



Symmetric Strong Diameter Two Property

Rainis Haller, Johann Langemets[✉], Vegard Lima
and Rihhard Nadel

Abstract. We study Banach spaces with the property that, given a finite number of slices of the unit ball, there exists a direction such that all these slices contain a line segment of length almost 2 in this direction. This property was recently named the *symmetric strong diameter two property* by Abrahamsen, Nygaard, and Pöldvere. The symmetric strong diameter two property is not just formally stronger than the strong diameter two property (finite convex combinations of slices have diameter 2). We show that the symmetric strong diameter two property is only preserved by ℓ_∞ -sums, and working with weak star slices we show that $\text{Lip}_0(M)$ have the weak star version of the property for several classes of metric spaces M .

Mathematics Subject Classification. Primary 46B20, 46B22.

Keywords. Strong diameter two property, almost square spaces, Lipschitz spaces.

1. Introduction

All Banach spaces considered in this paper are nontrivial and over the real field. The closed unit ball of a Banach space X is denoted by B_X and its unit sphere by S_X . The dual space of X is denoted by X^* and the bidual by X^{**} . By a *slice* of B_X , we mean a set of the form

$$S(B_X, x^*, \alpha) := \{x \in B_X : x^*(x) > 1 - \alpha\},$$

where $x^* \in S_{X^*}$ and $\alpha > 0$. If X is a dual space, then slices whose defining functional comes from (the canonical image of) the predual of X are called *weak* slices*.

This research belongs to the area of diameter two properties, which is a recent topic in geometry of Banach spaces and has received intensive attention in the last years (see [1–9, 12–15, 18]). Its central research objects are Banach spaces where certain subsets of the unit ball (slices, relatively weakly

R. Haller, J. Langemets, and R. Nadel were partially supported by institutional research funding IUT20-57 of the Estonian Ministry of Education and Research.

open subsets or convex combinations of slices) all have diameter equal to 2. Different subsets under consideration led to particular diameter properties.

Recall from [1] that a Banach space X is *almost square* (ASQ) if whenever $n \in \mathbb{N}$, and $x_1, \dots, x_n \in S_X$, there exists a sequence $(y_k)_{k=1}^\infty \subset S_X$ such that $\|x_i \pm y_k\| \rightarrow 1$ for every $i \in \{1, \dots, n\}$. If a Banach space X is ASQ, then every finite convex combination of slices of B_X has diameter two [1, Proposition 2.5], that is, X has the *strong diameter 2 property* (SD2P). Spaces which are ASQ include $c_0(X_n)$, where X_n are arbitrary Banach spaces, and Banach spaces X which are M-ideals in X^{**} (see [1]).

In this paper, we investigate the following property, which first appeared in [3], but was singled out and studied in [5].

Definition 1.1. A Banach space X has the *symmetric strong diameter 2 property* (SSD2P) if for every finite family $\{S_i\}_{i=1}^n$ of slices of B_X and $\varepsilon > 0$, there exist $x_i \in S_i$ and $y \in B_X$, independent of i , such that $x_i \pm y \in S_i$ for every $i \in \{1, \dots, n\}$ and $\|y\| > 1 - \varepsilon$.

It is known [3, Lemma 4.1] that if a Banach space has the SSD2P, then it has the SD2P. In fact, the SSD2P is strictly stronger than the SD2P. For example, $L_1[0, 1]$ has the SD2P, but not the SSD2P (see Remark 3.3 below). On the other hand, ASQ Banach spaces have the SSD2P (this can easily be observed from Theorem 2.1 (d) below). The converse fails, $C[0, 1]$ has the SSD2P (it is a Lindenstrauss space) and is not ASQ (this can be easily seen by considering the constant 1 function).

The following classes of spaces have the SSD2P:

- (a) Lindenstrauss spaces (this follows by inspecting the proof of Proposition 4.6 in [4]);
- (b) uniform algebras (see Theorem 4.2 in [3]);
- (c) ASQ-spaces, in particular, Banach spaces which are M-ideals in their bidual (see [1]);
- (d) Banach spaces with an infinite-dimensional centralizer (this follows by inspecting the proof of Proposition 3.3 in [6]);
- (e) somewhat regular linear subspaces of $C_0(L)$, whenever L is an infinite locally compact Hausdorff topological space [5];
- (f) Müntz spaces (this follows by inspecting the proof of Theorem 2.5 in [2]).

All of the above-listed spaces contain an almost isometric copy of c_0 . However, we do not know whether every space with the SSD2P contains c_0 . On the other hand, every Banach space containing a copy of c_0 can be equivalently renormed to have the SSD2P, in fact even to be ASQ (see [8]).

Let us summarize the results of the paper. We start our investigation in Sect. 2 by giving equivalent formulations of the SSD2P, which are often more convenient to use.

Recently in [13], it was proven that the SD2P is preserved by a lot of absolute normalized norms. However, in Sect. 3, we show that the only direct sums of Banach spaces that can have the SSD2P are the ℓ_∞ -sums. In Sect. 4, we prove that the SSD2P passes down from a superspace if a subspace is

an ai-ideal in it or if the quotient is strongly regular. For the other way, we show that if a subspace is an M-ideal in the superspace then the SSD2P will lift from the subspace to the superspace. In Sect. 5, we introduce the weak* version of the SSD2P and we study this property for Lipschitz spaces. At the end, we list some open problems.

2. Characterization of the SSD2P

Let $\mathcal{O}(x)$ denote the set of all relatively weakly open neighborhoods of x in B_X .

Theorem 2.1. *Let X be a Banach space. The following assertions are equivalent:*

- (a) X has the SSD2P.
- (b) Whenever $n \in \mathbb{N}$, U_1, \dots, U_n are relatively weakly open subsets of B_X and $\varepsilon > 0$, there exist $x_i \in U_i$, $i \in \{1, \dots, n\}$, and $y \in B_X$ such that $x_i \pm y \in U_i$ for every $i \in \{1, \dots, n\}$ and $\|y\| > 1 - \varepsilon$.
- (c) Whenever $n \in \mathbb{N}$, C_1, \dots, C_n are finite convex combinations of slices of B_X and $\varepsilon > 0$, there exist $x_i \in C_i$, $i \in \{1, \dots, n\}$, and $y \in B_X$ such that $x_i \pm y \in C_i$ for every $i \in \{1, \dots, n\}$ and $\|y\| > 1 - \varepsilon$.
- (d) Whenever $n \in \mathbb{N}$, $x_1, \dots, x_n \in S_X$, there exist nets $(y_\alpha^i) \subset S_X$ and $(z_\alpha) \subset S_X$ such that $y_\alpha^i \rightarrow x_i$ weakly, $z_\alpha \rightarrow 0$ weakly, and $\|y_\alpha^i \pm z_\alpha\| \rightarrow 1$ for all $i \in \{1, \dots, n\}$.
- (e) Whenever $n \in \mathbb{N}$, $x_1, \dots, x_n \in S_X$, $U_i \in \mathcal{O}(x_i)$, $i \in \{1, \dots, n\}$, $V \in \mathcal{O}(0)$, and $\varepsilon > 0$, there exist $y_i \in U_i \cap S_X$, $i \in \{1, \dots, n\}$, and $z \in V \cap S_X$ such that $\|y_i \pm z\| \leq 1 + \varepsilon$.

Proof. (a) \Rightarrow (b). Let $n \in \mathbb{N}$ and assume that U_1, \dots, U_n are relatively weakly open subsets of B_X and that $\varepsilon > 0$. By Bourgain’s lemma [11, Lemma II.1] each U_i contains a convex combination of slices, say $U_i \supset \sum_{j=1}^{n_i} \lambda_i^j S_i^j$, with $\sum_{j=1}^{n_i} \lambda_i^j = 1$ and $\lambda_i^j > 0$, for each $i \in \{1, \dots, n\}$. We apply the definition of the SSD2P to the family of all S_i^j to find $x_i^j \in S_i^j$ and $y \in B_X$ such that $x_i^j \pm y \in S_i^j$ and $\|y\| > 1 - \varepsilon$. Set $w_i := \sum_{j=1}^{n_i} \lambda_i^j x_i^j$. Then

$$w_i \in \sum_{j=1}^{n_i} \lambda_i^j S_i^j \subset U_i$$

and

$$w_i \pm y = \sum_{j=1}^{n_i} \lambda_i^j (x_i^j \pm y) \in \sum_{j=1}^{n_i} \lambda_i^j S_i^j \subset U_i.$$

This shows (a) \Rightarrow (b). The same proof also gives (a) \Rightarrow (c), while (b) \Rightarrow (a) and (c) \Rightarrow (a) are trivial.

(b) \Rightarrow (e). Let $n \in \mathbb{N}$, $x_1, \dots, x_n \in S_X$, $U_i \in \mathcal{O}(x_i)$, $i \in \{1, \dots, n\}$, $V \in \mathcal{O}(0)$, and $\varepsilon \in (0, 1)$. By choosing $0 < \delta < \varepsilon$ small enough there exist finite sets $A_i \subset S_X^*$ and $B \subset S_X^*$ such that

$$U_i \supset \bar{U}_i := \{x \in B_X : |x^*(x - x_i)| < \delta, x^* \in A_i\}$$

and

$$V \supset \bar{V} := \{x \in B_X : |x^*(x)| < \delta, x^* \in B\}.$$

Let

$$\bar{U}_i \supset \tilde{U}_i := \{x \in B_X : |x^*(x - x_i)| < \delta/2, x^* \in A_i\}$$

and

$$\bar{V} \supset \tilde{V} := \{x \in B_X : |x^*(x)| < \delta/2, x^* \in B\}.$$

For each $i \in \{1, \dots, n\}$, choose $x_i^* \in S_{X^*}$ such that $x_i^*(x_i) = 1$, and define $S_i := S(B_X, x_i^*, \delta/2)$. We apply (b) to the relatively weakly open sets $W_i = S_i \cap \tilde{U}_i$ and \tilde{V} and find $w_i \in W_i$ and $v \in \tilde{V}$ and $z \in B_X$ such that

$$w_i \pm z \in W_i, \quad v \pm z \in \tilde{V}, \quad \text{and} \quad \|z\| > 1 - \frac{\delta}{2}.$$

Define $u_i := \frac{w_i}{\|w_i\|}$. Since $w_i \in S_i$, we get

$$\|w_i\| > 1 - \frac{\delta}{2} \quad \text{and} \quad \|u_i - w_i\| < \frac{\delta}{2}.$$

From this and $w_i \in \tilde{U}_i$, we have $u_i \in U_i$.

Next, note that $-(v \pm z) \in \tilde{V}$ hence $z = \frac{1}{2}(-v + z) + \frac{1}{2}(v + z) \in \tilde{V}$ by convexity. Since $\|z\| > 1 - \delta/2$, we get that $y := \frac{z}{\|z\|} \in V$.

Finally, note that

$$\|u_i \pm y\| \leq \|w_i - u_i\| + \|w_i \pm z\| + \|z - y\| \leq \frac{\delta}{2} + 1 + \frac{\delta}{2} < 1 + \varepsilon.$$

(d) \Rightarrow (a). Let $n \in \mathbb{N}$, $S_1 := S(B_X, x_1^*, \alpha_1), \dots, S_n := S(B_X, x_n^*, \alpha_n)$ be slices of B_X and $\varepsilon \in (0, 1)$. Find a $\delta > 0$ such that

$$\frac{1}{1 + \delta} > 1 - \varepsilon \quad \text{and} \quad \frac{1 - 2\delta}{1 + \delta} > 1 - \alpha_i$$

for every $i \in \{1, \dots, n\}$. For every i choose an $x_i \in S_X \cap S(B_X, x_i^*, \delta)$. By (d) there are nets $(y_\alpha^i) \subset S_X$ and $(z_\alpha) \subset S_X$ such that $y_\alpha^i \rightarrow x_i$ weakly, $z_\alpha \rightarrow 0$ weakly, and $\|y_\alpha^i \pm z_\alpha\| \rightarrow 1$ for all $i \in \{1, \dots, n\}$.

Find an index α_0 such that

$$y_{\alpha_0}^i \in S(B_X, x_i^*, \delta), \quad \|x_i^*(z_{\alpha_0})\| < \delta, \quad \text{and} \quad \max \|y_{\alpha_0}^i \pm z_{\alpha_0}\| \leq 1 + \delta$$

for every $i \in \{1, \dots, n\}$. Finally, set $y_i := y_{\alpha_0}^i / (1 + \delta)$ and $z := z_{\alpha_0} / (1 + \delta)$. Then we have $y_i, y_i \pm z \in S_i$ and $z \in B_X$ with $\|z\| > 1 - \varepsilon$.

The implication (e) \Rightarrow (d) is straightforward. □

Theorem 2.1 (d) provides a condition that is easy to check. See for example Proposition 3.4 below or Proposition 3.3 in [6].

3. Direct Sums with the SSD2P

We recall that a norm N on \mathbb{R}^2 is *absolute* (see [10]) if

$$N(a, b) = N(|a|, |b|) \quad \text{for all } (a, b) \in \mathbb{R}^2$$

and *normalized* if

$$N(1, 0) = N(0, 1) = 1.$$

For $1 \leq p \leq \infty$, we denote the ℓ_p -norm on \mathbb{R}^2 by $\|\cdot\|_p$. Every norm $\|\cdot\|_p$ is absolute and normalized. Moreover, if N is an absolute normalized norm on \mathbb{R}^2 (see [10, Lemmata 21.1 and 21.2]), then

$$\|\cdot\|_\infty \leq N(\cdot) \leq \|\cdot\|_1$$

and if $(a, b), (c, d) \in \mathbb{R}^2$ and

$$|a| \leq |c| \quad \text{and} \quad |b| \leq |d|,$$

then

$$N(a, b) \leq N(c, d).$$

If X and Y are Banach spaces and N is an absolute normalized norm on \mathbb{R}^2 , then we denote by $X \oplus_N Y$ the product space $X \times Y$ with respect to the norm

$$\|(x, y)\|_N = N(\|x\|, \|y\|) \quad \text{for all } x \in X \text{ and } y \in Y.$$

In the special case, where N is the ℓ_p -norm, we write $X \oplus_p Y$.

We will now prove that the ℓ_∞ -norm is the only absolute normalized norm, which preserves the SSD2P.

Theorem 3.1. *Let X and Y be Banach spaces.*

- (a) $X \oplus_\infty Y$ has the SSD2P if and only if X or Y has the SSD2P.
- (b) If N is an absolute normalized norm different from the ℓ_∞ -norm, then $X \oplus_N Y$ does not have the SSD2P.

Proof. (a). Assume first that X has the SSD2P and denote by $Z := X \oplus_\infty Y$. For every $i \in \{1, \dots, n\}$, let W_i be a nonempty relatively weakly open subset of B_Z containing the element (u_i, v_i) , and $\varepsilon > 0$. Find nonempty relatively weakly open subsets $U_i \subset B_X$ and $V_i \subset B_Y$ such that

$$(u_i, v_i) \in U_i \times V_i \subset W_i.$$

Since X has the SSD2P, by Theorem 2.1 (b), we can find $x_i \in U_i$ and $x \in B_X$ such that $x_i, x_i \pm x \in U_i$ and $\|x\| > 1 - \varepsilon$. Set $z_i = (x_i, v_i)$ and $z = (x, 0)$. Then $z_i, z_i \pm z \in W_i$ and $\|z\| > 1 - \varepsilon$, which completes the proof.

Assume now that $X \oplus_\infty Y$ has the SSD2P. Suppose for contradiction that X and Y both fail to have the SSD2P.

Since X fails the SSD2P, there are nonempty relatively weakly open subsets $U_1, \dots, U_n \in B_X$ and an $\varepsilon > 0$ such that for all $x_i \in U_i$ and for all $x \in B_X$ with $\|x\| > 1 - \varepsilon$ there is an index i_0 such that $x_{i_0} + x \notin U_{i_0}$ or $x_{i_0} - x \notin U_{i_0}$. In addition, there are nonempty relatively weakly open subsets $V_1, \dots, V_m \in B_Y$ and a $\delta > 0$ such that for all $y_j \in V_j$ and for all $y \in B_Y$ with $\|y\| > 1 - \delta$ there is an index j_0 such that $y_{j_0} + y \notin V_{j_0}$ or $y_{j_0} - y \notin V_{j_0}$.

Set $W_{ij} := U_i \times V_j$ for every $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, m\}$. Then each W_{ij} is a nonempty relatively open subset of $B_{X \oplus_\infty Y}$ and by our assumption there should be $(x_{i_0}, y_{j_0}) \in W_{i_0 j_0}$ and $(x, y) \in B_Z$ such that $(x_{i_0}, y_{j_0}) \pm (x, y) \in W_{i_0 j_0}$ and $\|(x, y)\| > 1 - \max\{\delta, \varepsilon\}$, which is impossible.

(b). Denote $Z := X \oplus_N Y$. Note that $N(1, 1) > 1$, because N differs from the ℓ_∞ -norm. Let $a \in (0, 1)$ be such that $N(a, a) = 1$. Since $N(a, 1) > 1$ and $N(1, a) > 1$, there is a $\delta > 0$ such that if $N(u, v) \leq 1$ and $u > 1 - \delta$, then $v < a - \delta$ or if $v > 1 - \delta$, then $u < a - \delta$. Fix an $\varepsilon > 0$ with $a - \delta \leq (1 - \varepsilon)a$.

Consider slices $S_1 := S(B_Z, (x^*, 0), \delta)$ and $S_2 := S(B_Z, (0, y^*), \delta)$. Suppose for contradiction that Z has the SSD2P, then there are $z_1 = (x_1, y_1) \in S_1$, $z_2 = (x_2, y_2) \in S_2$, and $w = (u, v) \in B_Z$ such that

$$z_1 \pm w \in S_1, \quad z_2 \pm w \in S_2, \quad \text{and} \quad \|w\| > 1 - \varepsilon.$$

Therefore, $(x^*, 0)(z_1 \pm w) = x^*(x \pm u) > 1 - \delta$, which implies that $\|x_1 \pm u\| > 1 - \delta$. Similarly, we have that $\|y_2 \pm v\| > 1 - \delta$. Hence, $\|y_1 \pm v\| < a - \delta$ and $\|x_2 \pm u\| < a - \delta$. Now we see that

$$\|v\| \leq \frac{1}{2}(\|y_1 + v\| + \|y_1 - v\|) < a - \delta.$$

Similarly, one has that $\|u\| < a - \delta$. Thus,

$$\begin{aligned} 1 - \varepsilon &< \|w\| = N(\|u\|, \|v\|) \\ &\leq N(a - \delta, a - \delta) \\ &\leq N((1 - \varepsilon)a, (1 - \varepsilon)a) \\ &= (1 - \varepsilon)N(a, a) = 1 - \varepsilon, \end{aligned}$$

a contradiction. □

Remark 3.2. Note that Theorem 3.1 implies that almost squareness is also preserved only by ℓ_∞ -sums.

Remark 3.3. By Theorem 3.1, $L_1[0, 1]$ does not have the SSD2P, because $L_1[0, 1] = L_1[0, \frac{1}{2}] \oplus_1 L_1[\frac{1}{2}, 1]$.

If $(X_n)_{n=1}^\infty$ is a sequence of Banach spaces, then $\ell_\infty(X_n)$ is the Banach space of bounded sequences $(x_n)_{n=1}^\infty$, where $x_n \in X_n$, with norm $\|(x_n)\| = \sup_n \|x_n\|$.

Proposition 3.4. *Let $(X_n)_{n=1}^\infty$ be a sequence of Banach spaces. Then $\ell_\infty(X_n)$ has the SSD2P.*

Proof. Define $Z := \ell_\infty(X_n)$ and $Z_0 := c_0(X_n)$. Let $P_k: Z \rightarrow Z_0$, $z = (x_n) \mapsto P_k(z) = (0, \dots, 0, x_k, 0, \dots)$. Observe that $(P_k(z))_{k=1}^\infty$ is a weakly null sequence in Z , because it is a weakly null sequence in the subspace Z_0 , where $Z_0^* = \ell_1(X_n^*)$.

Let $z_1, \dots, z_m \in S_Z$. Choose $u = (u_n)_{n=1}^\infty \in S_Z$ such that $\|u_n\| = 1$ for all n . Define

$$y_k^i := z_i - P_k(z_i) \quad \text{and} \quad w_k := P_k(u).$$

Then $y_k^i \rightarrow z_i$ weakly and $w_k \rightarrow 0$ weakly since both $(P_k(z_i))_{k=1}^\infty$ and $(P_k(u))_{k=1}^\infty$ are weakly null. By definition $\|y_k^i \pm w_k\| = 1$. From Theorem 2.1 (d), we see that Z has the SSD2P. □

4. Subspaces with the SSD2P

We show that the SSD2P behaves similar to the SD2P by passing to subspaces.

Let X be a Banach space and Y a subspace of X . Following [4] we say that Y is an *almost isometric ideal* (ai-ideal) in X if for every finite-dimensional subspace E of X and every $\varepsilon > 0$ there exists a bounded linear operator $T : E \rightarrow Y$ such that $(1 - \varepsilon)\|e\| \leq \|Te\| \leq (1 + \varepsilon)\|e\|$ and $Te = e$ for all $e \in E \cap Y$.

Using ideas from [17] one can connect ai-ideals to Hahn–Banach extension operators. Recall that $\varphi : Y^* \rightarrow X^*$ is called a *Hahn–Banach extension operator* if $\varphi(y^*)(y) = y^*(y)$ and $\|\varphi(y^*)\| = \|y^*\|$ for all $y \in Y$ and $y^* \in Y^*$. The connection was made explicit in Theorem 1.4 in [4] which says that: *If Y is an ai-ideal in X , then there exists a Hahn–Banach extension operator $\varphi : Y^* \rightarrow X^*$ such that for every $\varepsilon > 0$, every finite-dimensional subspace $E \subset X$ and every finite-dimensional subspace $F \subset Y^*$ there exists $T : E \rightarrow Y$ which satisfies:*

- (a) $Te = e$ for all $e \in E \cap Y$.
- (b) $(1 - \varepsilon)\|e\| \leq \|Te\| \leq (1 + \varepsilon)\|e\|$ for all $e \in E$.
- (c) $\varphi(f)(e) = f(Te)$ for all $e \in E$ and $f \in F$.

The Principle of Local Reflexivity says that every Banach space is an ai-ideal in its bidual.

Proposition 4.1. *Let X be Banach space and Y be its closed subspace. If X has the SSD2P and Y is an ai-ideal in X , then Y has the SSD2P.*

Proof. Let $\varphi : Y^* \rightarrow X^*$ be a Hahn–Banach extension operator connected to the local projections. Let $y_1, \dots, y_n \in S_Y$, $U_i \in \mathcal{O}(y_i)$ and $V \in \mathcal{O}(0)$ in Y , and $\varepsilon > 0$.

Let $\delta > 0$ be so small that $(1 + \delta)^2 + \delta < 1 + \varepsilon$. By choosing δ even smaller if necessary there exist finite sets $A_i \subset S_{Y^*}$ and $B \subset S_{Y^*}$ such that

$$U_i \supset \bar{U}_i := \{y \in B_Y : |y^*(y - y_i)| < \delta, y^* \in A_i\}$$

and

$$V \supset \bar{V} := \{y \in B_Y : |y^*(y)| < \delta, y^* \in B\}.$$

Define corresponding neighborhoods in X by

$$\tilde{U}_i := \{x \in B_X : |\varphi(y^*)(x - y_i)| < \frac{\delta}{2}, y^* \in A_i\}$$

and

$$\tilde{V} := \{x \in B_X : |\varphi(y^*)(x)| < \frac{\delta}{2}, y^* \in B\}.$$

By Theorem 2.1 (e), there exist $x_i \in \tilde{U}_i \cap S_X$ and $z \in \tilde{V} \cap S_X$ such that $\|x_i \pm z\| \leq 1 + \delta$.

Define $E := \text{span}\{x_1, \dots, x_n, y_1, \dots, y_n, z\} \subset X$ and $F := \text{span}(A_1 \cup \dots \cup A_n \cup B) \subset Y^*$. Both E and F are finite-dimensional. Since Y is an ai-ideal in X there exists a bounded linear operator $T : E \rightarrow Y$ such that

- (a) $Te = e$ for all $e \in E \cap Y$.
- (b) $(1 - \frac{\delta}{2})\|e\| \leq \|Te\| \leq (1 + \frac{\delta}{2})\|e\|$ for all $e \in E$.
- (c) $\varphi(f)(e) = f(Te)$ for all $e \in E$ and $f \in F$.

Define $u_i := Tx_i/\|Tx_i\|$ and $v := Tz/\|Tz\|$. Then $\|u_i - Tx_i\| \leq \delta/2$ and $\|v - Tz\| \leq \delta/2$, hence

$$\begin{aligned} |y^*(u_i - y_i)| &\leq |y^*(Tx_i - y_i)| + \frac{\delta}{2} = |y^*(T(x_i - y_i))| + \frac{\delta}{2} \\ &= |\varphi(y^*)(x_i - y_i)| + \frac{\delta}{2} < \frac{\delta}{2} + \frac{\delta}{2} = \delta \end{aligned}$$

and similarly $|y^*(v)| < \delta$. This means that $u_i \in U_i \cap S_Y$, for all $i \in \{1, \dots, n\}$, and $v \in V \cap S_Y$. Finally,

$$\begin{aligned} \|u_i \pm v\| &= \|Tx_i - u_i\| + \|T(x_i \pm z)\| + \|Tz - w\| \\ &\leq \frac{\delta}{2} + \left(1 + \frac{\delta}{2}\right) \|x_i \pm z\| + \frac{\delta}{2} \leq (1 + \delta)^2 + \delta \leq 1 + \varepsilon. \end{aligned}$$

From Theorem 2.1 (e), we get that Y has the SSD2P. □

Recall that a Banach space X is *strongly regular* if every closed and convex subset of B_X has convex combinations of slices with arbitrarily small diameter. For a deeper discussion of strong regularity and related concepts, we refer the reader to [11].

Proposition 4.2. *Let X be a Banach space and Y a closed subspace. If X has the SSD2P and X/Y is strongly regular, then Y has the SSD2P too. In particular, SSD2P passes down to finite co-dimensional subspaces.*

We omit the proof of Proposition 4.2, because using Theorem 2.1 (b) it is similar to the proof of [9, Theorem 2.2 (iii)].

Recall that a subspace Y of a Banach space X is an *M-ideal* in X if there exists a bounded linear projection $P : X^* \rightarrow X^*$ such that $\ker P = Y^\perp$ and

$$\|x^*\| = \|Px^*\| + \|x^* - Px^*\|$$

for all $x^* \in X^*$.

Proposition 4.3. *Let Y be a proper closed subspace of a Banach space X . If Y is an M-ideal in X and Y has the SSD2P, then X has the SSD2P.*

Proof. The proof is modelled on the proof of [12, Proposition 3]. Let $S_i := S(B_X, x_i^*, \alpha_i)$, $i \in \{1, \dots, n\}$, be slices and let $\varepsilon > 0$.

Let $P : X^* \rightarrow X^*$ with $\ker P = Y^\perp$ be the M-ideal projection. Define

$$y_i^* := \frac{Px_i^*}{\|Px_i^*\|} \quad \text{and} \quad \beta_i := \frac{\varepsilon(1 - \|Px_i^*\|) + \varepsilon^2}{\|Px_i^*\|} > 0.$$

Since Y has the SSD2P there exist $u_i \in S(B_Y, y_i^*, \beta_i)$ and $v \in B_Y$ with $u_i \pm v \in S(B_Y, y_i^*, \beta_i)$ and $\|v\| > 1 - \varepsilon$. Note that we then have $|y_i^*(v)| < \beta_i$. The choice of β_i means that

$$Px_i^*(u_i) > (\|Px_i^*\| - \varepsilon)(1 + \varepsilon).$$

If we happen to have $Px_i^* = 0$ we just set $u_i = 0$ and use the v we get from the rest of the slices. And if $Px_i^* = 0$ for all i then use any $v \in S_Y$.

Find x_1, \dots, x_n such that

$$(x_i^* - Px_i^*)(x_i) > (\|x_i^* - Px_i^*\| - \varepsilon)(1 + \varepsilon).$$

By Proposition 2.3 in [19] for each i , there is a net $z_{\alpha,i}$ in Y such that $z_{\alpha,i} \rightarrow x_i$ in the $\sigma(X, Y^*)$ -topology and

$$\limsup \|y + (x_i - z_{\alpha,i})\| \leq 1$$

for all $y \in B_Y$. Hence, we may choose $z_i \in Y$ such that

$$\begin{aligned} \|u_i + x_i - z_i\| &< 1 + \varepsilon \\ \|u_i \pm v + x_i - z_i\| &< 1 + \varepsilon \\ |P(x_i^*)(x_i - z_i)| &< \varepsilon. \end{aligned}$$

Define

$$y_i := \frac{u_i + x_i - z_i}{1 + \varepsilon} \quad \text{and} \quad w := \frac{v}{1 + \varepsilon}.$$

Then

$$\begin{aligned} x_i^*(y_i) &= \frac{x_i^*(u_i + x_i - z_i)}{1 + \varepsilon} \\ &= \frac{Px_i^*(u_i) + (x_i^* - Px_i^*)(x_i) + Px_i^*(x_i - z_i)}{1 + \varepsilon} \\ &> \frac{(\|Px_i^*\| - \varepsilon)(1 + \varepsilon) + (\|x_i^* - Px_i^*\| - \varepsilon)(1 + \varepsilon) - \varepsilon}{1 + \varepsilon} \\ &> \|x_i^*\| - 3\varepsilon = 1 - 3\varepsilon. \end{aligned}$$

In addition, $1 \geq \|w\| \geq \frac{1-\varepsilon}{1+\varepsilon}$ and

$$\begin{aligned} x_i^*(y_i \pm w) &> 1 - 3\varepsilon \pm \frac{\|Px_i^*\|}{1 + \varepsilon} y_i^*(v) > 1 - 3\varepsilon - \frac{\|Px_i^*\|}{1 + \varepsilon} \beta_i \\ &= 1 - 3\varepsilon - \frac{\varepsilon - \varepsilon\|Px_i^*\| + \varepsilon^2}{1 + \varepsilon} > 1 - 4\varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we can choose it as small as we like so that $y_i \in S_i$, $y_i \pm w \in S_i$ and $\|w\|$ is as close to 1 as we like. □

The SD2P-version of the following result is [3, Theorem 4.10], its proof in [3] actually proves the SSD2P-version.

Theorem 4.4. *Let X be a Banach space and Y its proper closed subspace. If Y is an M -ideal in X , that is $X^* = Z \oplus_1 Y^\perp$ for some nonempty subspace Z of X^* , and moreover, if Z is 1-norming for X , then both X and Y have the SSD2P.*

*In particular, if X is non-reflexive and an M -ideal in X^{**} , then both X and X^{**} have the SSD2P.*

Remark 4.5. Similar results to Proposition 4.3 and Theorem 4.4 cannot hold for ASQ spaces, because c_0 is an M -ideal in $\ell_\infty = (c_0)^{**}$ and c_0 is ASQ, but ℓ_∞ is not ASQ.

5. Lipschitz Spaces with the w^* -SSD2P

Recall from [18] that a metric space M has the *long trapezoid property* (LTP) if for every finite subset $N \subset M$ and $\varepsilon > 0$, there exist $u, v \in M, u \neq v$, such that

$$(1 - \varepsilon)(d(x, y) + d(u, v)) \leq d(x, u) + d(y, v)$$

holds for all $x, y \in N$. In [18, Theorem 3.1], the authors prove that M has the LTP if and only if $\text{Lip}_0(M)$ has the w^* -SD2P, that is, every finite convex combination of weak* slices of $B_{\text{Lip}_0(M)}$ has diameter two. We show that for some M with the LTP the space $\text{Lip}_0(M)$ even has the weak* version of the SSD2P (see Theorem 5.7 below).

Definition 5.1. A dual Banach space X^* has the *weak* symmetric strong diameter 2 property* (w^* -SSD2P) if for every finite family $\{S_i\}_{i=1}^n$ of weak* slices of B_{X^*} and $\varepsilon > 0$ there exist $x_i^* \in S_i$ and $y^* \in B_{X^*}$, independent of i , such that $x_i^* \pm y^* \in S_i$ for every $i \in \{1, \dots, n\}$ and $\|y^*\| > 1 - \varepsilon$.

In a dual space, the (S)SD2P clearly implies the w^* -(S)SD2P. The space $(C[0, 1])^*$ has the w^* -SD2P and fails the SD2P (see [14, Example 1.1]). We do not know whether the w^* -SSD2P and the SSD2P for a dual space are really different. However, the w^* -SSD2P is stronger than the w^* -SD2P. Indeed, $\ell_\infty \oplus_1 \ell_\infty$ has the SD2P (see [3], hence also the w^* -SD2P), but ℓ_1 -sums never have the w^* -SSD2P (the proof is similar to the one of Theorem 3.1). We also note that a Banach space X has the SSD2P if and only if X^{**} has the w^* -SSD2P, because by Goldstine’s theorem B_X is w^* -dense in $B_{X^{**}}$ and the norm on X^{**} is w^* -lower semicontinuous.

Let M be a pointed metric space with metric d and a base point denoted by 0. The space $\text{Lip}_0(M)$ of all Lipschitz functions $f: M \rightarrow \mathbb{R}$ with $f(0) = 0$ is a Banach space with norm

$$\|f\| = \sup \left\{ \frac{|f(x) - f(y)|}{d(x, y)} : x, y \in M, x \neq y \right\}.$$

It is known that $\text{Lip}_0(M)$ is a dual space, whose canonical predual is the Lipschitz-free space $\mathcal{F}(M)$, the norm closed linear subspace of $\text{Lip}_0(M)^*$ spanned by the evaluation functionals δ_x with $x \in M$. If $\mu = \sum_{i=1}^n a_i \delta_{x_i}$ is an element in $\mathcal{F}(M)$ with $x_i \in M \setminus \{0\}$ and $a_i \neq 0$ for every $i \in \{1, \dots, n\}$, then we will denote the *support* of μ by $\text{supp}(\mu) := \{x_1, \dots, x_n\}$.

Proposition 5.2. *If M is an unbounded metric space, then $\text{Lip}_0(M)$ has the w^* -SSD2P.*

Proof. Let $n \in \mathbb{N}$, $S_i := S(B_{\text{Lip}_0(M)}, \mu_i, \alpha_i)$, $i \in \{1, \dots, n\}$, be weak* slices of $B_{\text{Lip}_0(M)}$, where $\mu_i \in \text{span}\{\delta_x : x \in M\}$, and $\varepsilon > 0$. We want to show that there exist $f_i \in S_i$ and $\varphi \in B_{\text{Lip}_0(M)}$ such that

$$f_i \pm \varphi \in S_i \quad \text{and} \quad \|\varphi\| > 1 - \varepsilon.$$

Choose $g_i \in S_i$ with $g_i(\mu_i) = 1$ for $i \in \{1, \dots, n\}$. Denote by $N := \{0\} \cup \bigcup_{i=1}^n \text{supp}(\mu_i)$. The main idea of the proof is to find norm preserving extensions f_i of $g_i|_N$ such that $f_i|_{M \setminus B(0, s)} = 0$ and $\varphi|_{B(0, t)} = 0$ for suitable

$0 < s < t$. Since N is a finite subset of M , there is an $r > 0$ such that $N \subset B(0, r)$. Let $s := 2r$. Then for every $x \in B(0, r)$ and $y \in M \setminus B(0, s)$ we have $d(x, 0) \leq d(x, y)$. Since M is unbounded there exists $u \in M \setminus B(0, s)$.

Let $\delta > 0$ be such that $(1 - \delta)^2 > \max\{1 - \varepsilon, 1 - \alpha_i\}$. Find a $t > 0$ such that for every $x \in B(0, s)$ and $y \in M \setminus B(0, t)$ one has

$$d(x, y) \geq (1 - \delta)(d(x, u) + d(u, y)).$$

For example, any t with $\delta t \geq 2(s + d(0, u))$ does the job.

Since M is unbounded there exists $v \in M \setminus B(0, t)$ such that

$$\frac{d(v, 0) - t}{d(v, 0)} > 1 - \delta, \text{ that is, } \delta \cdot d(v, 0) > t.$$

Define $\tilde{\varphi}: B(0, t) \cup \{v\} \rightarrow \mathbb{R}$ by $\tilde{\varphi}|_{B(0,t)} = 0$ and $\tilde{\varphi}(v) = d(v, 0) - t$. Then $\|\tilde{\varphi}\| \leq 1$, because for any $x \in B(0, t)$, we have

$$\frac{|\tilde{\varphi}(v) - \tilde{\varphi}(x)|}{d(v, x)} = \frac{|\tilde{\varphi}(v) - 0|}{d(v, x)} \leq \frac{d(v, 0) - t}{d(v, 0) - d(0, x)} \leq 1.$$

In addition, $\|\tilde{\varphi}\| > 1 - \delta$, because

$$\|\tilde{\varphi}\| \geq \frac{\tilde{\varphi}(v) - \tilde{\varphi}(0)}{d(v, 0)} = \frac{d(v, 0) - t}{d(v, 0)} > 1 - \delta.$$

For every $i \in \{1, \dots, n\}$ define $\tilde{f}_i: N \cup (M \setminus B(0, s)) \rightarrow \mathbb{R}$ by $\tilde{f}_i|_N = g_i$ and $\tilde{f}_i|_{M \setminus B(0,s)} = 0$. Then $\|\tilde{f}_i\| \leq 1$, because for any $x \in N$ and $y \in M \setminus B(0, s)$, we have

$$|\tilde{f}_i(x) - \tilde{f}_i(y)| = |\tilde{f}_i(x) - \tilde{f}_i(0)| \leq d(x, 0) \leq d(x, y).$$

Consider $f_i := (1 - \delta)\tilde{f}_i$ and $\varphi := (1 - \delta)\tilde{\varphi}$ and extend them norm preservingly to M . Observe that $\|f_i \pm \varphi\| \leq 1$, because for any $x \in B(0, s)$ and $y \in M \setminus B(0, t)$, we have

$$\begin{aligned} |(f_i \pm \varphi)(x) - (f_i \pm \varphi)(y)| &= |f_i(x) \pm \varphi(y)| \\ &\leq |f_i(x)| + |\varphi(y)| = |f_i(x) - f_i(u)| + |\varphi(y) - \varphi(u)| \\ &\leq (1 - \delta)d(x, u) + (1 - \delta)d(u, y) \leq d(x, y). \end{aligned}$$

Finally, note that $\|\varphi\| = (1 - \delta)\|\tilde{\varphi}\| > (1 - \delta)^2 > 1 - \varepsilon$, and

$$\begin{aligned} (f_i \pm \varphi)(\mu_i) &= f_i(\mu_i) = (1 - \delta)\tilde{f}_i(\mu_i) \\ &= (1 - \delta)g_i(\mu_i) = 1 - \delta \\ &> (1 - \delta)^2 > 1 - \alpha_i. \end{aligned}$$

□

Proposition 5.3. *If M is an infinite discrete metric space, then $Lip_0(M)$ has the w^* -SSD2P.*

Proof. Let $n \in \mathbb{N}$, $S_i := S(B_{Lip_0(M)}, \mu_i, \alpha_i)$, $i \in \{1, \dots, n\}$, be weak* slices of $B_{Lip_0(M)}$, where $\mu_i \in \text{span}\{\delta_x : x \in M\}$.

Let $N = \{0\} \cup \bigcup_{i=1}^n \text{supp}(\mu_i)$. For every $i \in \{1, \dots, n\}$ choose $g_i \in S_{Lip_0(N)}$ such that $g_i(\mu_i) = 1$ and let $x_i, y_i \in N$ be such that

$$g_i(x_i) - g_i(y_i) = d(x_i, y_i) = 1.$$

Fix any two different elements $u, v \in M \setminus N$. Define $f_i \in S_i$ and $\varphi \in S_{\text{Lip}_0(M)}$ by setting

$$f_i(x) := \begin{cases} g_i(x), & \text{if } x \in N, \\ \frac{g_i(x_i) + g_i(y_i)}{2}, & \text{if } x = u \text{ or } x = v, \\ 0 & \text{elsewhere,} \end{cases}$$

and

$$\varphi(x) := \begin{cases} \frac{g_i(x_i) - g_i(y_i)}{2} = \frac{1}{2}, & \text{if } x = u, \\ \frac{g_i(y_i) - g_i(x_i)}{2} = -\frac{1}{2}, & \text{if } x = v, \\ 0 & \text{elsewhere.} \end{cases}$$

Then $f_i, f_i \pm \varphi \in S_i$, and $\|\varphi\| = 1$. □

For $n \in \mathbb{N}$ denote by

$$K_n := \{x \in \ell_\infty : x(k) \in \{0, 1, \dots, n\} \text{ for all } k \in \mathbb{N}\}$$

with metric inherited from ℓ_∞ . Note that $\text{Lip}_0(K_n)$ has the w^* -SD2P, because K_n has the LTP (see Lemma 5.4). In addition, observe that Proposition 5.3 shows that $\text{Lip}_0(K_1)$ has the w^* -SSD2P. In Proposition 5.5 below, we prove that $\text{Lip}_0(K_2)$ has the w^* -SSD2P, and in Proposition 5.6 we prove that $\text{Lip}_0(K_n)$ has the w^* -SSD2P for every $n \geq 3$. It is unknown, whether every slice of the unit ball of $\text{Lip}_0(K_n)$ has diameter two for every $n \in \mathbb{N}$ (see [15, p. 114]).

Lemma 5.4. *Let $n \in \mathbb{N}$ and let N be a finite subset of K_n . Then there are $u, v \in K_n \setminus N$ satisfying*

- (a) $d(u, v) = 1$;
- (b) For all $x \in N$ one has $d(x, u) = d(x, v)$;
- (c) For all $x, y \in N$ one has $d(x, y) \leq d(x, u)$ or $d(x, y) \leq d(y, u)$.

In particular, K_n has the LTP.

Proof. Choose an $u \in K_n \setminus N$ such that $u(i) \in \{0, n\}$ for every $i \in \mathbb{N}$. For such an element u the condition (c) holds. We will construct a suitable $v \in K_n \setminus N$, such that it differs from u in only one coordinate i_0 . Let $I \subset \mathbb{N}$ be a finite subset such that for every $x \in N$ there is an $i \in I$ such that $d(x, u) = |x(i) - u(i)|$. Fix an $i_0 \in \mathbb{N} \setminus I$. Let $v(i_0)$ be such that $|u(i_0) - v(i_0)| = 1$. Hence, condition (a) holds and we will check condition (b). Let $x \in N$. Clearly, $d(x, v) \geq d(x, u)$, because $u(i) = v(i)$ for all $i \in I$. For the reverse inequality, observe that

$$|x(i_0) - v(i_0)| = \begin{cases} 1, & \text{if } x(i_0) = u(i_0), \\ |x(i_0) - u(i_0)| - 1, & \text{if } x(i_0) \neq u(i_0). \end{cases}$$

Hence, $d(x, u) \geq d(x, v)$, and condition (b) holds. □

Proposition 5.5. *The Banach space $\text{Lip}_0(K_2)$ has the w^* -SSD2P.*

Proof. Let $n \in \mathbb{N}$, $S_i := S(B_{\text{Lip}_0(K_2)}, \mu_i, \alpha_i)$, $i \in \{1, \dots, n\}$, be weak* slices of $B_{\text{Lip}_0(K_2)}$, where $\mu_i \in \text{span}\{\delta_x : x \in M\}$. We show that there exist $f_i \in S_i$ and $\varphi \in B_{\text{Lip}_0(K_2)}$ such that

$$f_i \pm \varphi \in S_i \quad \text{and} \quad \|\varphi\| = 1.$$

Let $N = \{0\} \cup \bigcup_{i=1}^n \text{supp}(\mu_i)$ and let $u, v \in K_2$ be as in Lemma 5.4 for N . For every μ_i choose $g_i \in S_{\text{Lip}_0(K_2)}$ such that $g_i(\mu_i) = 1$. The main idea of the proof is to define φ such that $\varphi = 0$ outside $\{u, v\}$ and f_i are norm preserving extensions of $g_i|_N$ satisfying $f_i(u) = f_i(v)$ and $|f_i(x) - f_i(y)| \leq 1$ for every $x, y \notin N$. For $j \in \{1, 2\}$ set

$$N_j := \{x \in N : d(x, u) = d(x, v) = j\}.$$

For every g_i define its norm preserving extension g_i^+ from N to $N \cup \{u, v\}$ by taking

$$\begin{aligned} g_i^+(u) &:= \min\{g_i(x) + d(x, u) : x \in N\}, \\ g_i^+(v) &:= \max\{g_i(x) - d(x, v) : x \in N \cup \{u\}\}. \end{aligned}$$

This means that g_i^+ is the maximal extension from N to u and then minimal extension to v preserving the Lipschitz constant (see [16] or [20, p. 18]). Note that $g_i^+(v) + 1 = g_i^+(u)$. Indeed, for every $k, l \in \{1, 2\}$,

$$\left(\min_{x \in N_k} g_i(x) + k \right) - \left(\max_{x \in N_l} g_i(x) - l \right) \geq 1,$$

that is,

$$\max_{x \in N_l} g_i(x) - \min_{x \in N_k} g_i(x) \leq k + l - 1,$$

because for $x \in N_l$ and $y \in N_k$

$$g_i(x) - g_i(y) \leq d(x, y) \leq \begin{cases} d(x, u) = l \leq k + l - 1 \\ \text{or} \\ d(y, u) = k \leq k + l - 1, \end{cases}$$

by Lemma 5.4.

If there is an element $x \in K_2 \setminus N$ such that

$$N_2^1(x) := \{y \in N_2 : d(x, y) = 1\} \neq \emptyset,$$

then choose arbitrarily a_i^x from the set

$$\left[\max_{y \in N_2^1(x)} g_i(y) - 1, \min_{y \in N_2^1(x)} g_i(y) + 1 \right] \cap [g_i^+(u) - 1, g_i^+(u)].$$

Note that the latter intersection is nonempty, because

$$\max_{y \in N_2^1(x)} g_i(y) - 1 \leq g_i^+(u) = g_i^+(v) + 1$$

and

$$\min_{y \in N_2^1(x)} g_i(y) + 1 \geq g_i^+(u) - 1.$$

Define

$$\varphi(x) := \begin{cases} \frac{1}{2}, & \text{if } x = u, \\ -\frac{1}{2}, & \text{if } x = v, \\ 0 & \text{elsewhere,} \end{cases}$$

and

$$f_i(x) := \begin{cases} g_i(x), & \text{if } x \in N, \\ a_i^x, & \text{if } x \in K_2 \setminus N \text{ and } d(x, N_2) = 1, \\ g_i^+(u) - \frac{1}{2} & \text{elsewhere.} \end{cases}$$

Then $f_i \in S_i$, $(f_i \pm \varphi)(\mu_i) = f_i(\mu_i) > 1 - \alpha_i$, and $\|\varphi\| = 1$. To check that $\|f_i \pm \varphi\| \leq 1$, we argue by cases:

- If $x \in N$ and $y = u$, then

$$\begin{aligned} |(f_i \pm \varphi)(x) - (f_i \pm \varphi)(u)| &= \left| g_i(x) - \left(g_i^+(u) - \frac{1}{2} \pm \frac{1}{2} \right) \right| \\ &= \begin{cases} |g_i(x) - g_i^+(u)| \leq d(x, u) \\ \text{or} \\ |g_i(x) - g_i^+(u) + 1| = |g_i(x) - g_i^+(v)| \leq d(x, v) = d(x, u). \end{cases} \end{aligned}$$

- If $x \in K_2 \setminus (N \cup \{u, v\})$ and $y = u$, then

$$\begin{aligned} |(f_i \pm \varphi)(x) - (f_i \pm \varphi)(u)| &= \begin{cases} |a_i^x - (g_i^+(u) - \frac{1}{2} \pm \frac{1}{2})| \leq 1 \leq d(x, u) \\ \text{or} \\ |(g_i^+(u) - \frac{1}{2}) - (g_i^+(u) - \frac{1}{2} \pm \frac{1}{2})| = \frac{1}{2} \leq d(x, u), \end{cases} \end{aligned}$$

because $a_i^x \in [g_i^+(u) - 1, g_i^+(u)]$.

- If $x \in K_2 \setminus (N \cup \{u, v\})$ and $y \in N$, then

$$|(f_i \pm \varphi)(x) - (f_i \pm \varphi)(y)| = |f_i(x) - f_i(y)| \leq d(x, y),$$

because $\|f_i\| \leq 1$.

- The other cases are trivial or similar to the ones above.

Hence, $\|f_i \pm \varphi\| \leq 1$, which completes the proof. □

Proposition 5.6. *The Banach space $Lip_0(K_n)$, $n \geq 3$, has the w^* -SSD2P.*

Proof. Let $n \geq 3$, $k \in \mathbb{N}$, $S_i := S(B_{Lip_0(K_n)}, \mu_i, \alpha_i)$, $i \in \{1, \dots, k\}$, be weak* slices of $B_{Lip_0(K_n)}$, where $\mu_i \in \text{span}\{\delta_x : x \in M\}$. We show that there exist $f_i \in S_i$ and $\varphi \in B_{Lip_0(K_n)}$ such that

$$f_i \pm \varphi \in S_i \quad \text{and} \quad \|\varphi\| = 1.$$

Set $N = \{0\} \cup \bigcup_{i=1}^k \text{supp}(\mu_i)$. Choose $u \in K_n$ such that $d(u, N) \geq 2$ and for every j one has $u(j) = 0$ or $u(j) = n$. (The geometrical idea behind choosing u is the following. There are uncountably many elements $u \in K_n$ with coordinates 0 or n . Closed balls of K_n with such centers u and with radius 1 do not intersect each other. Since N is finite, there are such balls

without elements from N . To visualize the K_n imagine an infinite dimensional Rubik’s cube.)

Define $\varphi(u) = 1$ and $\varphi = 0$ elsewhere. Now we deal with the f_i ’s. For every μ_i choose $g_i \in S_{\text{Lip}_0(K_n)}$ such that $g_i(\mu_i) = 1$. For every g_i let g_i^+ be its norm preserving extension from N to $K_n \setminus B(u, 1)$, where $B(u, 1)$ is the closed ball with center u and radius 1. Let

$$a_i = \frac{1}{2} \left(\max_{x \in S(u,2)} g_i^+(x) + \min_{x \in S(u,2)} g_i^+(x) \right).$$

Note that since $\text{Lip}(g_i^+) = 1$, the values of the function g_i^+ on the sphere $S(u, 2)$ differ by no more than 2. Moreover,

$$\min_{x \in S(u,2)} g_i^+(x) \leq n - 2$$

and

$$\max_{x \in S(u,2)} g_i^+(x) \geq -n + 2.$$

This is because $S(u, 2)$ contains a point of distance $n - 2$ from the origin. Therefore, $a_i \in [-n + 1, n - 1]$.

Define $f_i: K_n \rightarrow \mathbb{R}$ by

$$f_i(x) = \begin{cases} a_i, & \text{if } d(x, u) \leq 1, \\ g_i^+(x) & \text{elsewhere.} \end{cases}$$

Then $f_i \in S_i$ and $f_i \pm \varphi \in S_i$. Let us verify that $\|f_i \pm \varphi\| \leq 1$. Fix $x \in B(u, 1)$ and $y \in K_n \setminus B(u, 1)$.

If $x \neq u$, then $\varphi(x) = \varphi(y) = 0$ and therefore

$$|(f_i \pm \varphi)(x) - (f_i \pm \varphi)(y)| = |f_i(x) - f_i(y)| \leq d(x, y).$$

For $x = u$, fix $z \in S(u, 1)$ such that $d(u, y) = d(u, z) + d(z, y)$. Then

$$\begin{aligned} |(f_i \pm \varphi)(u) - (f_i \pm \varphi)(y)| &= |f_i(z) \pm \varphi(u) - f_i(y)| \\ &\leq |f_i(z) - f_i(y)| + 1 \\ &\leq d(z, y) + d(u, z) = d(u, y). \end{aligned}$$

□

We now collect the known examples of metric spaces M such that $\text{Lip}_0(M)$ has the w^* -SSD2P.

Theorem 5.7. *If M is an infinite metric space satisfying at least one of the following conditions:*

- (a) $\sup\{d(x, y) : x, y \in M, x \neq y\} = \infty$;
- (b) $\inf\{d(x, y) : x, y \in M, x \neq y\} = 0$;
- (c) M is a discrete metric space;
- (d) $M = K_n$, where $n \in \mathbb{N}$,

then $\text{Lip}_0(M)$ has the w^* -SSD2P.

Proof. (a), (c), and (d) are Propositions 5.2, 5.3, 5.5, and 5.6, respectively. An inspection of the proof of Theorem 2.4 in [7] shows (b). □

6. Questions

Let us end the paper with some questions that are suggested by the current work:

Question 6.1. *If a Banach space has the SSD2P must it then contain an isomorphic copy of c_0 ?*

Question 6.2. *Does there exist a dual Banach space with the w^* -SSD2P and without the SSD2P?*

Question 6.3. *If M has the LTP, does then $Lip_0(M)$ have the w^* -SSD2P?*

Acknowledgements

The authors wish to express their thanks to Indrek Zolk for his collaboration in proving Lemma 5.4

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

References

- [1] Abrahamsen, T.A., Langemets, J., Lima, V.: Almost square Banach spaces. *J. Math. Anal. Appl.* **434**(2), 1549–1565 (2016)
- [2] Abrahamsen, T.A., Leerand, A., Martiny, A., Nygaard, O.: Two properties of Müntz spaces. *Demonstr. Math.* **50**, 239–244 (2017)
- [3] Abrahamsen, T.A., Lima, V., Nygaard, O.: Remarks on diameter 2 properties. *J. Convex Anal.* **20**(2), 439–452 (2013)
- [4] Abrahamsen, T.A., Lima, V., Nygaard, O.: Almost isometric ideals in Banach spaces. *Glasgow Math. J.* **56**(2), 395–407 (2014)
- [5] Abrahamsen, T.A., Nygaard, O., Pöldvere, M.: New applications of extremely regular function spaces, to appear in *Pacific J. Math* (2017)
- [6] Acosta, M.D., Becerra-Guerrero, J., López-Pérez, G.: Stability results of diameter two properties. *J. Convex Anal.* **22**(1), 1–17 (2015)
- [7] Becerra Guerrero, J., López-Pérez, G., Rueda Zoca, A.: Octahedrality in Lipschitz-free Banach spaces, to appear in *Proc. Roy. Soc. Edinburgh Sect. A*
- [8] Becerra Guerrero, J., López-Pérez, G., Rueda Zoca, A.: Some results on almost square Banach spaces. *J. Math. Anal. Appl.* **438**(2), 1030–1040 (2016)
- [9] Becerra Guerrero, J., López-Pérez, G., Rueda Zoca, A.: Subspaces of Banach spaces with big slices. *Banach J. Math. Anal.* **10**, 771–782 (2016)
- [10] Bonsall, F.F., Duncan, J.: Numerical ranges. II, Cambridge University Press, New York-London, London Mathematical Society Lecture Notes Series, no. 10 (1973)
- [11] Ghoussoub, N., Godefroy, G., Maurey, B., Schachermayer, W.: Some topological and geometrical structures in Banach spaces. *Mem. Am. Math. Soc.* **70**(378), iv+116 (1987)
- [12] Haller, R., Langemets, J.: Two remarks on diameter 2 properties. *Proc. Est. Acad. Sci.* **63**(1), 2–7 (2014)

- [13] Haller, R., Langemets, J., Nadel, R.: Stability of average roughness, octahedrality, and strong diameter 2 properties of Banach spaces with respect to absolute sums. *Banach J. Math. Anal.* **12**(1), 222–239 (2018)
- [14] Haller, R., Langemets, J., Põldvere, M.: On duality of diameter 2 properties. *J. Convex Anal.* **22**, 465–483 (2015)
- [15] Ivakhno, Y.: Big slice property in the spaces of Lipschitz functions. *Visn. Khark. Univ., Ser. Mat. Prykl. Mat. Mekh.* **749**, 109–118 (2006)
- [16] McShane, E.J.: Extension of range of functions. *Bull. Am. Math. Soc.* **40**, 837–842 (1934)
- [17] Oja, E., Põldvere, M.: Principle of local reflexivity revisited. *Proc. Am. Math. Soc.* **135**(4), 1081–1088 (2007). (electronic)
- [18] Procházka, A., Rueda Zoca, A.: A characterisation of octahedrality in Lipschitz-free Banach spaces, to appear in *Ann. Inst. Fourier*
- [19] Werner, D.: M -ideals and the ‘basic inequality’. *J. Approx. Th.* **76**, 21–30 (1994)
- [20] Weaver, N.: *Lipschitz Algebras*. World Scientific, Singapore (1999)

Rainis Haller, Johann Langemets and Rihhard Nadel
Institute of Mathematics
University of Tartu
J. Liivi 2
50409 Tartu
Estonia
e-mail: johann.langemets@ut.ee

Rainis Haller
e-mail: rainis.haller@ut.ee

Rihhard Nadel
e-mail: nadel@ut.ee

Vegard Lima
NTNU, Norwegian University of Science and Technology
Aalesund, Postboks 1517
6025 Ålesund
Norway

Present Address

Vegard Lima
Department of Engineering Sciences
University of Agder
Postboks 422
4604 Kristiansand
Norway
e-mail: Vegard.Lima@uia.no

Received: April 5, 2018.

Revised: October 5, 2018.

Accepted: February 18, 2019.