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Existence and Uniqueness of Solution for Abstract Differential Equations with State-Dependent Time Impulses

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Abstract. We study the existence and uniqueness of mild and classical solutions for abstract impulsive differential equations with state-dependent time impulses and an example is presented.

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1. Introduction

Motivated by the developments in [14], in this work, we study the existence and uniqueness of a mild solution for the problem:

$$u'(t) = Au(t) + f(t, u(t), u(\gamma(t))), \qquad t \neq t_i, \ \ i = 1, \ldots, N, \tag{1.1}$$

$$u(t_j^+) = g_j(u(\sigma_j(u(t_j^+)))), \quad j = 1, \ldots, N, \tag{1.2}$$

$$u_0 = \varphi \in C([-p, 0]; X), \tag{1.3}$$

where $A : D(A) \subset X \mapsto X$ is the generator of an analytic semigroup of bounded linear operators $(T(t))_{t\geq 0}$ on a Banach space $(X, \|\cdot\|)$, $0 = t_0 < t_1 < t_2 < \cdots < t_N < t_{N+1} = a$ are pre-fixed numbers, and $g_i : X \mapsto X$, $f : [0, a] \times X \times X \mapsto X$, $\gamma : [0, a] \mapsto [-p, a]$ and $\sigma_i : X \mapsto [-p, a]$, $i = 1, \ldots, N$, are functions specified be later.

The study of state-dependent delay equations is motivated by applications and theory. For the related ODEs on finite-dimensional spaces, we cite the early works by Driver [4,5], Aiello et al. [1], the survey by Hartung et al. [7] and the references in there. For the case PDEs with state-dependent delay, we mention [8,9,18,19] and the recent works by Krisztin and Rezounenko [11], Yunfei et al. [16], Kosovalic et al. [12], and Hernandez et al. [10].

Concerning the theory of impulsive differential equations, their motivations and developments, we cite the books by Bainov and Covachev [2] and Lakshmikantham et al. [13] for the case of ordinary differential equations on finite-dimensional space and Benchohra et al. [3] for abstract differential equations and partial differential equations.

To the best of our knowledge, the papers by Hakl et al. [6] and Li and Wu [14] are the unique works treating on differential equations with state-dependent delayed impulses. As pointed in these papers, the study of this type of problems is motivated by applications arising in disease control, financial options, population dynamics, and control theory among some fields.

The problem of the existence and "uniqueness" of solutions for (1.1)–(1.3) is not trivial, since the function $u \mapsto g_i(u(\sigma_i(u(t_i^+))))$ is (in general) not Lipschitz. By noting that

$$\| u(\sigma_i(u(t_i^+))) - v(\sigma_i(v(t_i^+))) \| \leq (1 + [v]_{\mathcal{C}_{Lip}(X)}[\sigma_i]_{C_{Lip}}) \| u - v \|, \quad (1.4)$$

when the involved functions are Lipschitz, we work on spaces of sectionally Lipschitz functions, a hard problem in the framework of semigroup theory. We also note that the Lipschizianity of $T(\cdot)g_i(u(\sigma_i(u(t_i^+))))$ not depends on the Lipschizianity of $u(\cdot)$, an extra difficulty in our studies.

In Theorem 2.1, we prove the existence and uniqueness of solutions using the Banach principle. The existence of a mild solutions via the Schauder's fixed point theorem is established and proved in Theorem 2.2. In the last section, an example is presented. Next, we include some notations and results.

Let $(Z, \| \cdot \|_Z)$ and $(W, \| \cdot \|_W)$ be Banach spaces. We denote by $\mathcal{L}(Z, W)$ the space of bounded linear operators from Z into W endowed with operator norm denoted by $\| \cdot \|_{\mathcal{L}(Z,W)}$ and we write $\mathcal{L}(Z)$ and $\| \cdot \|_{\mathcal{L}(Z)}$ if $Z = W$. Moreover, if $X = Z = W$, we write simply $\| \cdot \|$ for the norms $\| \cdot \|_X$ and $\| \cdot \|_{\mathcal{L}(X)}$. In addition, $B_r(z, Z) = \{y \in Z : \| y - z \|_Z \leq r\}$.

Let $J \subset \mathbb{R}$ be a bounded interval. The spaces $C(J, Z)$ and $C_{Lip}(J, Z)$ and their norms denoted by $\| \cdot \|_{C(J,Z)}$ and $\| \cdot \|_{C_{Lip}(J,Z)}$ are the usual. We only note that $\| \cdot \|_{C_{Lip}(J;Z)}$ is given by $\| \cdot \|_{C_{Lip}(J;Z)} = \| \cdot \|_{C(J;Z)} + [\cdot]_{C_{Lip}(J;Z)}$, where $[\zeta]_{C_{Lip}(J;Z)} = \sup_{t,s \in J, t \neq s} \frac{\|\zeta(s) - \zeta(t)\|_Z}{|t-s|}$.

The notation $\mathcal{PC}(Z)$ is used for the space formed by all the bounded functions $u : [0, a] \mapsto Z$, such that $u(\cdot)$ is continuous at $t \neq t_i$, $u(t_i^-) = u(t_i)$ and $u(t_i^+)$ exists for all $i = 1, \ldots, N$, provided with the norm $\| u \|_{\mathcal{PC}(Z)} = \max_{i=0,1,\ldots,N} \| u \|_{C((t_i,t_{i+1}];Z)}$. In addition, $\mathcal{PC}_{Lip}(Z)$ represents the space of functions $u \in \mathcal{PC}(Z)$, such that $u_{|_{(t_i,t_{i+1}]}} \in C_{Lip}((t_i, t_{i+1}]; Z)$ for all $i = 0, 1, \ldots N$, endowed with the norm

$\| u \|_{\mathcal{PC}_{Lip}(Z)} = \max_{i=0,1,\ldots,N} \| u_{|_{(t_i,t_{i+1}]}} \|_{C_{Lip}((t_i,t_{i+1}];Z)}$.

We use the symbol $\mathcal{BPC}(Z)$ for the set of all the functions $u : [-p, a] \mapsto Z$, such that $u_{|_{[-p,t_1]}} \in C([-p, t_1]; Z)$ and $u_{|_{[0,a]}} \in \mathcal{PC}(Z)$. In addition, consider $\mathcal{BPC}_{Lip}(Z)$ the space formed by all the functions $u : [-p, a] \mapsto Z$, such that $u \in \mathcal{BPC}(Z)$, $u_{|_{[-p,0]}} \in C_{Lip}([-p, 0]; Z)$ and $u_{|_{[0,a]}} \in \mathcal{PC}_{Lip}(Z)$, endowed with the norm $\| u \|_{\mathcal{BPC}_{Lip}(Z)} = \max\{\| u_{|_{I_i}} \|_{C_{Lip}(I_i;Z)} : i = -1, 0, \ldots, N\}$, where $I_{-1} = [-p, 0]$.

For $u \in \mathcal{BPC}(Z)$ and $i \in \{-1, 0, 1, \cdots, N\}$, we denote by $\tilde{u}_i$ the function $\tilde{u}_i \in C([t_i, t_{i+1}]; Z)$ given by $\widetilde{u}_i(t) = u(t)$ for $t \in (t_i, t_{i+1}]$ and $\widetilde{u}_i(t) = u(t_i^+)$ for $t = t_i$. For $B \subseteq \mathcal{BPC}(Z)$ and $i \in \{-1, 0, 1, \cdots, N\}$, $\widetilde{B}_i$ is the set $\widetilde{B}_i = \{\tilde{u}_i : u \in B\}$. We note the next Ascoli–Arzela-type criteria.

Lemma 1.1. *A set $B \subseteq \mathcal{BPC}(Z)$ is relatively compact in $\mathcal{BPC}(Z)$ if and only if each set $\widetilde{B}_i$ is relatively compact in $C([t_i, t_{i+1}], Z)$.*

2. Existence of Solutions

Definition 2.1. A function $u \in \mathcal{BPC}(X)$ is called a mild solution of the problem (1.1)–(1.3) if $u_0 = \varphi$, $u(t_i^+) = g_i(u(\sigma_i(u(t_i^+))))$ for all $i = 1, \ldots, N$ and

$$u(t) = T(t)\varphi(0) + \int_0^t T(t-\tau) f(\tau, u(\tau), u(\gamma(\tau)))d\tau, \quad t \in [0, t_1],$$

$$u(t) = T(t-t_i)g_i(u(\sigma_i(u(t_i^+)))) + \int_{t_i}^t T(t-\tau) f(\tau, u(\tau), u(\gamma(\tau)))d\tau,$$
$$\forall t \in (t_i, t_{i+1}],\ i = 1, \ldots, N.$$

Definition 2.2. A function $u \in \mathcal{BPC}(X)$ is called a classical solution of (1.1)–(1.3) if $u_0 = \varphi$, $u(t_i^+) = g_i(u(\sigma_i(u(t_i^+))))$ for all $i = 1, \ldots, N$ and $u(\cdot)$ satisfy (1.1).

In this section, we assume that $(W, \| \cdot \|_W)$ is a Banach space continuously embedded in $(X, \| \cdot \|)$, such that $AT(\cdot) \in L^\infty([0, a]; \mathcal{L}(W, X))$. In addition, $C_0 \in \mathbb{R}$ is such that $\| T(t) \| \leq C_0$ for all $t \in [0, a]$. To prove our results, we introduce the following conditions.

- $\mathbf{H}_\gamma$ $\gamma \in C_{Lip}([0, a]; [-p, a])$, there is a function $k : \{1, \ldots, N\} \mapsto \{-1, 0, \ldots, N\}$, such that $\gamma(I_i) \subset I_{k(i)}$ and $k(i) \leq i$ for all i. Next, for convenience, we write $[\gamma]_{C_{Lip}}$ in place $[\gamma]_{C_{Lip}([0,a];[-p,a])}$.
- $\mathbf{H}_{\sigma_\mathbf{i}}$ There is a function $q : \{1, \ldots, N\} \mapsto \{-1, 0, 1, \ldots, N\}$, such that $q(i) \leq i$ and $\sigma_i \in C(X, I_{q(i)})$ for all $i \in \{1, \ldots, N\}$. Next, we write $[\sigma_i]_{C_{Lip}}$ in place $[\sigma_i]_{C_{Lip}(X; I_{q(i)})}$.
- $\mathbf{H}^{\mathbf{W}}_{\mathbf{g},\mathbf{X}}$ $g_i \in C_{Lip}(X; W)$ and $\mathcal{C}_{X,W}(g_i) = \| g_i \|_{C(X;W)} < \infty$ for all $i \in \{1, \ldots, N\}$. Next, $L_{Z,W}(g_i)$ denotes the Lipschitz constant of $g_i(\cdot)$, $L_{Z,W}(g) = \max_{i=1,\ldots,N} L_{Z,W}(g_i)$ and $\mathcal{C}_{Z,W}(g) = \max_{i=1,\ldots,N} \mathcal{C}_{Z,W}(g_i)$.
- $\mathbf{H_f}$ $f \in C_{Lip}([0, a] \times X \times X; X)$ and $C_X(f) = \| f \|_{C([0,a] \times X \times X;X)} < \infty$. Next, L_f denotes the Lipschitz constant of $f(\cdot)$.

Notations 1. We consider $b_i = t_{i+1} - t_i$, $b = \max_{i=1,\ldots,N} b_i$, $i_c : W \mapsto X$ is the inclusion map , $\Lambda_{X,W} = \max\{\| AT(\cdot) \|_{L^\infty([0,b],\mathcal{L}(W,X))}, C_0 \| i_c \|_{\mathcal{L}(W,X)}\}$, and

$$\Phi_{X,W} = \Lambda_{X,W}\mathcal{C}_{X,W}(g) + C_0(C_X(f) + bL_f) + [T(\cdot)\varphi(0)]_{C_{Lip}([0,a];X)} + [\varphi]_{C_{Lip}([-p,0];X)}.$$

The next useful result follows from the proof of [10, Lemma 1]. The proof is omitted.

Lemma 2.1. *Assume that the conditions $\mathbf{H}_\gamma$ and $\mathbf{H}_{\sigma_i}$ are satisfied, $u, v \in \mathcal{BPC}_{Lip}(X)$ and $u_0 = v_0$. Then, $u(\gamma(\cdot)) \in \mathcal{PC}_{Lip}(X)$, $[u(\gamma(\cdot))]_{\mathcal{PC}_{Lip}(X)} \leq [u]_{\mathcal{BPC}_{Lip}(X)}[\gamma]_{C_{Lip}}$ and*

$$\| u(\sigma_i(u(t_i^+))) - v(\sigma_i(v(t_i^+))) \| \leq (1 + [v]_{\mathcal{BPC}_{Lip}(X)}[\sigma_i]_{C_{Lip}}) \| u - v \|_{\mathcal{PC}(X)} . \tag{2.1}$$

Theorem 2.1. *Assume that the conditions* $\mathbf{H}_\gamma$, $\mathbf{H}_{\sigma_i}$, $\mathbf{H}^{\mathbf{W}}_{\mathbf{g},\mathbf{X}}$, *and* $\mathbf{H}_\mathbf{f}$ *are satisfied,* $T(\cdot)\varphi(0) \in C_{Lip}([0,a];X)$ *and* $\varphi \in C_{Lip}([-p,0];X)$*. Then, there exists a unique classical solution* $u \in \mathcal{BPC}_{Lip}(X)$ *of* (1.1)–(1.3) *provided that*

$$2C_0 bL_f(3+[\gamma]_{C_{Lip}}) + \Lambda_{X,W}L_{X,W}(g)(1+4\max_{j=1,\ldots,N}[\sigma_j]_{C_{Lip}}\Phi_{X,W}) < 1. \quad (2.2)$$

Proof. First of all, we consider the polynomial $P : \mathbb{R} \to \mathbb{R}$ given by the following:

$$P(x) = \Phi_{X,W} + (C_0 bL_f(3+[\gamma]_{C_{Lip}}) + \Lambda_{X,W}L_{X,W}(g) - 1)x + \Lambda_{X,W}L_{X,W}(g)\max_i[\sigma_i]_{C_{Lip}}x^2. \quad (2.3)$$

From (2.2) and noting that $C_0bL_f(3+[\gamma]_{C_{Lip}}) + \Lambda_{X,W}L_{X,W}(g) < 1$, we infer that $P(\cdot)$ has a root $0 < R_1$. We select now $0 < R$, such that $P(R) < 0$. From the fact that $P(R) < 0$, we note that

$$\Phi_{X,W} + C_0bL_f(1+[\gamma]_{C_{Lip}})R < R, \quad (2.4)$$

$$\Lambda_{X,W}L_{X,W}(g)(1+R\max_{i=1,\ldots,N}[\sigma_i]_{C_{Lip}}) + 2C_0bL_f < 1. \quad (2.5)$$

Let $\mathcal{S}(R)$ be the space $\mathcal{S}(R) = \{u \in \mathcal{BPC}_{Lip}(X)\,;\, u_0 = \varphi,\, [u_{|_{[0,a]}}]_{\mathcal{PC}_{Lip}(X)} \leq R\}$, endowed with the metric $d(u,v) = \| u - v \|_{\mathcal{PC}(X)}$ and $\Gamma : \mathcal{S}(R) \to \mathcal{BPC}(X)$ be the map given by $(\Gamma u)_0 = \varphi$:

$$\Gamma u(t) = T(t)\varphi(0) + \int_0^t T(t-\tau)f(s,u(s),u(\gamma(s)))ds, \quad t \in [0,t_1],$$

$$\Gamma u(t) = T(t-t_i)g_i(u(\sigma_i(u(t_i^+)))) + \int_{t_i}^t T(t-s)f(s,u(s),u(\gamma(s)))ds,$$
$$t \in (t_i,t_{i+1}],\ i = 1,\ldots,N.$$

To prove that Γ is a $\mathcal{S}(R)$-valued function, for $t \in (t_i,t_{i+1})$, $i \in \{1,\ldots,N\}$ and $h > 0$ such that $t+h \in (t_i,t_{i+1}]$, we have that

$$\begin{aligned}&\| \Gamma u(t+h) - \Gamma u(t) \| \\ &\quad\leq \int_{t-t_i}^{t-t_i+h} \| AT(s)g_i(u(\sigma_i(u(t_i^+)))) \| ds \\ &\qquad+ \int_{t_i}^{t_i+h} \| T(t+h-s) \| \| f(s,u(s),u(\gamma(s))) \| ds \\ &\qquad+ \int_{t_i}^{t} \| T(t-s) \| \| f(s+h,u(s+h),u(\gamma(s+h))) \\ &\qquad-f(s,u(s),u(\gamma(s))) \| ds\end{aligned}$$

$$\leq \| AT(\cdot) \|_{L^\infty([0,b];\mathcal{L}(W,X))} \mathcal{C}_{X,W}(g)h + C_0\mathcal{C}_X(f)h$$
$$+ \int_{t_i}^{t} \| T(t-s) \| L_f(1 + [u]_{\mathcal{BPC}_{Lip}(X)} + [u(\gamma(\cdot))]_{C_{Lip}(I_i;X)})hds,$$

and hence, $[(\Gamma u)_{|_{I_i}}]_{C_{Lip}(I_i;X)} \leq \Phi_{X,W} + C_0bL_f(1+[\gamma]_{C_{Lip}})R < R$. In a similar way, we prove that

$$[(\Gamma u)_{|_{[0,t_1]}}]_{C_{Lip}([0,t_1];X)} \leq [T(\cdot)\varphi(0)]_{C_{Lip}([0,a];X)}$$
$$+ C_0(\mathcal{C}_X(f) + bL_f + bL_f(1 + [\gamma]_{C_{Lip}}))R \leq R.$$

From the above remarks and (2.4), we infer that $[(\Gamma u)_{|_{[0,a]}}]_{\mathcal{PC}_{Lip}(X)} \leq R$, which implies that Γ is a $\mathcal{S}(R)$-valued function.

On the other hand, from (2.1), for $u, v \in \mathcal{S}(R)$, $i = 1, \ldots, N$, and $t \in (t_i, t_{i+1}]$, we get

$$\| \Gamma u(t) - \Gamma v(t) \|$$
$$\leq \| T(t-t_i) \|_{\mathcal{L}(W,X)} L_{X,W}(g) \| u(\sigma_i(u(t_i^+))) - v(\sigma_i(v(t_i^+))) \|$$
$$+ \int_{t_i}^{t} \| T(t-s) \| L_f \| (\| u(\cdot) - v(\cdot) \|_{C(I_i;X)} + u(\gamma(\cdot)) - v(\gamma(\cdot)) \|_{C(I_i;X)})ds$$
$$\leq \left(C_0 \| i_c \|_{\mathcal{L}(W,X)} L_{X,W}(g)(1 + [v]_{\mathcal{BPC}_{Lip}(X)}[\sigma_i]_{C_{Lip}}) + 2C_0bL_f\right) d(u,v).$$

In addition, for $t \in [0, t_1]$, we note that $\| \Gamma u(t) - \Gamma v(t) \| \leq 2C_0bL_f d(u,v)$. From the above:

$$d(\Gamma u, \Gamma v) \leq \left(\Lambda_{X,W} L_{X,W}(g) \left(1 + R \max_{i=1,\ldots,N} [\sigma_i]_{C_{Lip}}\right) + 2C_0bL_f\right) d(u,v),$$

which implies that $\Gamma(\cdot)$ is a contraction and there exists a unique mild solution $u \in \mathcal{S}(R)$ of (1.1)–(1.3).

We prove now that $u(\cdot)$ is a classical solution. Let $\widetilde{u}_i$, $i \geq 1$, be defined as in the introduction. It is easy to see that $\widetilde{u}_i(\cdot)$ is the mild solution of the problem:

$$w'(t) = Aw(t) + f(t, u(t), u(\gamma(t))), \qquad t \in I_i = [t_i, t_{i+1}], \tag{2.6}$$
$$w(t_i) = g_i(u(\sigma_i(u(t_i^+)))). \tag{2.7}$$

Since $f(\cdot, u(\cdot), u(\gamma(\cdot)))$ is Lipschitz on I_i and the semigroup is analytic, from [17], it is easy to infer that $\widetilde{u}_i$ is a classical solution of (2.6)–(2.7) on I_i. The same argument proves that $\widetilde{u}_0$ is a classical solution of (2.6) on $[0, t_1]$ with the initial condition $u(0) = \varphi(0)$. From the above remarks, we obtain that $u \in \mathcal{BPC}_{Lip}(X)$ is a classical solution of (1.1)–(1.3). □

Next, using the ideas in the proof of Theorem 2.1, we discuss briefly the case in which the functions $f(\cdot)$, $g_i(\cdot)$ are unbounded and (or) locally Lipschitz. To begin, we include the next conditions.

$\mathcal{H}_{g,X}^{W}$ Each function g_i is continuous from X into W, which takes bounded sets into bounded sets, and there is $L_{X,W}(g_i, \cdot) \in C(\mathbb{R};\mathbb{R})$, such that $\| g_i(x) - g_i(y) \|_W \leq L_{X,W}(g_i, r) \| x - y \|$ for all $x, y \in B_r(0, X)$ and every $r > 0$. Next, $L_{X,W}(g, r) = \max_{i=1\ldots,N} L_{X,W}(g_i, r)$ and $\mathcal{C}_{X,W}(g, r) = \max_{i=1\ldots,N} \mathcal{C}_{X,W}(g_i, r)$, where $\mathcal{C}_{X,W}(g_i, r) = \| g_i \|_{C(B_r(0,X);W)}$.

$\mathcal{H}_f$ $f(\cdot)$ is continuous from $I \times X$ into X, which takes bounded sets into bounded sets, and there is $L_f(\cdot) \in C(\mathbb{R};\mathbb{R})$, such that $\| f(t,x) - f(s,y) \| \leq L_f(r)(| t-s | + \| x-y \|)$ for all $x, y \in B_r(0,X)$, $t, s \in [0,a]$ and every $r > 0$. Next, for $r > 0$, we use the notation $\mathcal{C}_X(f,r) = \| f \|_{C([0,a]\times B_r(0,X);X)}$.

Notations 2. For $r > 0$, we define $\Phi_{X,W}(r) = \Lambda_{X,W}\mathcal{C}_{X,W}(g,r) + C_0 C_X(f,r) + C_0 b L_f(r) + [T(\cdot)\varphi(0)]_{C_{Lip}([0,a];X)} + [\varphi]_{C_{Lip}([-p,0];X)}$.

The proof of Proposition 2.1 below follows from the proof of Theorem 2.1.

Proposition 2.1. *Let conditions $\mathbf{H}_\gamma, \mathbf{H}_{\sigma_i}$, $\mathcal{H}^W_{g,X}$ and $\mathcal{H}_f$ hold. Suppose that $T(\cdot)\varphi(0)$ belongs to $C_{Lip}([0,a];X)$, $\varphi \in C_{Lip}([-p,0];X)$, and there is $r > 0$, such that $\| \varphi \|_{C([-p,0];X)} < r$ and (2.2) is valid with $L_f(r), \Phi_{X,W}(r)$ and $L_{X,W}(g,r)$ in place $L_f, \Phi_{X,W}$ and $L_{X,W}(g)$, also $C_0(\| \varphi(0) \| + b\mathcal{C}_X(f,r)) + \Gamma_{X,W}(r)\mathcal{C}_{X,W}(g,r) < r$. Then, there exists a unique classical solution $u \in \mathcal{BPC}_{Lip}(X)$ of (1.1)–(1.3).*

To complete this section, we study the existence of solution using the Schauder's fixed point Theorem. The next lemma follows arguing as in the proof of [15, Proposition 4.2.1].

Lemma 2.2. *Let $\alpha \in (0,1)$, $\xi \in L^\infty([b,c];X)$, and $v : [b,c] \mapsto X$ be the function defined by $v(t) = \int_b^t T(t-s)\xi(s)ds$. Then, $[v]_{C^\alpha([b,c];X)} \leq \| \xi \|_{L^\infty([b,c];X)} (c-b)^{1-\alpha}(C_0 + \frac{C_1}{\alpha(1-\alpha)})$ and $v \in C^\alpha([b,c];X)$.*

Theorem 2.2. *Assume that the conditions $\mathbf{H}_\gamma$ and $\mathbf{H}_{\sigma_i}$ are satisfied, there is a Banach space $(Y, \| \cdot \|_Y) \hookrightarrow (X, \| \cdot \|)$, such that $\| T(t) - I \|_{\mathcal{L}(Y,X)} \to 0$ as $t \to 0$, $g_i \in C(X;Y)$ for all i, $f \in C([0,a] \times X \times X; X)$, the functions $g_i(\cdot), f(\cdot)$ are bounded, and $(T(t))_{t\geq 0}$ is compact. Then, there exists a mild solution of the problem (1.1)–(1.3).*

Proof. Let $\mathcal{C}_{X,Y}(g) = \max_{i=1,\ldots,N} \| g_i \|_{C(X;Y)}$, $C_X(f) = \| f \|_{C([0,a]\times X\times X;X)}$ and $\alpha \in (0,1)$. Let $\mathcal{BPC}_\varphi(X) = \{u \in \mathcal{BPC}(X) : u_0 = \varphi\}$ endowed with the metric $d(u,v) = \| u - v \|_{\mathcal{BPC}(X)}$ and $\Gamma : \mathcal{BPC}_\varphi(X) \mapsto \mathcal{BPC}(X)$ be defined as in the proof of Theorem 2.1. It is easy to prove that Γ is continuous. Next, we show that Γ is completely continuous.

Let $i \in \{1,\ldots,N\}$. From Lemma 2.2, for $t \in (t_i, t_{i+1})$, $h > 0$ with $t + h \in (t_i, t_{i+1}]$, we get the following:

$$\begin{aligned}
\| \Gamma u(t+h) - \Gamma u(t) \| &\leq \| (T(t+h-t_i) - T(t-t_i)) g_i(u(\sigma_i(u(t_i^+)))) \| \\
&\quad + \| \int_{t_i}^{t+h} T(t+h-s) f(s,u(s),u(\gamma(s)))ds \\
&\quad - \int_{t_i}^{t} T(t-s) f(s,u(s),u(\gamma(s)))ds \| \\
&\leq \| (T(t+h-t_i) - T(t-t_i)) \|_{\mathcal{L}(Y,X)} \, \mathcal{C}_{X,Y}(g) \\
&\quad + \mathcal{C}_X(f) a^{1-\alpha} \left(C_0 + \frac{C_1}{\alpha(1-\alpha)} \right) h^\alpha,
\end{aligned}$$

which implies that the set of functions $\{(\Gamma u)_{|_{I_i}} : u \in \mathcal{BPC}_\varphi(X)\}$ is right equicontinuous at $t \in (t_i, t_{i+1})$, since $\| (T(t+h-t_i) - T(t-t_i)) \|_{\mathcal{L}(Y,X)} \to 0$ as $h \to 0$. A similar argument proves that $\{(\Gamma u)_{|_{I_i}} : u \in \mathcal{BPC}_\varphi(X)\}$ is left equicontinuous at $t \in (t_i, t_{i+1}]$. From the above, we allows us to infer that $\{(\Gamma u)_{|_{I_i}} : u \in \mathcal{BPC}_\varphi(X)\}$ is equicontinuous on I_i. In addition, for $u \in \mathcal{BPC}_\varphi(X)$ and $0 < h < \delta$, we note that

$$\begin{aligned}
\| \widetilde{\Gamma u}(t_i + h) - \widetilde{\Gamma u}(t_i) \| &= \| (T(h) - I) g_i(u(\sigma_i(u(t_i^+)))) \| \\
&\quad + \int_{t_i}^{t_i+h} \| T(t_i + h - s) f(s, u(s), u(\gamma(s))) \| ds \\
&\leq \| T(h) - I \|_{\mathcal{L}(Y,X)} \mathcal{C}_{X,Y}(g) + \mathcal{C}_X(f) a^{1-\alpha} \left(C_0 + \frac{C_1}{\alpha(1-\alpha)} \right) h^\alpha,
\end{aligned}$$

which proves that $\Gamma \widetilde{\mathcal{BPC}_\varphi(X)}_i = \{(\widetilde{\Gamma u})_i : u \in \mathcal{BPC}_\varphi(X)\}$ is right equicontinuous at t_i. From the above, it follows that $\{\widetilde{(\Gamma u)}_i : u \in \mathcal{BPC}_\varphi(X)\}$ is equicontinuous on $[t_i, t_{i+1}]$.

We prove now that $\{(\widetilde{\Gamma u})_i(t) : u \in \mathcal{BPC}_\varphi(X)\}$ is relatively compact in X for all $t \in [t_i, t_{i+1}]$. Since the semigroup is compact, $(Y, \| \cdot \|_Y) \hookrightarrow (X, \| \cdot \|)$ and $g_i(\cdot)$ is bounded with values in Y, we have that $U = \{g_j(u(\sigma_j(u(t_j^+)))) : u \in \mathcal{BPC}_\varphi(X),\ j = 1, \ldots, N\}$ is relatively compact in X. For $t \in (t_i, t_{i+1}]$ and $0 < \varepsilon < t - t_i$, we note that

$$\begin{aligned}
(\widetilde{\Gamma u})_i(t) &= T(t - t_i) g_i(u(\sigma_i(u(t_i^+)))) \\
&\quad + T(\varepsilon) \int_{t_i}^{t-\varepsilon} T(t - \varepsilon - s) f(s, u(s), u(\gamma(s))) ds \\
&\quad + \int_{t-\varepsilon}^{t} T(t - s) f(s, u(s), u(\gamma(s))) ds \\
&\in T(t - t_i) U + T(\varepsilon) C_0 (t - \varepsilon - t_i) \mathcal{C}_X(f) B_1(0, X) + \varepsilon C_0 \mathcal{C}_X(f) B_1(0, X),
\end{aligned}$$

so that, $\{(\widetilde{\Gamma u})_i(t) : u \in \mathcal{BPC}_\varphi(X)\} \subset K_\varepsilon + D_\varepsilon$, where K_ε is relatively compact and the diameter of D_ε converges to zero as $\varepsilon \to 0$. This proves that the set $\Gamma \mathcal{BPC}_\varphi(X)(t)$ is relatively compact in X. Moreover, $\Gamma \widetilde{\mathcal{BPC}_\varphi(X)}_i(t_i) = \{g_i(u(\sigma_i(u(t_i^+)))) : u \in \mathcal{BPC}_\varphi(X)\}$ is relatively compact in X. From the above remarks, we have that $(\Gamma \widetilde{\mathcal{BPC}_\varphi(X)})_i$ is relatively compact in $C([t_i, t_{i+1}]; X)$. In a similar way, we prove that $(\Gamma \mathcal{BPC}_\varphi(X))_1 = \{(\Gamma u)_{|_{[0,t_1]}} : u \in \mathcal{BPC}_\varphi(X))\}$ is relatively compact in $C([0, t_1]; X)$.

From the above remarks and Lemma 1.1, it follows that Γ is completely continuous and noting that the functions $f(\cdot)$ and $g_i(\cdot)$ are bounded, we infer that there exists $r > 0$, such that $\Gamma(\mathcal{BPC}_\varphi(X)) \subset B_r(0, \mathcal{BPC}_\varphi(X))$. Thus, Γ is completely continuous from $B_r(0, \mathcal{BPC}_\varphi(X))$ into $B_r(0, \mathcal{BPC}_\varphi(X))$, and there exists a mild solution $u \in B_r(0, \mathcal{BPC}_\varphi(X))$ of (1.1)–(1.3). $\square$

3. Example

Consider the following problem motivated by equations arising in population dynamics:

$$u_t(t,x) = u_{xx}(t)(x) + \alpha u(t,x)(1 - u(\gamma(t),x)),\ x \in \Omega,\ t \in (t_i, t_{i+1}], \quad (3.1)$$
$$u(t_i^+, x) = \alpha_i u(\sigma_i(u(t_i^+)), x), \quad (3.2)$$
$$u(\theta, x) = \varphi(\theta, x), \qquad \theta \in [-p, 0], \quad (3.3)$$

We note that, in this problem, the impulses depending on the state can be justified by a control population decision.

To treat this problem, we take $X = C(\Omega; \mathbb{R})$, where $\Omega \subset \mathbb{R}^n$ is bounded and $A : D(A) \subset X \mapsto X$ is the realization of the second-order derivative in X. We assume that $\alpha, \alpha_i \in \mathbb{R}$, $\mathbf{H}_\gamma$ and $\mathbf{H}_{\sigma_\mathbf{i}}$ are satisfied, $T(\cdot)\varphi(0) \in C_{Lip}([0,a]; X)$ and $\varphi \in C_{Lip}([-p,0]; X)$. We define $g_i : X \mapsto X$ and $f : [0,a] \times X \times X \mapsto X$ by $f(t,x,y)(\xi) = \alpha x(\xi)(1 - y(\xi))$ and $g_i(t,x)(\xi) = \alpha_i x(\xi)$. It is trivial to see that

$$\| f(t,x_1,y_1) - f(s,x_2,y_2) \| \leq |\ \alpha\ | \left((1+r) \| x_1 - x_2 \| + r \| y_1 - y_2 \|\right),$$
$$\| f(t,x,y) \| \leq |\ \alpha\ |\ r(1+r),$$
$$\| g_i(x) - g_i(y) \| \leq |\ \alpha_i\ | \| x - y \|, \quad \| A g_i(z) \| \leq |\ \alpha_i\ | \| Az \|,$$

for all $t, s \in [0,a]$, $x, y \in B_r(0; X)$, and $z \in D(A)$.

Proposition 3.1. *Suppose that there is $r > \| \varphi \|_{C([-p,0];X)}$, such that the inequality (2.2) is verified and*

$$C_0(\| \varphi(0) \| + b \mid \alpha \mid (1+2r)) + \Lambda_{X,W} \left(\max_{i=1,\ldots,N} \mid \alpha_i \mid \right) < r.$$

Then, there exists a unique classical solution $u \in \mathcal{BPC}_{Lip}(X)$ of (3.1)–(3.3).

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