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Numerical Solution of Fredholm Fractional Integro-differential Equation with Right-Sided Caputo's Derivative Using Bernoulli Polynomials Operational Matrix of Fractional Derivative

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Abstract. In this article, fractional integro-differential equation (FIDE) of Fredholm type involving right-sided Caputo's fractional derivative with multi-fractional orders is considered. Analytical expressions of the expansion coefficient c_k by Bernoulli polynomials approximation have been derived for both approximation of single- and double-variable function. The Bernoulli polynomials operational matrix of right-sided Caputo's fractional derivative $\mathbf{P}^{\alpha}_{-;B}$ is derived. By approximating each term in the Fredholm FIDE with right-sided Caputo's fractional derivative in terms of Bernoulli polynomials basis, the equation is reduced to a system of linear algebraic equation of the unknown coefficients c_k. Solving for the coefficients produces the approximate solution for this special type of FIDE.

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Keywords. Fredholm fractional integro-differential equation, Right-sided Caputo's fractional derivative, Bernoulli polynomials.

1. Introduction

In the past decade, we have seen an increasing amount of works made in the field of FIDEs. One may find such equations in mathematical models that describe a variety of physical phenomena such as wind ripple in the desert, nono-hydrodynamics, dropwise consideration, glass-forming process [1], viscoelasticity [2], and epidemic processes [3]. Numerous numerical methods are carried over naturally from those methods adopted for fractional differential equations (FDE). Some of the methods applied to solve FIDE numerically are Adomian Decomposition Method [4] and Homotopy Analysis Method [5].

One of the most widely used methods is the method based on polynomials which include the use of operational matrix. Examples of works that adopt polynomial-based method are Legendre polynomials [6], Chebyshev polynomials [7], Jacobi polynomials [8], and Bernstein polynomials [9].

On the other hand, the use of Bernoulli polynomials to solve FIDE is not new. Bhrawy et al. [10] uses matrix based on Bernoulli polynomials in solving high-order linear and nonlinear FIDE of Fredholm type. Meanwhile, [11] uses Bernoulli polynomials to solve nonlinear FIDE of Volterra type where the integral is approximated by Legendre–Gaussian quadrature rule. Other recent works involving Bernoulli polynomials can be found in [12]. It is found that many of these works mainly considers only FIDEs with left-sided Caputo's fractional derivative. Equations that involve right-sided Caputo's fractional derivatives receive much less attention. Saatmandi et al. [13] have derived the operational matrix of left-sided Caputo's fractional derivative and solve numerically the one-dimensional space fractional diffusion equation via tau method. Recently, Bhrawy et al. further derived the operational matrix of right-sided Caputo's fractional derivative based on shifted Legendre polynomials and combined with spectral-tau method to solve fractional advection–dispersion equation [14] and fractional diffusion-wave equation [15]. Bhrawy et al. [16] also derived the operational matrix of left-sided and right-sided Caputo's fractional derivative based on Chebyshev polynomials of first kind and applied the operational matrices in combination with spectral-tau method to solve two-sided space–time-fractional telegraph equation. These mentioned works focus on solving problems involving FDEs with right-sided Caputo's derivative, while the corresponding problem in FIDE is not considered. Moreover, such methods only focus on the use of orthogonal polynomials.

Therefore, it is the main aim of this paper to consider FIDE that involves right-sided Caputo's fractional derivatives and find numerical solutions to it by Bernoulli polynomials operational matrix method. This Bernoulli polynomials have some advantages over the other classical orthogonal polynomials when approximating an arbitrary function, as shown in [17]. The FIDE involving right-sided Caputo's fractional derivative is defined as follows:

$$\sum_{r=1}^{l} q_r\, {}^{C}D_{*}^{\alpha_r} f(x) = h(x) + \int_0^1 K(x,t) f(t)\mathrm{d}t$$
$$f^{(i)}(0) = d_i, \quad i = 0, 1, \ldots, m-1, \tag{1.1}$$

where $f(x)$ is the unknown solution, $q_r \in \mathbb{R}, r = 0, \ldots, l$ are constants, ${}^{C}D_{*}^{\alpha_r}$ can be either the left-sided Caputo's derivative ${}^{C}D_{a+}^{\alpha_r}$ or the right-sided Caputo's derivative ${}^{C}D_{b-}^{\alpha_r}$, $\alpha_r \geq 0$ are real derivative orders, $h(x)$ is the forced term known a-priori, and $K(x,t)$ is the Fredholm integral kernel function. Here, we assume that the solution of this FIDE exist. This study focuses on function approximation by Bernoulli polynomials, the derivation of the Bernoulli expansion coefficients, and, more importantly, the Bernoulli polynomials operational matrix of fractional derivative. The derived expressions are used to find numerical solution for FIDEs that involves not just the left-sided Caputo's fractional derivative ${}^{C}D^{\alpha}_{a+}$ but also the right-sided Caputo's fractional derivative ${}^{C}D^{\alpha}_{b-}$.

The paper is organized as follows. Section 2 gives basic properties of fractional calculus and Bernoulli polynomials. Section 3 discusses about function approximation by Bernoulli polynomials for single- and double-variable function. In Sect. 4, Bernoulli polynomials operational matrix for both left-sided and right-sided Caputo's fractional derivatives has been derived. In Sect. 5, we discuss about error and convergence analysis of proposed method. Section 6 describes the general procedure of solving single FIDE using Bernoulli polynomials approximation and examples are given in Sect. 7. Section 8 states the conclusion of this paper.

2. Preliminaries and Basic Concepts

2.1. Classical Caputo's Fractional Derivative with Singular Kernel

In this section, we give the definition and basic properties of Caputo's fractional derivatives for both left-sided and right-sided. For more details of Caputo's fractional derivative, the reader is advised to refer to [18–20].

Definition 2.1 ([18] *Theorem 2.1*). Let $\Re(\alpha) \geq 0$ and let n be given by the following:

$$n = \begin{cases} \lceil \Re(\alpha) \rceil & , \ \alpha \notin \mathbb{N}_0 \\ \alpha & , \ \alpha \in \mathbb{N}_0, \end{cases} \tag{2.1}$$

where $\lceil \Re(\alpha) \rceil$ denotes the smallest integer greater or equal to $\Re(\alpha)$. If $y(x) \in AC^n[a,b]$, then the *Caputo's fractional derivatives* are the following:

(1) The *left-sided Caputo's fractional derivative* ${}^{C}D^{\alpha}_{a+;x}f(x)$ of order $\alpha \in \mathbb{C}$ $(\Re(\alpha) > 0)$ exists almost everywhere on $[a,b]$ and is defined as follows:

$${}^{C}D^{\alpha}_{a+;x}f(x) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{f^{(n)}(t)}{(x-t)^{\alpha-n+1}} \mathrm{d}t, & \alpha \notin \mathbb{N}_0, n = [\Re(\alpha)] + 1, x > a \\ D^n f(x), & \alpha \in \mathbb{N}_0 \end{cases} \tag{2.2}$$

(2) The *right-sided Caputo's fractional derivative* ${}^{C}D^{\alpha}_{b-;x}f(x)$ of order $\alpha \in \mathbb{C}$ $(\Re(\alpha) > 0)$ exists almost everywhere on $[a,b]$ and is defined as follows:

$${}^{C}D^{\alpha}_{b-;x}f(x) = \begin{cases} \frac{(-1)^n}{\Gamma(n-\alpha)} \int_x^b \frac{f^{(n)}(t)}{(t-x)^{\alpha-n+1}} \mathrm{d}t, & \alpha \notin \mathbb{N}_0, n = [\Re(\alpha)] + 1, x < b \\ (-1)^n D^n f(x), & \alpha \in \mathbb{N}_0. \end{cases} \tag{2.3}$$

Below are some important properties of these operators: ${}^{C}D^{\alpha}_{a+}, {}^{C}D^{\alpha}_{b-}$. All these operators satisfy the linearity property:

$${}^{C}D^{\alpha}_{*}(k_1 f_1(x) + k_2 f_2(x)) = k_1({}^{C}D^{\alpha}_{*} f_1(x)) + k_2({}^{C}D^{\alpha}_{*} f_2(x)), \tag{2.4}$$

where ${}^{C}D^{\alpha}_{*}$ applies to both left-sided and right-sided corresponding operators:

$$^{C}D_{a+}^{\alpha}(x-a)^{\beta}=\frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(x-a)^{\beta-\alpha} \tag{2.5}$$

$$^{C}D_{b-}^{\alpha}(b-x)^{\beta}=\frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(b-x)^{\beta-\alpha} \tag{2.6}$$

$$^{C}D_{a+}^{\alpha}(x-a)^{k}=0,\ ^{C}D_{b-}^{\alpha}(b-x)^{k}\quad=0,\ k=0,1,\ldots,n-1. \tag{2.7}$$

2.2. Bernoulli Polynomials: Definitions and Basic Properties

There are many ways to represent the Bernoulli polynomials and it is constructed from Bernoulli numbers. Various important properties and relations with polynomials of the same class such as Genocchi, Euler can be found in the works by [21].

Definition 2.2. [21] The Bernoulli Polynomials $B_n(x)$ of order n are defined via its generating function given by the following:

$$\mathcal{G}_B(x,t)=\frac{te^{xt}}{e^t-1}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!}. \tag{2.8}$$

The Bernoulli Polynomials $B_n(x)$ of order n can also be expressed using Bernoulli numbers b_n by the following explicit formula:

$$B_n(x)=\sum_{k=0}^{n}\binom{n}{k}b_{n-k}x^k,\ n=0,1,2,\ldots,\in\mathbb{N}_0, \tag{2.9}$$

where the Bernoulli numbers are defined by the generating function, $\mathcal{G}_b(t)=\sum_{k=0}^{\infty}b_n\frac{t^n}{n!},\ n=0,1,2,\ldots,\in\mathbb{N}_0$. The first few Bernoulli numbers are $b_0=1$, $b_1=-1/2$, $b_2=1/6$, $b_3=0\ldots$. In particular, $b_{2k+1}=0,\ k=1,2,3,\ldots$.

Bernoulli polynomials satisfy important properties which are stated below [21]:

$$\begin{aligned}B_{n+1}(x)-B_n(x)&=nx^{n-1}\\ B_{n+1}(x)&=\sum_{k=0}^{n}\binom{n}{k}B_k(x)\\ B_n(1-x)&=(-1)^nB_n(x)\\ B_n(x)&=n\int_0^x B_{n-1}(t)\mathrm{d}t+b_n\end{aligned} \tag{2.10}$$

$$\int_0^1 B_n(x)\mathrm{d}x=\delta_{0,n},\ n\geq 0 \tag{2.11}$$

$$B_{2k+1}(1)+B_{2k+1}(0)=0,\ k=0,1,2,\ldots \tag{2.12}$$

The following property will be useful later:

$$\begin{aligned}B_n(1)-B_n(0)=B_n(1)-b_n&=\delta_{1,n},\ n\geq 1\\ B_0(1)=B_0(0)&=1.\end{aligned} \tag{2.13}$$

The integral of product of two Bernoulli polynomials is given by the following:

$$\beta_{n,m} = \int_0^1 B_n(x)B_m(x)\mathrm{d}x = \begin{cases} 1, & n = m = 0 \\ 0, & n + m = 1 \\ (-1)^{n-1}\frac{m!n!}{(m+n)!}b_{n+m}, & n + m \geq 2. \end{cases} \tag{2.14}$$

The first derivative property are similar to that of Genocchi and Euler:

$$\frac{\mathrm{d}B_n(x)}{\mathrm{d}x} = nB_{n-1}(x), \; n \geq 1. \tag{2.15}$$

3. Function Approximation by Bernoulli Polynomials

Given an arbitrary continuously differentiable function $f(x) \in C^\infty(I)$ over a region $I = [a, b] \in \mathbb{R}$, one may expand the function in a series of Bernoulli polynomials acting as basis:

$$f(x) = \sum_{n=0}^{\infty} c_n B_n(x), \tag{3.1}$$

where c_n are Bernoulli expansion coefficients to be determined. Practically, one needs to truncate the above series to obtain the approximate function $f^*(x)$ up to order $N-1$ as follows:

$$f(x) \approx f^*(x) = \sum_{n=0}^{N-1} c_n B_n(x). \tag{3.2}$$

Rewriting the equation in matrix notation, it takes the familiar form:

$$\begin{gathered} f(x) \approx f^*(x) = \mathbf{C}^T\mathbf{B}(x) \\ \mathbf{C} = [c_0 \; c_1 \; \cdots \; c_{N-1}]^T, \mathbf{B}(x) = [B_0(x) \; B_1 \; \cdots \; B_{N-1}]^T, \end{gathered} \tag{3.3}$$

where $\mathbf{C}$ is the *coefficient vector* and $\mathbf{B}(x)$ is the *Bernoulli polynomials basis vector*. Approximation by polynomials exists and has been guaranteed by the important Weierstrass approximation theorem which can be found in [22].

3.1. Computation of Bernoulli Coefficients for Single-Variable Function

To determine the values of the expansion coefficients c_n for Bernoulli polynomials, one may follow one of the following two approaches:

First Approach:

Step 1: From (3.1), take inner products with respect to Bernoulli polynomials basis over the interval $[0, 1]$:

$$\begin{aligned} \langle f(x), B_j(x) \rangle &\approx \sum_{i=0}^{N-1} c_i \langle B_i(x), B_j(x) \rangle \\ \int_0^1 f(x)B_j(x)\mathrm{d}x &\approx \sum_{i=0}^{N-1} c_i \left(\int_0^1 B_i(x)B_j(x)\mathrm{d}x \right). \end{aligned} \tag{3.4}$$

Representing the above in matrix:

$$\mathbf{F}_B \approx \mathbf{C}^T \mathbf{T}_B, \tag{3.5}$$

where $\mathbf{F}_B = \left[\int_0^1 f(x)B_j(x)\mathrm{d}x\right]^T$, and the matrix $\mathbf{T}_B = [T_{i,j;B}]^T$, and $T_{i,j;B} = \int_0^1 B_i(x)B_j(x)\mathrm{d}x$ has elements which are integral of products of two Bernoulli polynomials computed using (2.14).

Step 2: Taking the inverse of $\mathbf{T}_B$ at both sides of (3.5) gives the coefficient vector C with coefficients c_n as its elements:

$$\mathbf{C}^T = \mathbf{F}_B \mathbf{T}_B^{-1}. \tag{3.6}$$

Second Approach: Another approach which we will derive here requires that $f(x) \in C^N([0,1])$.

Step 1: Assume that a function $f(x)$ is expanded in Bernoulli polynomials:

$$f(x) = \sum_{n=0}^{N-1} c_n B_n(x). \tag{3.7}$$

Then, taking k-order derivatives and using the first derivative property of (2.15) repeatedly give:

$$f^{(k)}(x) = \frac{d^k}{\mathrm{d}x^k} f(x) = \sum_{n=k}^{N-1} c_n (n)^{(k)} B_{n-k}(x), k = 0, 1, 2, \ldots, \tag{3.8}$$

where $(n)^{(k)} = n(n-1)\cdots(n-k+1)$ is the falling factorial.

Step 2: Using property of (2.13),

$$\begin{aligned} f^{(k)}(1) - f^{(k)}(0) &= \sum_{n=k}^{N-1} c_n (n)^{(k)} (B_{n-k}(1) - B_{n-k}(0)), k = 0, 1, 2, \ldots \\ &= \sum_{n=k}^{N-1} c_n (n)^{(k)} \delta_{1,n-k} = c_{k+1}(k+1)^{(k)} = c_{k+1}(k+1)! \end{aligned} \tag{3.9}$$

since $\delta_{1,n-k} = \begin{cases} 1, & n = k+1 \\ 0, & n \neq k+1 \end{cases}$, and thus,

$$c_{k+1} = \frac{1}{(k+1)!}(f^{(k)}(1) - f^{(k)}(0)), k = 0, 1, 2, \ldots \tag{3.10}$$

$$c_k = \frac{1}{k!}(f^{(k-1)}(1) - f^{(k-1)}(0)), k = 1, 2, 3, \ldots \tag{3.11}$$

Step 3: Finally, to compute coefficient c_0:

$$\int_0^1 f(x)\mathrm{d}x = \sum_{n=0}^{N} c_n \int_0^1 B_n(x)\mathrm{d}x. \tag{3.12}$$

By (2.11):

$$c_0 = \int_0^1 f(x)\mathrm{d}x. \tag{3.13}$$

3.2. Computation of Bernoulli Coefficients for Two-Variable Function

Given a function with two variables $K(x,t)$, it can be expanded in Bernoulli polynomials in variables x,t by the following series:

$$K(x,t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} k_{i,j}B_i(x)B_j(t). \tag{3.14}$$

Again, in real practice, only approximation can be made by truncating the above series using Bernoulli polynomials up to order N for both x and t:

$$K(x,t) \approx \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} k_{i,j}B_i(x)B_j(t) = \mathbf{B}^T(x)\mathbf{K}\mathbf{B}(t), \tag{3.15}$$

where $\mathbf{K} = [k_{i,j}]_{N\times N}$ is the two-variable coefficient matrix with coefficients $k_{i,j}$ as its elements.

The following procedure shows the first approach to compute the $k_{i,j}$:

First Approach:

Step 1: From (3.15), take inner products with respect to Bernoulli polynomials basis over the region $[0,1]\times[0,1]$ with respect to x and t:

$$\langle\langle K(x,t), B_k(x)\rangle, B_l(x)\rangle \approx \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} k_{i,j}\langle B_i(x), B_k(x)\rangle\langle B_j(t), B_l(t)\rangle. \tag{3.16}$$

This is equivalent in matrix notation as follows:

$$\begin{gathered}\int_0^1\int_0^1 \mathbf{B}(x)K(x,t)\mathbf{B}^T(t)\mathrm{d}x\mathrm{d}t \approx \left(\int_0^1 \mathbf{B}(x)\mathbf{B}^T(x)\mathrm{d}x\right)\mathbf{K}\left(\int_0^1 \mathbf{B}(t)\mathbf{B}^T(t)\mathrm{d}t\right)\\ \mathcal{K}_B = \mathbf{T}_B\mathbf{K}\mathbf{T}_B,\end{gathered} \tag{3.17}$$

where $\mathbf{T}_B$ is the same as given in (3.5) and

$$\mathcal{K}_B = \int_0^1\int_0^1 \mathbf{B}(x)K(x,t)\mathbf{B}^T(t)\mathrm{d}x\mathrm{d}t = \left[\int_0^1\int_0^1 K(x,t)B_k(x)B_l(t)\mathrm{d}x\mathrm{d}t\right]. \tag{3.18}$$

Step 2: Finally, taking the inverses for both $\mathbf{T}_B$ at both sides of (3.17) gives the two-variable coefficient matrix $\mathbf{K}$:

$$\mathbf{K} = \mathbf{T}_B^{-1}\mathcal{K}_B\mathbf{T}_B^{-1}. \tag{3.19}$$

It is clear that this approach requires $K(x,t)$ to be a continuous integrable function and does not require the strict condition $K(x,t) \in C^{\infty}([0,1]^2)$.

Second Approach: Second approach similar to the concepts used previously for single-variable function requires that $K(x,t) \in C^{N-2}([0,1]^2)$. We have the following theorem:

Theorem 3.1. *Given a function $K(x,t) \in C^{N-2}([0,1]^2)$, it can be approximated by Bernoulli polynomials in variables x,t by the following truncated series of order N:*

$$K(x,t) = \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} k_{i,j}B_i(x)B_j(t). \tag{3.20}$$

In matrix notation, the series takes the form of $K(x,t) = \mathbf{B}^T(x)\mathbf{K}\mathbf{B}(t)\mathbf{K} = [k_{i,j}]$ *where* $\mathbf{K}$ *is the two-variable coefficient matrix and the* $k_{i,j}$ *is given by the following:*

$$
\begin{aligned}
k_{i,j} &= \frac{1}{i!j!}\left(\frac{\partial^{i+j-2}}{\partial x^{i-1}\partial t^{j-1}}K(1,1) - \frac{\partial^{i+j-2}}{\partial x^{i-1}\partial t^{j-1}}K(0,1)\right.\\
&\qquad \left. - \frac{\partial^{i+j-2}}{\partial x^{i-1}\partial t^{j-1}}K(1,0) + \frac{\partial^{i+j-2}}{\partial x^{i-1}\partial t^{j-1}}K(0,0)\right) \quad , \ i \geq 1, \ j \geq 1\\
k_{0,j} &= \frac{1}{j!}\int_0^1 \left(\frac{\partial^{j-1}}{\partial t^{j-1}}K(x,1) - \frac{\partial^{j-1}}{\partial t^{j-1}}K(x,0)\right)\mathrm{d}x \quad , \ i = 0, \ j \geq 1\\
k_{i,0} &= \frac{1}{i!}\int_0^1 \left(\frac{\partial^{i-1}}{\partial x^{i-1}}K(1,t) - \frac{\partial^{i-1}}{\partial x^{i-1}}K(0,t)\right)\mathrm{d}t \quad , \ i \geq 1, \ j = 0\\
k_{0,0} &= \int_0^1 \int_0^1 K(x,t)\mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{3.21}
$$

Proof. From (3.20), let $\phi_i(t) = \sum_{j=0}^{N-1} k_{i,j}B_j(t)$, then, $K(x,t) = \sum_{i=0}^{N-1} \phi_i(t)$ $B_i(x)$. Thus, by applying (3.11) w.r.t. variable x, we have the following:

$$
\phi_i(t) = \frac{1}{i!}\left(\frac{\partial^{i-1}}{\partial x^{i-1}}K(1,t) - \frac{\partial^{i-1}}{\partial x^{i-1}}K(0,t)\right), \ i \geq 1, \ j \geq 1. \tag{3.22}
$$

Applying (3.11) to $\phi_i(t) = \sum_{j=0}^{N-1} k_{i,j}B_j(t)$ w.r.t. variable t and using (3.22), we obtain the expression $k_{i,j}$ for $i \geq 1, \ j \geq 1$ as follows:

$$
\begin{aligned}
k_{i,j} &= \frac{1}{j!}\left(\frac{\partial^{j-1}}{\partial t^{j-1}}\phi_i(1) - \frac{\partial^{j-1}}{\partial t^{j-1}}\phi_i(0)\right)\\
&= \frac{1}{j!}\frac{1}{i!}\left\{\frac{\partial^{j-1}}{\partial t^{j-1}}\left(\frac{\partial^{i-1}}{\partial x^{i-1}}K(1,t) - \frac{\partial^{i-1}}{\partial x^{i-1}}K(0,t)\right)|_{t=1}\right.\\
&\qquad \left. - \frac{\partial^{j-1}}{\partial t^{j-1}}\left(\frac{\partial^{i-1}}{\partial x^{i-1}}K(1,t) - \frac{\partial^{i-1}}{\partial x^{i-1}}K(0,t)\right)|_{t=0}\right\}\\
&= \frac{1}{i!j!}\left(\frac{\partial^{i+j-2}}{\partial x^{i-1}\partial t^{j-1}}K(1,1) - \frac{\partial^{i+j-2}}{\partial x^{i-1}\partial t^{j-1}}K(0,1)\right.\\
&\qquad \left. - \frac{\partial^{i+j-2}}{\partial x^{i-1}\partial t^{j-1}}K(1,0) + \frac{\partial^{i+j-2}}{\partial x^{i-1}\partial t^{j-1}}K(0,0)\right), \ i \geq 1, \ j \geq 1.
\end{aligned} \tag{3.23}
$$

For the case $i = 0, \ j \geq 1$, take integral over $[0,1]$ w.r.t. x for (3.20), we have the following:

$$
\int_0^1 K(x,t)\mathrm{d}x = \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} k_{i,j}B_j(t)\int_0^1 B_i(x)\mathrm{d}x = \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} k_{i,j}B_j(t)\delta_{i,0}. \tag{3.24}
$$

Thus, $\int_0^1 K(x,t)\mathrm{d}x = \sum_{j=0}^{N-1} k_{0,j} B_j(t)$, where

$$\begin{aligned} k_{0,j} &= \frac{1}{j!}\left\{\frac{\partial^{j-1}}{\partial t^{j-1}}\left(\int_0^1 K(x,t)\mathrm{d}x\right)|_{t=1} - \frac{\partial^{j-1}}{\partial t^{j-1}}\left(\int_0^1 K(x,t)\mathrm{d}x\right)|_{t=0}\right\} \\ &= \frac{1}{j!}\int_0^1 \left(\frac{\partial^{j-1}}{\partial t^{j-1}}K(x,1) - \frac{\partial^{j-1}}{\partial t^{j-1}}K(x,0)\right)\mathrm{d}x. \end{aligned} \tag{3.25}$$

Now, for $i \geq 1,\ j = 0$, take integral over $[0,1]$ w.r.t. t for (3.20), we have the following:

$$\int_0^1 K(x,t)\mathrm{d}t = \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} k_{i,j}\left(\int_0^1 B_j(t)\mathrm{d}t\right) B_i(x) = \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} k_{i,j}\left(\delta_{j,0}\right) B_i(x). \tag{3.26}$$

Thus, $\int_0^1 K(x,t)\mathrm{d}t = \sum_{i=0}^{N-1} k_{i,0} B_i(x)$, where

$$\begin{aligned} k_{i,0} &= \frac{1}{i!}\left\{\frac{\partial^{i-1}}{\partial x^{i-1}}\left(\int_0^1 K(x,t)\mathrm{d}t\right)|_{x=1} - \frac{\partial^{i-1}}{\partial x^{i-1}}\left(\int_0^1 K(x,t)\mathrm{d}t\right)|_{x=0}\right\} \\ &= \frac{1}{i!}\int_0^1 \left(\frac{\partial^{i-1}}{\partial x^{i-1}}K(1,t) - \frac{\partial^{i-1}}{\partial x^{i-1}}K(0,t)\right)\mathrm{d}t. \end{aligned} \tag{3.27}$$

For $i = 0,\ j = 0$, take integrals over $[0,1]$ w.r.t. x and t for (3.20), we have the following:

$$\begin{aligned} \int_0^1\int_0^1 K(x,t)\mathrm{d}x\mathrm{d}t &= \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} k_{i,j}\left(\int_0^1 B_j(t)\mathrm{d}t\right)\left(\int_0^1 B_i(x)\mathrm{d}x\right) \\ &= \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} k_{i,j}\delta_{i,0}\delta_{j,0}. \end{aligned} \tag{3.28}$$

Thus, $k_{0,0} = \int_0^1\int_0^1 K(x,t)\mathrm{d}x\mathrm{d}t$ □

4. Bernoulli Polynomials Operational Matrix of Caputo's Fractional Derivative

In general, most articles that adopted the polynomial-based method of operational matrices only considered problems that involve left-sided Caputo ${}^C D^{\alpha}_{a+} f(x)$ and very less attention is given to equations that involve right-sided Caputo's fractional derivative ${}^C D^{\alpha}_{b-}$. The Bernoulli polynomials operational matrix of left-sided Caputo's fractional derivative has been applied by others and its implicit form in matrix can be found in [10,11]. This section derives the explicit form of the elements of Bernoulli polynomials operational matrix of left-sided as well as right-sided Caputo's fractional derivative. The Bernoulli polynomials operational matrix $\mathbf{P}^{\alpha}_{*;B}$ of Caputo's fractional derivatives ${}^C D^{\alpha}_{*}$ is the $N \times N$ matrix that approximates the operation of Caputo's fractional derivatives acting on the Bernoulli polynomials basis vector

$\mathbf{B}(x) = [B_i(x)]^T$ and takes the following general form:

$$
{}^C D_*^{\alpha} \mathbf{B}(x) = \mathbf{P}_{*;B}^{\alpha} \mathbf{B}(x)
$$

$$
{}^C D_*^{\alpha} \begin{bmatrix} B_0(x) \\ B_1(x) \\ \vdots \\ B_{N-1}(x) \end{bmatrix} = \begin{bmatrix} \rho_{0,0} & \rho_{0,1} & \cdots & \rho_{0,N-1} \\ \rho_{1,0} & \rho_{1,1} & \cdots & \rho_{1,N-1} \\ \vdots & \cdots & \cdots & \vdots \\ \rho_{N-1,0} & \rho_{N-1,1} & \cdots & \rho_{N-1,N-1} \end{bmatrix} \begin{bmatrix} B_0(x) \\ B_1(x) \\ \vdots \\ B_{N-1}(x) \end{bmatrix}, \tag{4.1}
$$

where $\rho_{i,j}$ is the expansion coefficient of the fractional derivative ${}^C D_*^{\alpha} B_i(x)$ of order-i Bernoulli polynomials by the order-j Bernoulli polynomials, that is

$$
{}^C D_*^{\alpha} B_i(x) = \sum_{j=0}^{N-1} \rho_{i,j} B_j(x). \tag{4.2}
$$

Let ${}^C D_*^{\alpha}$ represents either the left-sided Caputo's fractional derivative ${}^C D_{a+}^{\alpha}$ or the right-sided Caputo's fractional derivative ${}^C D_{b-}^{\alpha}$, while $\mathbf{P}_{*;B}^{\alpha}$ represents either the Bernoulli polynomials operational matrix of left-sided Caputo's fractional derivative $\mathbf{P}_{+;B}^{\alpha}$ or the Bernoulli polynomials operational matrix of right-sided Caputo's fractional derivative $\mathbf{P}_{-;B}^{\alpha}$.

4.1. Derivation of Operational Matrix of Fractional Derivative

This section briefly explains the usual approach to find the polynomial-based operational matrix particularly when the basis polynomials are non-orthogonal. It uses the concept of least-square approximation by taking inner products w.r.t. the basis polynomials and solving the "*normal equation*" generated which can be expressed in Gram matrices. To do so, we start the procedure by consider ${}^C D_*^{\alpha} \mathbf{B}(x) = \mathbf{P}_{*;B}^{\alpha} \mathbf{B}(x)$. Hence, take the inner product by multiplying with Bernoulli polynomials basis $\mathbf{B}(x)$ and integrate over interval $[0, 1]$:

$$
\begin{aligned}
\int_0^1 \left({}^C D_*^{\alpha} \mathbf{B}(x) \right) \mathbf{B}^T(x) \mathrm{d}x &= \mathbf{P}_{*;B}^{\alpha} \left(\int_0^1 \mathbf{B}(x) \mathbf{B}^T(x) \mathrm{d}x \right) \\
\Theta_{N;B} &= \mathbf{P}_{*;B}^{\alpha} \mathbf{T}_B,
\end{aligned} \tag{4.3}
$$

where

$$
\Theta_{N;B} = \left[\int_0^1 \left\{ {}^C D_*^{\alpha} (B_i(x)) \right\} B_j(x) \mathrm{d}x \right] \tag{4.4}
$$

and $\mathbf{T}_B = [T_{i,j;B}] = \int_0^1 \mathbf{B}(x) \mathbf{B}^T(x) \mathrm{d}x$. Hence, after taking inverse of $\mathbf{T}_B$ at both sides, we have $\mathbf{P}_{*;B}^{\alpha} = \Theta_{N;B} \mathbf{T}_B^{-1}$.

By doing so, it leads to the following theorem on computation of Bernoulli polynomials operational matrix of Caputo's fractional derivative.

Theorem 4.1. *Given a set* $B_i(x), i = 0, 1, \ldots N-1$ *of* N *Bernoulli polynomials, the Bernoulli polynomials operational matrix of Caputo's fractional derivative of order* $\alpha \neq \lceil \alpha \rceil$ *over the interval* $[0, 1]$ *is the* $N \times N$ *matrix* $\mathbf{P}_{*;B}^{\alpha}$*:*

$$
\mathbf{P}_{*;B}^{\alpha} = \Theta_{N;B} \mathbf{T}_B^{-1}, \tag{4.5}
$$

where $\Theta_{N;B}$ *is given in* (4.4) *and* $\mathbf{T}_B = [T_{i,j;B}]$ *is computed using* (3.5). ${}^C D_*^\alpha$ *represents either* ${}^C D_{a+}^\alpha$ *or* ${}^C D_{b-}^\alpha$*, and* $\mathbf{P}_{*;B}^\alpha$ *represents either* $\mathbf{P}_{+;B}^\alpha$ *or* $\mathbf{P}_{-;B}^\alpha$*.*

This method has the merit of relaxing strict requirements of the differentiability of the function $f(x)$ to be approximated. However, this method does not produce an explicit form for the expression of the elements $\rho_{i,j}$ of the operational matrix $\mathbf{P}_{*;B}^\alpha$.

4.2. Bernoulli Polynomials Operational Matrix of Left-Sided Caputo's Fractional Derivative

In this section, we derive the analytical expression of the Bernoulli polynomials operational matrix of left-sided Caputo's fractional derivative using second approach different from the usual approach described in Sect. 4.1. Here, the analytical expression of each element $\rho_{i,j}$ of the operational matrix $\mathbf{P}_{+;B}^\alpha$ will be derived. Before this, we need the following Lemma.

Lemma 4.2. *The left-sided Caputo's fractional derivative of fractional order* α *of a Bernoulli polynomials of order* i *is given by the following:*

$$ {}^C D_{a+}^\alpha B_i(x) = \begin{cases} \sum_{r=n}^{i} \sum_{v=n}^{r} \frac{i! b_{i-r} a^{r-v} (x-a)^{v-\alpha}}{(i-r)!(r-v)!\Gamma(v-\alpha+1)}, \ n = \lceil \alpha \rceil, \ i \geq \alpha \\ 0, \qquad i < \alpha. \end{cases} \tag{4.6} $$

Proof. For $n-1 < \alpha \leq n, n = \lceil \alpha \rceil$, we have ${}^C D_{a+}^\alpha B_i(x) = \sum_{r=0}^{i} \binom{i}{r} b_{i-r} {}^C D_{a+}^\alpha x^r$. Using (2.5) and ${}^C D_{a+}^\alpha B_i(x) = 0, \ i < \alpha$:

$$ \begin{aligned} {}^C D_{a+}^\alpha B_i(x) &= \sum_{r=n}^{i} \binom{i}{r} b_{i-r} {}^C D_{a+}^\alpha (x-a+a)^r, \ i \geq n = \alpha \\ &= \sum_{r=n}^{i} \binom{i}{r} b_{i-r} \sum_{v=n}^{r} \binom{r}{v} a^{r-v} {}^C D_{a+}^\alpha (x-a)^v \\ &= \sum_{r=n}^{i} \binom{i}{r} b_{i-r} \sum_{v=n}^{r} \binom{r}{v} a^{r-v} \frac{\Gamma(v+1)}{\Gamma(v-\alpha+1)} (x-a)^{v-\alpha} \\ &= \sum_{r=n}^{i} \binom{i}{r} b_{i-r} \sum_{v=n}^{r} \frac{r! a^{r-v}}{(r-v)!\Gamma(v-\alpha+1)} (x-a)^{v-\alpha} \\ &= \sum_{r=n}^{i} \sum_{v=n}^{r} \frac{i! b_{i-r} a^{r-v}}{(i-r)!(r-v)!\Gamma(v-\alpha+1)} (x-a)^{v-\alpha}. \end{aligned} $$

□

Then, using Lemma 4.2, we have the following theorem.

Theorem 4.3. *For* ${}^C D_{a+}^\alpha \mathbf{B}(x) = \mathbf{P}_{+;B}^\alpha \mathbf{B}(x)$*, the element of* $\mathbf{P}_{+;B}^\alpha$ *is given by the following:*

$$ \rho_{i,j} = \begin{cases} \frac{1}{j!} \sum_{r=n}^{i} \sum_{v=n}^{r} \frac{i! b_{i-r} a^{r-v}}{(i-r)!(r-v)!\Gamma(v-\alpha-j+2)} ((1-a)^{v-\alpha-j+1} - (-a)^{v-\alpha-j+1}), \ j \geq 1 \\ \sum_{r=n}^{i} \sum_{v=n}^{r} \frac{i! b_{i-r} a^{r-v}}{(i-r)!(r-v)!\Gamma(v-\alpha+2)} ((1-a)^{v-\alpha+1} - (-a)^{v-\alpha+1}), \qquad j = 0 \end{cases} \tag{4.7} $$

Proof. Let ${}^{C}D^{\alpha}_{a+}B_i(x) = \sum_{j=0}^{N-1} \rho_{i,j}B_j(x)$, where as usual ${}^{C}D^{\alpha}_{a+}B_i(x) = 0,\ i < \alpha$. Using (3.8) w.r.t. the column index j and (4.6), For $i \geq \alpha, j \geq 1$:

$$\begin{aligned}
\rho_{i,j} &= \frac{1}{j!}\left(\frac{d^{j-1}}{\mathrm{d}x^{j-1}}({}^{C}D^{\alpha}_{a+}B_i(x))|_{x=1} - \frac{d^{j-1}}{\mathrm{d}x^{j-1}}({}^{C}D^{\alpha}_{a+}B_i(x))|_{x=0}\right),\ j \geq 1 \\
&= \frac{1}{j!}\sum_{r=n}^{i}\sum_{v=n}^{r}\frac{i!b_{i-r}a^{r-v}}{(i-r)!(r-v)!\Gamma(v-\alpha+1)}\frac{\Gamma(v-\alpha+1)}{\Gamma(v-\alpha-j+2)} \\
&\quad \times ((x-a)^{v-\alpha-j+1}|_{x=1} - (x-a)^{v-\alpha-j+1}|_{x=0}) \\
&= \frac{1}{j!}\sum_{r=n}^{i}\sum_{v=n}^{r}\frac{i!b_{i-r}a^{r-v}((1-a)^{v-\alpha-j+1} - (-a)^{v-\alpha-j+1})}{(i-r)!(r-v)!\Gamma(v-\alpha-j+2)}.
\end{aligned}$$

For $i \geq \alpha, j = 0$, using (3.13) and (4.6):

$$\begin{aligned}
\rho_{i,0} &= \int_0^1 {}^{C}D^{\alpha}_{a+}B_i(x)\mathrm{d}x \\
&= \sum_{r=n}^{i}\sum_{v=n}^{r}\frac{i!b_{i-r}a^{r-v}}{(i-r)!(r-v)!\Gamma(v-\alpha+1)}\int_0^1 (x-a)^{v-\alpha}\mathrm{d}x \\
&= \sum_{r=n}^{i}\sum_{v=n}^{r}\frac{i!b_{i-r}a^{r-v}}{(i-r)!(r-v)!\Gamma(v-\alpha+2)}((1-a)^{v-\alpha+1} - (-a)^{v-\alpha+1}).
\end{aligned}$$

□

4.3. Bernoulli Polynomials Operational Matrix of Right-Sided Caputo's Fractional Derivative

In this section, we derive the analytical expression of the Bernoulli polynomials operational matrix of right-sided Caputo's fractional derivative using the second approach different from the usual approach described in Sect. 4.1. Here, the analytical expression of each element $\rho_{i,j}$ of the operational matrix $\mathbf{P}^{\alpha}_{-;B}$ will be derived. Before this, we need the following Lemma.

Lemma 4.4. *The right-sided Caputo's fractional derivative of fractional order α of a Bernoulli polynomials of order i is given by the following:*

$$ {}^{C}D^{\alpha}_{b-}B_i(x) = \begin{cases} \sum_{r=n}^{i}\sum_{v=n}^{r}\frac{(-1)^v i! b_{i-r}(b)^{r-v}(b-x)^{v-\alpha}}{(i-r)!(r-v)!\Gamma(v-\alpha+1)}, & n = \lceil\alpha\rceil, i \geq \alpha \\ 0, \quad i < \alpha. \end{cases} \tag{4.8}$$

Proof. For $n-1 < \alpha \leq n, n = \lceil\alpha\rceil$, we have ${}^{C}D^{\alpha}_{b-}B_i(x) = \sum_{r=0}^{i}\binom{i}{r}b_{i-r}{}^{C}D^{\alpha}_{b-}x^r$. Using (2.6) and ${}^{C}D^{\alpha}_{b-}B_i(x) = 0,\ i < \alpha$:

$$\begin{aligned}
{}^{C}D^{\alpha}_{b-}B_i(x) &= \sum_{r=n}^{i}\binom{i}{r}b_{i-r}{}^{C}D^{\alpha}_{b-}(x-b+b)^r \\
&= \sum_{r=n}^{i}\binom{i}{r}b_{i-r}\sum_{v=n}^{r}\binom{r}{v}b^{r-v}(-1)^{v}{}^{C}D^{\alpha}_{b-}(b-x)^v
\end{aligned}$$

$$= \sum_{r=n}^{i} \binom{i}{r} b_{i-r} \sum_{v=n}^{r} (-1)^v \binom{r}{v} b^{r-v} \frac{\Gamma(v+1)}{\Gamma(v-\alpha+1)} (b-x)^{v-\alpha}$$

$$= \sum_{r=n}^{i} \binom{i}{r} b_{i-r} \sum_{v=n}^{r} \frac{(-1)^v r! b^{r-v}}{(r-v)!\Gamma(v-\alpha+1)} (b-x)^{v-\alpha}$$

$$= \sum_{r=n}^{i} \sum_{v=n}^{r} \frac{(-1)^v i! b_{i-r}(b)^{r-v}}{(i-r)!(r-v)!\Gamma(v-\alpha+1)} (b-x)^{v-\alpha}. \qquad \square$$

Then, using Lemma 4.4, we have the following theorem.

Theorem 4.5. *For* ${}^{C}D^{\alpha}_{b-}\mathbf{B}(x) = \mathbf{P}^{\alpha}_{-;B}\mathbf{B}(x)$*, the element of* $\mathbf{P}^{\alpha}_{-;B}$ *is given by the following:*

$$\rho_{i,j} = \begin{cases} \frac{1}{j!}\sum_{r=n}^{i}\sum_{v=n}^{r} \frac{(-1)^v i! b_{i-r}(b)^{r-v}((b-1)^{v-\alpha-j+1}-(b)^{v-\alpha-j+1})}{(i-r)!(r-v)!\Gamma(v-\alpha-j+2)}, & j \geq 1 \\ \sum_{r=n}^{i}\sum_{v=n}^{r} \frac{(-1)^v i! b_{i-r}(b)^{r-v}((b)^{v-\alpha+1}-(b-1)^{v-\alpha+1})}{(i-r)!(r-v)!\Gamma(v-\alpha+2)}, & j = 0. \end{cases} \tag{4.9}$$

Proof. Let ${}^{C}D^{\alpha}_{b-}B_i(x) = \sum_{j=0}^{N-1} \rho_{i,j}B_j(x)$ where as usual ${}^{C}D^{\alpha}_{b-}B_i(x) = 0,\ i < \alpha$. Using (3.11) w.r.t. the column index j and (4.8), for $i \geq \alpha, j \geq 1$:

$$\rho_{i,j} = \frac{1}{j!}\left(\frac{d^{j-1}}{\mathrm{d}x^{j-1}}({}^{C}D^{\alpha}_{b-}B_i(x))|_{x=1} - \frac{d^{j-1}}{\mathrm{d}x^{j-1}}({}^{C}D^{\alpha}_{b-}B_i(x))|_{x=0}\right),\ j \geq 1$$

$$= \frac{1}{j!}\sum_{r=n}^{i}\sum_{v=n}^{r} \frac{(-1)^v i! b_{i-r}(b)^{r-v}}{(i-r)!(r-v)!\Gamma(v-\alpha+1)} \frac{\Gamma(v-\alpha+1)}{\Gamma(v-\alpha-j+2)}$$
$$\times ((b-x)^{v-\alpha-j+1}|_{x=1} - (b-x)^{v-\alpha-j+1}|_{x=0})$$

$$= \frac{1}{j!}\sum_{r=n}^{i}\sum_{v=n}^{r} \frac{(-1)^v i! b_{i-r}(b)^{r-v}((b-1)^{v-\alpha-j+1} - (b)^{v-\alpha-j+1})}{(i-r)!(r-v)!\Gamma(v-\alpha-j+2)}.$$

For $i \geq \alpha, j = 0$, using (4.8):

$$\rho_{i,0} = \int_0^1 {}^{C}D^{\alpha}_{b-}B_i(x)\mathrm{d}x$$

$$= \sum_{r=n}^{i}\sum_{v=n}^{r} \frac{(-1)^v i! b_{i-r}(b)^{r-v}}{(i-r)!(r-v)!\Gamma(v-\alpha+1)} \int_0^1 (b-x)^{v-\alpha}\mathrm{d}x$$

$$= \sum_{r=n}^{i}\sum_{v=n}^{r} \frac{(-1)^v i! b_{i-r}(b)^{r-v}}{(i-r)!(r-v)!\Gamma(v-\alpha+1)} \left[\frac{-(b-x)^{v-\alpha+1}}{v-\alpha+1}\right]_{x=0}^{x=1}$$

$$= \sum_{r=n}^{i}\sum_{v=n}^{r} \frac{(-1)^v i! b_{i-r}(b)^{r-v}}{(i-r)!(r-v)!\Gamma(v-\alpha+2)} ((b)^{v-\alpha+1} - (b-1)^{v-\alpha+1}).$$

$\square$

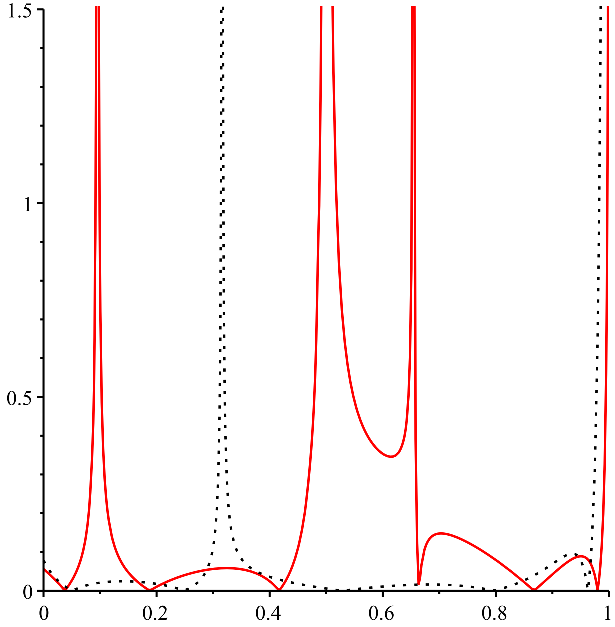

Figure 1. Relative error of Right-SLOMFD(red) and Right-BerOMFD(black) for $\alpha = 0.5$ and $N = 5$

5. Error and Convergence Analysis

In this section, we will discuss the error estimation, and error upper bound and convergence analysis for the right-sided Bernoulli operational matrix that we derived in Sect. 4.3. First, we compute the relative error ϵ_N for approximating the right-sided Caputo's fractional derivative by operational matrix based on shifted Legendre polynomials given in [15] and operational matrix based on Bernoulli polynomials. We define the relative error as follows:

$$\left| \frac{{}^{C}D_{*}^{\alpha} Y_i(x) - [\mathbf{P}_{*;B}^{\alpha}\mathbf{Y}](i)}{D^{\alpha} Y_i(x)} \right|, \tag{5.1}$$

where $Y_i(x)$ is either the Shifted Legendre polynomials $\tilde{L}_i(x)$ or the Bernoulli polynomials $B_i(x)$. ${}^{C}D_{*}^{\alpha}$ represents either the left-sided Caputo's fractional derivative or the right-sided Caputo's fractional derivative. $\mathbf{P}_{*;B}^{\alpha}$ represents either the operational matrix of left-sided Caputo's fractional derivative $\mathbf{P}_{+;B}^{\alpha}$ or the right-sided Caputo's fractional derivative $\mathbf{P}_{-;B}^{\alpha}$.

In Fig. 1, we compare the relative error between operational matrix of right-sided Caputo's fractional derivative based on shifted Legendre polynomials (Right-SLOMFD) derived in [15] and operational matrix of right-sided Caputo's fractional derivative based on Bernoulli polynomials (Right-BerOMFD). From the figure given, it is clear that the overall relative errors for Right-BerOMFD are much smaller than that of Right-SLOMFD. This gives the motivation of proposing Bernoulli polynomials operational matrix of right-sided Caputo's fractional derivative as a method to obtain numerical solution of FIDE.

Here, we show the error upper bound and convergence analysis for the method using Bernoulli operational matrix of right-sided Caputo's fractional derivative. To show this, we need the following theorem.

Theorem 5.1. *Suppose that H is a Hilbert space and Y is a closed subspace of H, such that $dimY < \infty$ and $y_1, y_2, \ldots, y_n$ is any basis for Y. Let f be an arbitrary element in H and y_0 be the unique best approximation of f out of Y. Then*

$$\|f - y_0\|_2^2 = \frac{\text{Gram}(f, y_1, \ldots, y_n)}{\text{Gram}(y_1, y_2, \ldots, y_n)},$$

where $\text{Gram}(f, y_1, \ldots, y_n) = \begin{vmatrix} \langle f, f\rangle & \langle f, y_1\rangle & \cdots & \langle f, y_n\rangle \\ \langle y_1, f\rangle & \langle y_1, y_1\rangle & \cdots & \langle y_1, y_n\rangle \\ \vdots & \vdots & \cdots & \vdots \\ \langle y_n, f\rangle & \langle y_n, y_1\rangle & \cdots & \langle y_n, y_n(t)\rangle \end{vmatrix}.$

Lemma 5.2. *Suppose that $f \in L^2[0,1]$ is approximated by f_{N-1}, such that $f_{N-1}(x) = \sum_{n=0}^{N-1} c_n B_n(x) = \boldsymbol{C}^T \boldsymbol{B}(x)$. Then, we have $\lim_{N-1\to\infty} \|f(x) - f_{N-1}(x)\|_2 = 0$.*

The error vector E^α of the operational matrix $\mathbf{P}^\alpha_{-;B}$ is given by $E^\alpha = \mathbf{P}^\alpha_{-;B}\mathbf{B}(x) - D^\alpha\mathbf{B}(x)$, where $E^\alpha = [e_1^\alpha, e_2^\alpha, ..., e_N^\alpha]$. By (3.2) and Theorem 5.1, we have the following:

$$\|(b-x)^{v-\alpha} - \sum_{j=0}^{N-1} c_j B_j(x)\|_2$$
$$= \left(\frac{\text{Gram}((b-x)^{v-\alpha}, B_0(x), B_1(x), \ldots, B_{N-1}(x))}{\text{Gram}(B_0(x), B_1(x), B_2(x), \ldots, B_{N-1}(x))}\right)^{\frac{1}{2}}.$$

Thus, according to (3.2),(4.8) and (4.9), we get

$$\begin{aligned} \|e_i^\alpha\|_2 &= |D^\alpha_{b-} B_i(x) - \mathbf{P}^\alpha_{-;B}\mathbf{B}(x)|_2 \\ &\leq \sum_{r=n}^{i}\sum_{v=n}^{r} \frac{(-1)^v i! b_{i-r} b^{r-v}}{(i-r)!(r-v)!\Gamma(v-\alpha+1)} |(b-x)^{v-\alpha} - \sum_{j=0}^{N-1} c_j B_j(x)|_2 \\ &\leq \sum_{r=n}^{i}\sum_{v=n}^{r} \frac{(-1)^v i! b_{i-r} b^{r-v}}{(i-r)!(r-v)!\Gamma(v-\alpha+1)} \\ &\quad\times \left(\frac{\text{Gram}((b-x)^{v-\alpha}, B_0(x), \ldots, B_{N-1}(x))}{\text{Gram}(B_0(x), B_1(x), \ldots, B_{N-1}(x))}\right)^{\frac{1}{2}}, i = 0, 1, 2, \ldots, N-1, \end{aligned} \tag{5.2}$$

where

$$c_j = \begin{cases} \frac{1}{j!}\frac{((b-1)^{v-\alpha-j+1} - (b)^{v-\alpha-j+1})B(v-\alpha+1, j)}{\Gamma(j)(v-\alpha+j+1)}, & j \geq 1 \\ \frac{((b)^{v-\alpha+1} - (b-1)^{v-\alpha+1})}{v-\alpha+1}, & j = 0 \end{cases} \tag{5.3}$$

By considering Lemma 5.2 and (5.2), we can conclude that, by increasing the number of the Bernoulli bases, the vector e_i^{α} tends to zero. The convergence analysis for operational matrix for left-sided derivative is similar to the right-sided one.

6. Solving FIDE with Right-Sided Caputo's Derivative via Bernoulli Operational Matrix

Here, we consider solving the following types of FIDE of Fredholm type involving l number of right-sided Caputo's fractional derivatives:

$$\sum_{r=1}^{l} q_r \, {}^{C}D_{*}^{\alpha_r} f(x) = h(x) + \int_0^1 K(x,t) f(t) \mathrm{d}t, \tag{6.1}$$

where $f^{(i)}(0) = d_i, \;\; i = 0, 1, \ldots, m-1$, $f(x)$ is the unknown solution, $q_r \in \mathbb{R}, r = 0, \ldots, l$ are constants, ${}^{C}D_{*}^{\alpha_r}$ can be either the left-sided Caputo's derivative ${}^{C}D_{a+}^{\alpha_r}$ or the right-sided Caputo's derivative ${}^{C}D_{b-}^{\alpha_r}$, $\alpha_r \geq 0$ are real derivative orders, $h(x)$ is the forced term known a-priori, and $K(x,t)$ is the Fredholm integral kernel function. The general procedure of spectral-tau method using Bernoulli polynomials approximation is given as follows:
Step 1: As in a typical method using operational matrix of fractional derivative, one begins with

$$f(x) \approx \mathbf{C}^T \mathbf{B}(x) \tag{6.2}$$

and applies the right-sided Caputo's fractional derivative to (6.2) and approximates using Bernoulli polynomials operational matrix $\mathbf{P}_{-;B}^{\alpha_r}$:

$${}^{C}D_{b-}^{\alpha_r} f(x) \approx \mathbf{C}^T \mathbf{P}_{-;B}^{\alpha_r} \mathbf{B}(x). \tag{6.3}$$

Step 2: For $h(x)$, approximate in Bernoulli polynomials:

$$h(x) \approx \mathbf{H}^T \mathbf{B}(x) \quad = [h_i]^T, \tag{6.4}$$

where the coefficients h_i are computed using (3.11).
Step 3: For the kernel function $K(x,t)$ which has two variables, it is approximated in Bernoulli polynomials basis by the following:

$$K(x,t) \approx \mathbf{B}^T(x) \mathbf{K} \mathbf{B}(t) \mathbf{K} \quad = [k_{i,j}], \tag{6.5}$$

where the coefficients $k_{i,j}$ are computed using Theorem (3.1).
Step 4: By substituting (3.5), (6.3), (6.2), and(6.5) into

$$\sum_{r=1}^{l} q_r \mathbf{B}^T(x) (\mathbf{P}_{-;B}^{\alpha_r})^T \mathbf{C} = \mathbf{B}^T(x)\mathbf{H} + \mathbf{B}^T(x)\mathbf{K} \left(\int_0^1 \mathbf{B}(t)\mathbf{B}^T(t) \mathrm{d}t \right) \mathbf{C} \tag{6.6}$$

and after rearranging, we obtain the following:

$$\mathbf{B}^T(x) \left(\sum_{r=1}^{l} q_r (\mathbf{P}_{-;B}^{\alpha_r})^T \mathbf{C} - \mathbf{H} - \mathbf{K}\mathbf{T}_B \mathbf{C} \right) = 0. \tag{6.7}$$

Thus, the residual is as follows:

$$\mathcal{R}(x) = \mathbf{B}^T(x)\left(\sum_{r=1}^{l} q_r(\mathbf{P}^{\alpha_r}_{-;B})^T\mathbf{C} - \mathbf{H} - \mathbf{K}\mathbf{T}_B\mathbf{C}\right). \tag{6.8}$$

Step 5: Since the set of Bernoulli polynomials basis, $\mathbf{B}(x) = [B_0(x) \ \cdots \ B_{N-1}(x)]$ are linearly independent, and hence:

$$\sum_{r=1}^{l} q_r{\mathbf{P}^{\alpha_r}_{-;B}}^T\mathbf{C} - \mathbf{H} - \mathbf{K}\mathbf{T}_B\mathbf{C} = 0. \tag{6.9}$$

Thus, this produces a system of N algebraic equations.
Step 6: The initial condition given is approximated in Bernoulli polynomials:

$$f^{(i)}(0) = d_i, \ i = 0, \ldots, m-1(-1)^i\mathbf{B}^T(0)(\mathbf{P}^{i}_{-;B})^T\mathbf{C} = d_i. \tag{6.10}$$

Step 7: Selecting $N - m$ equations from (6.9) combining with the initial conditions from (6.10), one has a system of N linear algebraic equations to be solved for $\mathbf{C}$ using any suitable numerical methods such as Gaussian elimination method.
Step 8: After obtaining $\mathbf{C}$, we get the approximate solution, $f^*(x) = \mathbf{C}^T\mathbf{B}(x)$.

7. Numerical Examples

In this section, we will give five examples of FIDEs involving right-sided Caputo's fractional derivative and solved using our proposed method. Ones may consider the maximum error function of the approximate function $f^*_N(x)$ of order N for each numerical solution obtained:

$$e_\infty(N) = \|f(x) - f^*_N(x)\|_\infty = \max|f(x) - f^*_N(x)|, a \le x \le b, \tag{7.1}$$

where $f(x)$ is the exact solution and $f^*_N(x)$ is the approximate solution of order N. In addition, one may compute the absolute error of N-order approximation at a particular point $x \in [0,1]$ as $e_N(x) = \|f(x) - f^*_N(x)\|$. However, in the examples that we shown, using suitable N, we able to obtain exact solution.

7.1. Linear FIDE

Example 1.

$${}^{C}D^{1/2}_{1-}f(x) = h(x) + \int_0^1 (x^2 + t^2)f(t)\mathrm{d}t, \tag{7.2}$$

where $h(x) = \frac{8\sqrt{1-x}}{3\sqrt{\pi}}(x-1) + \frac{23}{3}x^2 + \frac{79}{30}$ and its exact solution is $f(x) = x^2 - 2x - 7$.

Using $N = 4$ Bernoulli polynomials basis, the operational matrix is as follows:

$$\mathbf{P}^{1/2}_{-;B} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -\frac{4}{3\sqrt{\pi}} & \frac{16}{9\sqrt{\pi}} & \frac{8}{7\sqrt{\pi}} & \frac{16}{9\sqrt{\pi}} \\ -\frac{4}{15\sqrt{\pi}} & -\frac{208}{231\sqrt{\pi}} & \frac{8}{3\sqrt{\pi}} & \frac{80}{33\sqrt{\pi}} \\ \frac{2}{105\sqrt{\pi}} & \frac{8}{117\sqrt{\pi}} & -\frac{100}{77\sqrt{\pi}} & \frac{392}{117\sqrt{\pi}} \end{bmatrix}. \tag{7.3}$$

On the other hand:

$$\mathbf{H}^T = \begin{bmatrix} -\frac{467\sqrt{\pi}+96}{90\sqrt{\pi}} \\ \frac{2(-288+805\sqrt{\pi})}{63\sqrt{\pi}} - \frac{1771\sqrt{\pi}-640}{99\sqrt{\pi}} \\ \frac{161\sqrt{\pi}+32}{21\sqrt{\pi}} \\ \frac{2(-288+805\sqrt{\pi})}{9\sqrt{\pi}} - \frac{10(1771\sqrt{\pi}-640)}{99\sqrt{\pi}} \end{bmatrix}, \tag{7.4}$$

while $\mathbf{T}_B = [T_{i,j;B}]^T$ with using $T_{i,j;B} = \int_0^1 B_i(x)B_j(x)\mathrm{d}x$ and getting $\mathbf{K}$ using (3.19). After substitution of Bernoulli expansion in matrix for each term, the residual system of $N = 4$ algebraic equations is as follows:

$$\begin{gathered} \mathbf{P}^{1/2\ T}_{-;B}\mathbf{C} - \mathbf{H} - \mathbf{K}\mathbf{T}_B\mathbf{C} = 0 \\ -\left(\frac{4}{3\sqrt{\pi}}+\frac{1}{12}\right)c_1 - \left(\frac{4}{15\sqrt{\pi}}+\frac{1}{180}\right)c_2 + \left(\frac{2}{105\sqrt{\pi}}+\frac{1}{120}\right)c_3 - \frac{467\sqrt{\pi}+96}{90\sqrt{\pi}} - \frac{2}{3}c_0 = 0 \\ \frac{16}{9\sqrt{\pi}}c_1 - \frac{208}{231\sqrt{\pi}}c_2 + \frac{8}{117\sqrt{\pi}}c_3 - \frac{2(-288+805\sqrt{\pi})}{63\sqrt{pi}} + \frac{1771\sqrt{\pi}-640}{99\sqrt{\pi}} - c_0 = 0 \\ \frac{8}{7\sqrt{\pi}}c_1 + \frac{8}{3\sqrt{\pi}}c_2 - \frac{100}{77\sqrt{\pi}}c_3 - \frac{161\sqrt{\pi}+32}{21\sqrt{\pi}} - c_0 = 0 \\ \frac{16}{9\sqrt{\pi}}c_1 + \frac{80}{33\sqrt{\pi}}c_2 + \frac{392}{117\sqrt{\pi}}c_3 - \frac{2(-288+805\sqrt{\pi})}{9\sqrt{\pi}} + \frac{10(1771\sqrt{\pi}-640)}{99\sqrt{\pi}} = 0. \end{gathered} \tag{7.5}$$

Solving (7.5), the expansion coefficients are obtained as $\mathbf{C} = [c_0\ c_1\ c_2\ c_3]^T = \left[-\frac{23}{3}\ \ -1\ \ -1\ 0\right]^T$. Finally, the approximate solution is obtained as $f^*(x) = \mathbf{C}^T\mathbf{B}(x) = x^2 - 2x - 7$ which is the exact solution.

Example 2. Consider also the following linear Fredholm FIDE with multi-fractional orders and involve both left-sided Caputo's derivative and right-sided Caputo's derivative without the initial conditions:

$$-\frac{1}{4}{}^C D^{2/3}_{1-}f(x) - \frac{3}{4}{}^C D^{1/3}_{0+}f(x) + \frac{1}{5}{}^C D^{1}_{1-}f(x) = h(x) + \int_0^1 K(x,t)f(t)\mathrm{d}t \tag{7.6}$$

with a singular kernel $K(x,t) = \frac{x^2}{t^{1/2}}$, where

$$h(x) = \frac{\Gamma(2/3)\sqrt{3}}{8\pi}\left(-\frac{8}{35} - \frac{18}{7}x - \frac{27}{14}x^2 - \frac{81}{14}x^3\right)(1-x)^{1/3}$$
$$-\frac{3}{4\Gamma(2/3)}\left(-\frac{81}{44}x^{11/3} - \frac{18}{35}x^{5/3} + \frac{3}{5}x^{2/3}\right) + \frac{2}{3}x^3 + \frac{31}{945}x^2 + \frac{4}{35}x - \frac{2}{25}. \tag{7.7}$$

The exact solution is $f(x) = -\frac{5}{6}x^4 - \frac{2}{7}x^2 + \frac{2}{5}x$.

Assume we use $N = 5$ Bernoulli polynomials. The operational matrices involved fractional operators are as follows:

$$\mathbf{P}^{1/3}_{-;B} = \frac{\sqrt{3}\Gamma(2/3)}{\pi}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ -\frac{9}{8} & \frac{207}{182} & \frac{27}{13} & \frac{45}{26} & \frac{405}{104} \\ -\frac{9}{56} & -\frac{9}{8} & \frac{3105}{988} & \frac{9}{4} & \frac{4455}{988} \\ \frac{9}{560} & \frac{225}{3458} & -\frac{135}{176} & \frac{2475}{988} & \frac{567}{176} \\ \frac{27}{910} & \frac{27}{616} & \frac{81}{2470} & -\frac{27}{11} & \frac{2673}{1235} \end{bmatrix}$$

$$\mathbf{P}^{2/3}_{+;B} = \frac{1}{\Gamma(2/3)}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ \frac{9}{10} & \frac{63}{44} & -\frac{783}{748} & \frac{9}{11} & -\frac{567}{374} \\ -\frac{9}{40} & \frac{279}{748} & \frac{27}{11} & -\frac{441}{374} & \frac{81}{44} \\ -\frac{9}{880} & \frac{9}{280} & \frac{1323}{17204} & \frac{9}{4} & -\frac{51597}{34408} \\ \frac{27}{616} & -\frac{135}{4301} & \frac{81}{2860} & \frac{6615}{8602} & \frac{1215}{572} \end{bmatrix},$$

where $\mathbf{H} = [0.080730245\ 1.57226359\ 4.54462943\ 3.59901587\ 4.48753077]$ and

$$\mathbf{T}_B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{12} & 0 & -\frac{1}{120} & 0 \\ 0 & 0 & \frac{1}{180} & 0 & -\frac{1}{630} \\ 0 & -\frac{1}{120} & 0 & \frac{1}{840} & 0 \\ 0 & 0 & -\frac{1}{630} & 0 & \frac{1}{2100} \end{bmatrix}. \tag{7.8}$$

In this case, since the integral kernel $K(x,t) = \frac{x^2}{t^{1/2}}$ is singular, the expansion coefficient $k_{i,j}$ for the kernel matrix $\mathbf{K}$ cannot be computed using (3.1), because $K(x,t)$ is not differentiable for variable t at $t = 0$. Therefore, one has to resolve to the method by inner products given in (3.19). Thus, we obtain the following:

$$K(x,t) = \frac{x^2}{t^{1/2}} = \mathbf{B}^T(x)\mathbf{K}\mathbf{B}(t)$$
$$\mathbf{K} = \begin{bmatrix} \frac{2}{3} & -\frac{8}{3} & \frac{52}{3} & -\frac{40}{3} & \frac{140}{3} \\ 2 & -8 & 52 & -40 & 140 \\ 2 & -8 & 52 & -40 & 140 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{7.9}$$

After substitution of Bernoulli expansion in matrix for each term, the residual system of $N = 5$ algebraic equations is as follows:

$$\begin{aligned}
-0.3773971414c_1 &+ 0.1323935356c_2 - 0.0036842558c_3 - 0.0245232951c_4 \\
&- 0.0807302457 - 0.6666666667c_0 = 0 \\
-0.6719807945c_1 &- 0.4632845312c_2 - 0.0489945421c_3 + 0.02507709482c_4 \\
&- 1.5722635995 - 2c_0 = 0 \\
0.5254764302c_1 &- 2.0127154313c_2 - 0.5184781227c_3 - 0.0059340198c_4 \\
&- 4.5446294395 - 2c_0 = 0 \\
-0.7761958099c_1 &+ 0.2331456370c_2 - 1.7137459457c_3 - 0.7678079788c_4 \\
&- 3.5990158733 = 0 \\
0.1128612141c_1 &- 1.8612021718c_2 + 0.2292764414c_3 \\
&- 1.5804421095c_4 - 4.4875307705 = 0.
\end{aligned} \tag{7.10}$$

Solving (7.10) above, the coefficients are obtained as $\mathbf{C} = [c_0\ c_1\ c_2\ c_3\ c_4]^T = \left[-\frac{13}{210}\ -\frac{151}{210}\ -\frac{41}{21}\ -\frac{5}{3}\ -\frac{5}{6}\right]^T$. Finally, the approximate solution is obtained as $f^*(x) = \mathbf{C}^T\mathbf{B}(x) = -\frac{5}{6}x^4 - \frac{2}{7}x^2 + \frac{2}{5}x$, which is the exact solution.

7.2. Nonlinear FIDE

Now, we solve nonlinear Fredholm FIDE with right-sided Caputo's derivatives with the proposed Bernoulli polynomial method. Instead of combining with spectral-tau method, collocation method is being used here as it is more feasible than spectral-tau method in the presence of nonlinear terms in the FIDE. The results of the following examples are compared with the operational matrix of right-sided Caputo's fractional derivative based on Shifted Legendre polynomial given in [15].

Example 3. Consider the following nonlinear Fredholm FIDE with the initial condition where there is a single right-sided Caputo's derivative and a nonlinear term $[f(x)]^2$:

$$\begin{aligned}
{}^C D_{1-}^{1/2} f(x) + [f(x)]^2 &= h(x) + \int_0^1 K(x,t) f(t)\mathrm{d}t \\
f(0) &= -2
\end{aligned} \tag{7.11}$$

with kernel $K(x,t) = x^2 + t^2$, where $h(x) = \frac{10\sqrt{1-x}}{\sqrt{\pi}} + \frac{59x^2}{2} + 20x + \frac{71}{12}$. The exact solution is $f(x) = -5x - 2$.

Assume that we use $N = 6$ Bernoulli polynomials. In this case, we have a nonlinear term $[f(x)]^2$ which is approximated as follows:

$$f(x)^2 \approx \mathbf{C}^T\mathbf{B}(x)\mathbf{B}^T(x)\mathbf{C}. \tag{7.12}$$

Instead of using spectral-tau, we turned to adopt collocation method to solve the residual equation. Using Maple and fixing the digits of accuracy up to 10, the approximate solution obtained is as follows:

Table 1. Example 3: comparison of absolute errors between order $N = 6$ Bernoulli polynomials and $N = 6, 9$ shifted Legendre polynomial

x	Bernoulli, $e_{6,B}(x)$	Legendre, $e_{6,\text{Leg}}(x)$	Legendre, $e_{9,\text{Leg}}(x)$
0.0	1.00000E−09	5.00000E−09	1.00000E−09
0.1	7.10000E−08	1.14182E+01	3.00000E−09
0.2	7.90000E−08	9.43561E+00	2.00000E−09
0.3	5.90000E−08	3.28575E+00	2.00000E−09
0.4	3.70000E−08	1.43486E+00	2.00000E−09
0.5	3.60000E−08	2.15129E+00	2.00000E−09
0.6	6.10000E−08	1.30504E+00	1.00000E−09
0.7	9.50000E−08	7.31162E+00	0.00000E+00
0.8	8.40000E−08	1.30701E+01	0.00000E+00
0.9	6.20000E−08	1.52217E+01	2.00000E−09
1.0	4.79000E−07	1.04622E+01	2.00000E−09

$$\begin{aligned} f^*(x) = &-1.999999999 - 5.000001122x + 4.482295544 \times 10^{-6}x^2 \\ &- 3.52519041 \times 10^{-6}x^3 - 5.63317826 \times 10^{-6}x^4 \\ &+ 6.275503624 \times 10^{-6}x^5. \end{aligned} \tag{7.13}$$

Table 1 compares the absolute error between the proposed Bernoulli method with the operational matrix method of [15] which is using Shifted Legendre polynomial. The results show that the proposed Bernoulli method is able to achieve a higher accuracy than Shifted Legendre polynomial for the same N. The shifted Legendre polynomial will achieve a higher accuracy than Bernoulli using a larger value of N which implies that more number of basis polynomials with higher degrees are required for Shifted Legendre to achieve a superior accuracy than Bernoulli polynomial method.

Example 4. Consider the following nonlinear Fredholm FIDE with nonlinear terms involving right-sided Caputo's derivative with the initial condition:

$$({}^C D_{1-}^{2/3} f(x))^2 - 2[f(x)]^2 = h(x) + \int_0^1 K(x,t) f(t) \mathrm{d}t f(0) = 0 \tag{7.14}$$

with kernel $K(x,t) = x^2 + t^2$, where

$$\begin{aligned} h(x) = &\frac{2187}{256} \frac{\Gamma\left(\frac{2}{3}\right)^2 (1-x)^{2/3} x^2}{\pi^2} - \frac{1215}{128} \frac{\Gamma\left(\frac{2}{3}\right)^2 (1-x)^{2/3} x}{\pi^2} \\ &+ \frac{675}{256} \frac{\Gamma\left(\frac{2}{3}\right)^2 (1-x)^{2/3}}{\pi^2} - \frac{9\,x^4}{8} + 3x^3 - \frac{9}{4}x^2 - \frac{1}{10}. \end{aligned} \tag{7.15}$$

The exact solution is $f(x) = -\frac{3}{4}x^2 + x$.

Assume that we use $N = 4$ Bernoulli polynomials. The nonlinear term $({}^C D_{1-}^{2/3} f(x))^2$ is approximated as follows:

$$({}^C D_{1-}^{2/3} f(x))^2 \approx \mathbf{C}^T \mathbf{B}(x) \mathbf{P}_{-;B}^{2/3} (\mathbf{P}_{-;B}^{2/3})^T \mathbf{B}^T(x) \mathbf{C}. \tag{7.16}$$

Table 2. Example 4: comparison of absolute errors between order $N = 4$ Bernoulli polynomials and $N = 4$ shifted Legendre polynomial

x	Bernoulli,$e_{4,B}(x)$	Legendre, $e_{4,\mathrm{Leg}}(x)$
0.0	0.00000E+00	0.00000E+00
0.1	2.91915E−03	3.39648E−01
0.2	2.58579E−03	3.35111E−01
0.3	1.90118E−04	1.87877E−01
0.4	3.07765E−03	3.50857E−02
0.5	6.02732E−03	4.41974E−02
0.6	7.46866E−03	2.27145E−02
0.7	6.21149E−03	8.12565E−02
0.8	1.06558E−03	2.10173E−01
0.9	9.15926E−03	2.73495E−01
1.0	2.56532E−02	1.53960E−01

Similarly, collocation method is used here. Using Maple and fixing the digits of accuracy up to 10, the approximate solution obtained is as follows:

$$f^*(x) = 1.049421454x - 0.9721361433x^2 + 0.1983679384x^3. \tag{7.17}$$

Table 2 compares the absolute error between the proposed Bernoulli method with Shifted Legendre polynomial for $N = 4$. The results show that the proposed Bernoulli method is able to achieve a higher accuracy than Shifted Legendre polynomial for Example 4 of a nonlinear Fredholm FIDE.

Example 5. Consider another nonlinear Fredholm FIDE with the initial condition where there is a nonlinear term $[f(x)]^2$:

$${}^{C}D_{1-}^{1/2} f(x) + [f(x)]^2 = h(x) + \int_0^1 K(x,t) f(t) \mathrm{d}t f(0) = -7 \tag{7.18}$$

with kernel $K(x,t) = x - t$, where

$$h(x) = -\frac{8}{3}\frac{x\sqrt{1-x}}{\sqrt{\pi}} + \frac{8}{3}\frac{\sqrt{1-x}}{\sqrt{\pi}} + x^4 - 4x^3 - 10x^2 + \frac{107x}{3} + \frac{541}{12}. \tag{7.19}$$

The exact solution is $f(x) = x^2 - 2x - 7$.

Assume we use $N = 6$ Bernoulli polynomials. In this case, we have a nonlinear term $[f(x)]^2$ which is approximated as follows:

$$f(x)^2 \approx \mathbf{C}^T \mathbf{B}(x) \mathbf{B}^T(x) \mathbf{C}. \tag{7.20}$$

Table 3. Example 5: comparison of absolute errors between order $N = 6$ Bernoulli polynomials and $N = 6$ shifted Legendre polynomial

x	Bernoulli, $e_{6,B}(x)$	Legendre, $e_{6,\mathrm{Leg}}(x)$
0.0	0.00000E+00	0.00000E+00
0.1	1.00000E−09	2.18781E−02
0.2	6.00000E−09	1.05917E−02
0.3	8.00000E−09	5.89246E−03
0.4	6.00000E−09	1.81807E−02
0.5	5.00000E−09	2.75772E−02
0.6	1.00000E−08	3.82080E−02
0.7	1.90000E−08	4.91458E−02
0.8	1.70000E−08	4.65342E−02
0.9	1.80000E−08	4.28770E−03
1.0	1.34000E−07	1.66661E−01

Instead of using spectral-tau, we turned to adopting collocation method to solve the residual equation. Using Maple and fixing the digits of accuracy up to 10, the approximate solution obtained is as follows:

$$\begin{aligned} f^*(x) =& -7.000000000 - 1.999999931x + 0.9999987946x^2 \\ &+ 4.860561970 \times 10^{-6}x^3 - 7.482818204 \times 10^{-6}x^4 \\ &+ 3.892003543 \times 10^{-6}x^5. \end{aligned} \tag{7.21}$$

Table 3 compares the absolute error between the proposed Bernoulli method with the operational matrix method of [15] which is based on shifted Legendre polynomial for $N = 6$. The results show that the proposed Bernoulli method is able to achieve a higher accuracy than shifted Legendre polynomial for Example 5 of a nonlinear Fredholm FIDE.

8. Conclusion

In this paper, the analytical expression of the expansion coefficients for a single-variable as well as double-variable function approximation by Bernoulli polynomials has been derived. In addition, the operational matrix of right-sided Caputo's fractional derivative in Bernoulli polynomials basis has been derived. The derived expressions for Bernoulli polynomials expansion coefficients and Bernoulli polynomials operational matrix are applied in solving linear Fredholm FIDE involving multi-orders of right-sided Caputo's fractional derivative as well as nonlinear Fredholm FIDE with right-sided Caputo's fractional derivative. Specifically, the numerical results have shown that the operational matrix based on Bernoulli polynomial performs better than Shifted Legendre polynomial in terms of accuracy. In particular, the double-variable expansion coefficient is used to approximate the Fredholm integral kernel by the kernel matrix. The illustrative examples given show that the Bernoulli polynomials operational matrix method provides high accuracy in solving FIDEs with right-sided Caputo's fractional derivative.

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