [19].

Theorem 4.1 [19, Theorems 3.1 and 3.2]. *The operator A is with compact resolvent. Further, we have*

$$\sigma(A) = \{\lambda \in \mathbb{C} \text{ such that } \Gamma(\lambda) = 0\},$$

where

$$\Gamma(\lambda) := [(1+k_2)\lambda + \delta](2\lambda + \gamma)e^{\lambda} + \gamma[(1-k_2)\lambda - \delta]e^{-\lambda}.$$

◇

Proof. Suppose that $0 \in \sigma_p(A)$. Then, there exists $(u,v) \neq (0,0) \in \mathcal{D}(A)$ such that $A(u,v) = (0,0)$, i.e.,

$$\begin{cases} v(x) = 0 \\ u''(x) = 0 \\ u'(0) = \gamma u(0) \\ u'(1) = -\delta u(1). \end{cases} \tag{4.2}$$

The solution of the system (4.2) is given by

$$u(x) = a + bx.$$

Substituting it into the boundary conditions, we get that the system (4.2) has only the zero solution. Hence, 0 is not an eigenvalue of A.

Now, we consider the inhomogeneous equation $AF = G$, where $F = (u,v) \in \mathcal{D}(A)$ and $G = (f,g) \in \mathcal{X}$, i.e.,

$$\begin{cases} v(x) = f(x) \\ u''(x) = g(x) \\ u'(0) - \gamma u(0) = f(0) \\ u'(1) + \delta u(1) = -k_2 f(1). \end{cases}$$

It is easy to see that $u_1(x) = 1$ and $u_2(x) = x$ are the solutions of the homogeneous equation $u''(x) = 0$. An elementary calculation by the variation-of-constants reveals that the general solution to the equation $u''(x) = g(x)$ is given by

$$\begin{aligned} u(x) &= au_1(x) + bu_2(x) + \int_0^x \frac{-u_1(x)u_2(t) + u_2(x)u_1(t)}{u_1(t)u_2'(t) - u_2(t)u_1'(t)}\ g(t)\ \mathrm{d}t \\ &= a + bx + \int_0^x (-t + x)\ g(t)\ \mathrm{d}t. \end{aligned}$$

Using the boundary conditions, we get

$$\begin{aligned} a = \frac{1}{\Delta}\Bigg[&-f(0)(u_2'(1) + \delta u_2(1)) - k_2 f(1) + \int_0^1 \frac{(u_1'(1) + \delta u_1(1))u_2(t)}{u_1(t)u_2'(t) - u_2(t)u_1'(t)}\ g(t)\ \mathrm{d}t \\ &- \int_0^1 \frac{(u_2'(1) + \delta u_2(1))u_1(t)}{u_1(t)u_2'(t) - u_2(t)u_1'(t)}\ g(t)\ \mathrm{d}t\Bigg] \\ = \frac{1}{\Delta}\Bigg[&-f(0)(1 + \delta) - k_2 f(1) + \int_0^1 (\delta t - 1 - \delta)\ g(t)\ \mathrm{d}t\Bigg] \end{aligned}$$

and

$$\begin{aligned} b = \frac{1}{\Delta}\Bigg[&f(0)(u_1'(1) + \delta u_1(1)) - k_2 f(1) + \int_0^1 \frac{(u_1'(1) + \delta u_1(1))u_2(t)}{u_1(t)u_2'(t) - u_2(t)u_1'(t)}\ g(t)\ \mathrm{d}t \\ &- \int_0^1 \frac{(u_2'(1) + \delta u_2(1))u_1(t)}{u_1(t)u_2'(t) - u_2(t)u_1'(t)}\ g(t)\ \mathrm{d}t\Bigg] \\ = \frac{1}{\Delta}\Bigg[&f(0)\delta - k_2 f(1) + \int_0^1 (\delta t - 1 - \delta)\ g(t)\ \mathrm{d}t\Bigg], \end{aligned}$$

where

$$\Delta = (u_1'(1) + \delta u_1(1)) + \gamma(u_2'(1) + \delta u_2(1)) = \delta + \gamma(1 + \delta).$$

Since $0 \notin \sigma_p(A)$, hence $\Delta \neq 0$. Consequently, $A^{-1}(f, g) = (u, v)$. So, we obtain

$$\begin{aligned} \|A^{-1}(f, g)\|^2 &= \|(u, v)\|^2 \\ &= \int_0^1 |u'(x)|^2\ \mathrm{d}x + \gamma|u(0)|^2 + \delta|u(1)|^2 + \int_0^1 |v(x)|^2\ \mathrm{d}x \\ &= \int_0^1 \left|b + \int_0^x g(t)\ \mathrm{d}t\right|^2 \mathrm{d}x + \frac{1}{\gamma}|b - f(0)|^2 + \frac{1}{\delta}\left|-k_2 f(1) - b - \int_0^1 g(t)\ \mathrm{d}t\right|^2 \\ &\quad + \int_0^1 |f(x)|^2 \mathrm{d}x \\ &= \int_0^1 \left|b + \int_0^x g(t)\ \mathrm{d}t\right|^2\ \mathrm{d}x + \frac{1}{\gamma}|b - f(0)|^2 + \frac{1}{\delta}\left|k_2 f(1) + b + \int_0^1 g(t)\ \mathrm{d}t\right|^2 \\ &\quad + \int_0^1 \left|\int_0^x f'(t)\ \mathrm{d}t\right|^2\ \mathrm{d}x \\ &\le \int_0^1 \left[\left(\frac{4}{\delta} + 2\right)\left(b^2 + \left(\int_0^1 |g(t)|\ \mathrm{d}t\right)^2\right) + \frac{2}{\gamma}\left(b^2 + |f(0)|^2\right) + \frac{2k_2^2}{\delta}|f(1)|^2 \right. \\ &\quad \left. + \left(\int_0^1 |f'(t)|\ \mathrm{d}t\right)^2\right] \mathrm{d}x \end{aligned}$$

$$\leq \int_0^1 \left[\left(\frac{4}{\delta}+2\right)\left(b^2+\int_0^1 |g(t)|^2 \, \mathrm{d}t\right) + \frac{2}{\gamma}\left(b^2+|f(0)^2|\right) + \frac{2k_2^2}{\delta}|f(1)|^2 + \int_0^1 |f'(t)|^2 \, \mathrm{d}t \right] \mathrm{d}x.$$

Hence, we have

$$\|A^{-1}(f,g)\|^2 \leq C \int_0^1 \left[\int_0^1 |f'(t)|^2 \, \mathrm{d}t + \gamma|f(0)|^2 + \delta|f(1)|^2 + \int_0^1 |g(t)|^2 \, \mathrm{d}t \right] \mathrm{d}x$$
$$= C\|(f,g)\|^2,$$

where C is a positive constant. Then, we get $\|A^{-1}\| \leq C$. So, $0 \in \rho(A)$. Further, it follows from the Sobolev embedding theorem that A^{-1} is compact. Thus the resolvent set of A is compact.

On the other hand, let $\lambda \in \mathbb{C}$ and we consider the eigenvalue problem

$$\begin{cases} (A-\lambda)F = 0, \\ F = (u,v) \in \mathcal{D}(A). \end{cases} \tag{4.3}$$

The system (4.3) is equivalent to

$$\begin{cases} v(x) - \lambda u(x) = 0 \\ u''(x) - \lambda v(x) = 0 \\ u'(0) = \gamma u(0) + v(0) \\ u'(1) = -\delta u(1) - k_2 v(1). \end{cases} \tag{4.4}$$

Clearly, we have $v(x) = \lambda u(x)$. Substituting it into the system (4.4), we get

$$\begin{cases} u''(x) - \lambda^2 u(x) = 0 \\ u'(0) = \gamma u(0) + \lambda u(0) \\ u'(1) = -\delta u(1) - k_2 \lambda u(1). \end{cases} \tag{4.5}$$

The solution of the system (4.5) is formally given by

$$u(x) = ae^{\lambda x} + be^{-\lambda x}$$

and satisfies the following boundary conditions

$$\begin{cases} u'(0) = a\lambda - b\lambda = (\gamma+\lambda)(a+b) \\ u'(1) = a\lambda e^{\lambda} - b\lambda e^{-\lambda} = (-\delta - k_2\lambda)(ae^{\lambda} + be^{-\lambda}). \end{cases} \tag{4.6}$$

Hence, the system (4.6) can be written as

$$\begin{cases} \gamma a = -(2\lambda+\gamma)b \\ a[(1+k_2)\lambda + \delta]e^{\lambda} - b[(1-k_2)\lambda - \delta]e^{-\lambda} = 0. \end{cases}$$

So, we get that the necessary and sufficient condition to obtain a non-zero solution is

$$[(1+k_2)\lambda + \delta](2\lambda+\gamma)e^{\lambda} + \gamma[(1-k_2)\lambda - \delta]e^{-\lambda} = 0.$$

□

Let $\sigma(A) = (\lambda_n)_{n\in\mathbb{N}^*}$. For each eigenvalue $\lambda_n \in \sigma(A)$, the corresponding eigenvector of A is

$$\varphi_n = \left(\lambda_n^{-1}\left[e^{\lambda_n x} - \frac{\gamma}{(2\lambda_n+\gamma)} e^{-\lambda_n x} \right], \left[e^{\lambda_n x} - \frac{\gamma}{(2\lambda_n+\gamma)} e^{-\lambda_n x} \right] \right),$$

and the eigenvector of A^* corresponding to $\overline{\lambda_n}$ is

$$\varphi_n^* = \xi_n \left(\overline{\lambda_n^{-1}} \left[e^{\overline{\lambda_n} x} - \frac{\gamma}{(2\overline{\lambda_n} + \gamma)} e^{-\overline{\lambda_n} x} \right], - \left[e^{\overline{\lambda_n} x} - \frac{\gamma}{(2\overline{\lambda_n} + \gamma)} e^{-\overline{\lambda_n} x} \right] \right),$$

where

$$\xi_n^{-1} = \frac{4\gamma}{(2\lambda_n + \gamma)} + \frac{\gamma}{\lambda_n^2} \left[1 - \frac{\gamma}{(2\lambda_n + \gamma)} \right]^2 + \frac{\delta}{\lambda_n^2} \left[e^{\lambda_n} - \frac{\gamma}{(2\lambda_n + \gamma)} e^{-\lambda_n} \right]^2.$$

Evidently,

$$\langle \varphi_n, \varphi_n^* \rangle_{\mathcal{X}} = 1, \qquad \langle \varphi_m, \varphi_n^* \rangle_{\mathcal{X}} = 0, \ m \neq n.$$

For each $\lambda_n \in \sigma(A)$ and for $F \in \mathcal{X}$ the corresponding eigenprojection P_n is

$$P_n F = \langle F, \varphi_n^* \rangle_{\mathcal{X}} \varphi_n$$

and

$$\|P_n\| = \|\varphi_n^*\| \|\varphi_n\| = |\xi_n| \|\varphi_n\|^2. \tag{4.7}$$

Proposition 4.1 [19, p. 253]. *The eigenvectors of A fail to be a basis for $\mathcal{X}$. Furthermore, we have*

$$\|P_n\| \approx \frac{|\lambda_n|^2}{2|\Re(\lambda_n)|\delta} \leq \begin{cases} M_1 e^{-4\Re(\lambda_n)}, & \text{as } k_2 \neq 1, \\ M_1 e^{-2\Re(\lambda_n)}, & \text{as } k_2 = 1, \end{cases} \tag{4.8}$$

where M_1 is a positive constant.

Proof. An elementary calculation reveals that

$$\lim_{n \to \infty} |\lambda_n^{-4} e^{-2\lambda_n} \xi_n| = \frac{4}{\gamma^2 \delta} \tag{4.9}$$

and

$$\lim_{n \to \infty} |\lambda_n^2 \Re\lambda_n e^{2\Re\lambda_n}| \|\varphi_n\|^2 = \frac{\gamma^2}{8}. \tag{4.10}$$

Then, combining Eqs. (4.9) and (4.10) we get

$$\lim_{n \to \infty} |\lambda_n^{-2} \Re\lambda_n| |\xi_n| \|\varphi_n\|^2 = \frac{1}{2\delta}. \tag{4.11}$$

Hence, Eqs. (4.7) and (4.11) imply that $\sup_n \|P_n\| = \infty$ and consequently the eigenvectors of A fail to be a basis for $\mathcal{X}$. On the other hand, $\Gamma(\lambda) = 0$ yields

$$e^{2\lambda} = \frac{\gamma(k_2 - 1)\lambda + \gamma\delta}{2\lambda^2(1 + k_2) + [\gamma(1 + k_2) + 2\delta]\lambda + \gamma\delta}.$$

So, for $k_2 \neq 1$ we obtain

$$\begin{aligned} |e^{2\lambda}| = e^{2\Re\lambda} &= \frac{|\gamma(k_2 - 1)\lambda + \gamma\delta|}{|2\lambda^2(1 + k_2) + [\gamma(1 + k_2) + 2\delta]\lambda + \gamma\delta|} \\ &\leq \frac{|\gamma(k_2 - 1)\lambda + \gamma\delta|}{|2\lambda^2(1 + k_2)|} \\ &\leq \frac{|\gamma(k_2 - 1)| + \frac{\gamma\delta}{|\lambda|}}{2|\lambda|(1 + k_2)}. \end{aligned} \tag{4.12}$$

Thus, it follows from Eq. (4.12) that

$$|\lambda| \le D_1 e^{-2\Re\lambda}, \tag{4.13}$$

where D_1 is a positive constant. Further, for $k_2 = 1$ we have

$$|e^{2\lambda}| = e^{2\Re\lambda} \le \frac{\gamma\delta}{4|\lambda^2|}.$$

Hence, we get

$$|\lambda^2| \le \frac{\gamma\delta}{4} e^{-2\Re\lambda}. \tag{4.14}$$

Consequently, Eqs. (4.7), (4.11), (4.13) and (4.14) imply that

$$\|P_n\| \approx \frac{|\lambda_n|^2}{2|\Re(\lambda_n)|\delta} \le \begin{cases} M_1 e^{-4\Re(\lambda_n)}, & \text{as } k_2 \neq 1, \\ M_1 e^{-2\Re(\lambda_n)}, & \text{as } k_2 = 1, \end{cases}$$

where M_1 is a positive constant. □

The following result hold.

Proposition 4.2. *The family* $\{P_n, w_n\}_{n\in\mathbb{N}^*}$ *forms a non-orthogonal fusion frame for* $\mathcal{X}$, *where* $w_n = \frac{1}{|\lambda_n|^{\frac{5}{2}+\frac{\xi}{2}}}$, $\xi > 0$. ♢

Proof. It is clear here that $w_n > 0$. Now, let $f \in \mathcal{X}$. In view of Eq. (4.8), we have

$$\begin{aligned} \sum_{n=1}^{\infty} w_n^2 \|P_n(f)\|^2 &\le \|f\|^2 \sum_{n=1}^{\infty} w_n^2 \|P_n\|^2 \\ &\le \|f\|^2 \sum_{n=1}^{\infty} \left[\frac{|\lambda_n|^2}{2|\Re(\lambda_n)|\delta} \frac{1}{|\lambda_n|^{\frac{1}{2}(5+\xi)}} \right]^2 \\ &\le \|f\|^2 \underbrace{\sum_{n=1}^{\infty} \left[\frac{1}{2|\Re(\lambda_n)|\delta} \frac{1}{|\lambda_n|^{\frac{1}{2}(1+\xi)}} \right]^2}_{<\infty}. \end{aligned} \tag{4.15}$$

Indeed, $\Gamma(\lambda)$ is an entire function of exponential type. Then, it follows from Remark 2.1 that $\Gamma(\lambda)$ is an entire function of finite order at most 1. Moreover, $(\lambda_n)_{n\ge1}$ are the zeros of $\Gamma(\lambda)$. Hence, Theorem 2.3 implies that the series $\sum_{n\ge1} \frac{1}{|\lambda_n|^{1+\xi}}$ is convergent whenever $\xi > 0$. Consequently, we have

$$\sum_{n=1}^{\infty} \left[\frac{1}{2|\Re(\lambda_n)|\delta} \frac{1}{|\lambda_n|^{\frac{1}{2}(1+\xi)}} \right]^2 < \infty.$$

On the other hand, we have P_n is a bounded operator with closed range. Hence, in view of [1, p. 372], P_n admits a pseudo-inverse $P_n^\dagger$. Moreover,

$$\inf \left\{ \|P_n g\| \text{ such that } \|g\| = 1, g \in N(P_n)^\perp \right\} = \frac{1}{\|P_n^\dagger\|} > 0.$$

Hence,

$$\begin{aligned}\sum_{n=1}^{\infty} w_n^2\|P_n(f)\|^2 &\geq \underbrace{\sum_{n=1}^{\infty}\|f_n\|^2 w_n^2\|P_n^{\dagger}\|^{-2}}_{>0}, \quad f_n \in N(P_n)^{\perp}\\ &= \sum_{n=1}^{\infty}\left(\|f\|^2-\|g_n\|^2\right) w_n^2\|P_n^{\dagger}\|^{-2}, \quad g_n \in N(P_n)\\ &\geq \|f\|^2 \underbrace{\sup_{n\in\mathbb{N}^*}\left(1-\frac{\|g_n\|^2}{\|f\|^2}\right)\sum_{n=1}^{\infty} w_n^2\|P_n^{\dagger}\|^{-2}}_{>0}. \end{aligned} \tag{4.16}$$

As a consequence, Eqs. (4.15) and (4.16) imply that the family $\{P_n, w_n\}_{n\in\mathbb{N}^*}$ forms a non-orthogonal fusion frame for $\mathcal{X}$. □

Now, let us consider the following operator:

$$A_k(u,v) := (-1)^k(v,u'), \quad (u,v) \in \mathcal{D}(A_k),$$

where

$$\mathcal{D}(A_k) := \left\{(u,v) \in H^1(0,1) \times L^2(0,1)\right\}.$$

Let $(u,v) \in \mathcal{D}(A)$. We have

$$\begin{aligned}\|A_k(u,v)\|^2 &= \int_0^1 |v'(x)|^2\mathrm{d}x + \gamma|v(0)|^2 + \delta|v(1)|^2 + \int_0^1 |u'(x)|^2\mathrm{d}x\\ &\leq \int_0^1 |v'(x)|^2\mathrm{d}x + \gamma|v(0)|^2 + \delta|v(1)|^2 + \int_0^1 |u''(x)|^2\mathrm{d}x\\ &+ \int_0^1 |u'(x)|^2\mathrm{d}x + \gamma|u(0)|^2 + \delta|u(1)|^2 + \int_0^1 |v(x)|^2\mathrm{d}x\\ &= \|A(u,v)\|^2 + \|(u,v)\|^2.\end{aligned}$$

Consequently,

$$\|A_k(u,v)\| \leq \|A(u,v)\| + \|(u,v)\|. \tag{4.17}$$

Using the results described above, we can now prove the objective of this part.

Proposition 4.3. *For $|\varepsilon| < 1$, the series $A + \sum_{k\geq 1}\varepsilon^k A_k F$ converges for all $F = (u,v) \in \mathcal{D}(A)$. If we designate its sum by $A(\varepsilon)F$, we define a linear operator $A(\varepsilon)$ with domain $\mathcal{D}(A)$. For $|\varepsilon| < \frac{1}{2}$, the operator $A(\varepsilon)$ is closed.* ♢

Proof. The proof follows immediately from Theorem 3.1 and Eq. (4.17). □

Theorem 4.2. *For $|\varepsilon_n|$ enough small and $|\varepsilon|$ enough small there exist two sequences of projections $\{P_n(\varepsilon_n)\}_{n\in\mathbb{N}^*}$ and $\{P_n(\varepsilon)\}_{n\in\mathbb{N}^*}$ of $T(\varepsilon)$ having the form*

$$\begin{aligned}P_n(\varepsilon_n) &= P_n + \varepsilon_n P_{n,1} + \varepsilon_n^2 P_{n,2} + \cdots\\ P_n(\varepsilon) &= P_n + \varepsilon P_{n,1} + \varepsilon^2 P_{n,2} + \cdots\end{aligned}$$

such that the systems

(i) $\{P_n(\varepsilon_n), w'_n\}_{n\in\mathbb{N}^*}$
(ii) $\{P_n(\varepsilon), w'_n\}_1^N \cup \{P_n(\varepsilon_n), w'_n\}_{N+1}^\infty$
(iii) $\{P_n(\varepsilon), w'_n\}_1^N \cup \{P_n, w'_n\}_{N+1}^\infty$

form non-orthogonal fusion frames for $\mathcal{X}$. ♢

Proof. The proof is a direct implication from Theorems 3.2, 3.3 and 4.1, Propositions 4.2 and 4.3 and Eq. (4.17). □

Theorem 4.3. *For $|\varepsilon|$ enough small, there exists a sequence of projections $\{P_n(\varepsilon)\}_{n\in\mathbb{N}^*}$ having the form*

$$P_n(\varepsilon) = P_n + \varepsilon P_{n,1} + \varepsilon^2 P_{n,2} + \cdots$$

such that the family $\{P_n(\varepsilon), w'_n\}_{n\in\mathbb{N}^*}$ forms a non-orthogonal fusion frame for $\mathcal{X}$. ♢

Proof. Let $n \in \mathbb{N}^*$, λ_n the n^{th} eigenvalue of A and $r_n = \min\left\{\frac{\left||\lambda_n| - \frac{\pi}{3}|\lambda_{n-1}|\right|}{2}, \frac{\left||\lambda_{n+1}| - \frac{\pi}{3}|\lambda_n|\right|}{2}\right\}$.
As $\{z \in \mathbb{C},\ |z - \lambda_n| \le r_n\} \cap \sigma(A) = \{\lambda_n\}$, then let $\mathcal{C}_n = \mathcal{C}(\lambda_n, r_n)$ be the closed circle with center λ_n and radius r_n and $z \in \mathcal{C}_n$.

It is easy to verify that the operator A is normal. Hence, it follows from [14, p. 60] that

$$\|R_z\| = \|(A - zI)^{-1}\| = \frac{1}{d(z, \sigma(A))}.$$

Consequently, we obtain

$$\begin{aligned} \alpha_n &= aM_n + bN_n \\ &= a\max_{z\in\mathcal{C}_n}\|R_z\| + b\max_{z\in\mathcal{C}_n}\|AR_z\| \\ &= \frac{a}{r_n} + b\max_{z\in\mathcal{C}_n}\|I + zR_z\| \\ &\le \frac{a}{r_n} + b\max_{z\in\mathcal{C}_n}(1 + |z|\|R_z\|). \end{aligned}$$

Thus, we get

$$\alpha_n \le \frac{a}{r_n} + b\left(2 + \frac{|\lambda_n|}{r_n}\right). \tag{4.18}$$

If $r_n = \frac{\left||\lambda_n| - \frac{\pi}{3}|\lambda_{n-1}|\right|}{2}$, then Eq. (4.18) yields

$$\begin{aligned} \alpha_n &\le \left(\frac{2a}{\left||\lambda_n| - \frac{\pi}{3}|\lambda_{n-1}|\right|} + b\left(2 + \frac{2|\lambda_n|}{\left||\lambda_n| - \frac{\pi}{3}|\lambda_{n-1}|\right|}\right)\right) \\ &\le \frac{2a}{\left||\lambda_n| - \frac{\pi}{3}|\lambda_{n-1}|\right|} + b\left(2 + \frac{2}{\left|1 - \frac{\pi}{3}\frac{|\lambda_{n-1}|}{|\lambda_n|}\right|}\right). \end{aligned} \tag{4.19}$$

As $\Gamma(\lambda)$ is an entire function of finite order at most 1, Theorem 2.2 implies that

$$n(r) \leq Cr^{\kappa}$$

for some constant C and all values of r, where $1 < \kappa < 1+\xi$. If we choose $r = |\lambda_n|$ then $n(r) \geq n$ and hence

$$n^{\frac{1}{\kappa}} \leq C^{\frac{1}{\kappa}} |\lambda_n|.$$

Then, it follows from Eq. (4.19) that

$$\sup_{n\geq 1} \alpha_n \leq 2a \frac{C^{\frac{1}{\kappa}}}{\left|1-\frac{\pi}{3}\right|} + b\left(2+\frac{2}{\left|1-\frac{\pi}{3}\right|}\right) < \infty.$$

For $r_n = \frac{\left||\lambda_{n+1}|-\frac{\pi}{3}|\lambda_n|\right|}{2}$, we show by a similar way as the above that

$$\begin{aligned}\sup_{n\geq 1} \alpha_n &\leq \sup_{n\geq 1} \frac{2a}{\left||\lambda_{n+1}|-\frac{\pi}{3}|\lambda_n|\right|} + b\sup_{n\geq 1}\left(2+\frac{2}{\left|\frac{|\lambda_{n+1}|}{|\lambda_n|}-\frac{\pi}{3}\right|}\right)\\ &\leq 2a \frac{C^{\frac{1}{\kappa}}}{\left|1-\frac{\pi}{3}\right|} + b\left(2+\frac{2}{\left|1-\frac{\pi}{3}\right|}\right) < \infty.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\sum_{n=1}^{\infty} (w_n' r_n M_n \alpha_n)^2 &\leq \left(\sup_{n\geq 1}\alpha_n\right)^2 \sum_{n=1}^{\infty} (w_n')^2\\ &\leq \left(\sup_{n\geq 1}\alpha_n\right)^2 \sum_{n=1}^{\infty} 2w_n^2\\ &\leq \left(\sup_{n\geq 1}\alpha_n\right)^2 \sum_{n=1}^{\infty} \frac{2}{|\lambda_n|^{(5+\xi)}} < \infty.\end{aligned}$$

Consequently, the family $\{P_n(\varepsilon), w_n'\}_{n\in\mathbb{N}^*}$ forms a non-orthogonal fusion frame for $\mathcal{X}$. □

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