Mediterr. J. Math. (2018) 15:214
https://doi.org/10.1007/s00009-018-1261-2
1660-5446/18/060001-15
published online November 8, 2018

Mediterranean Journal of Mathematics

CrossMark

Periodic Solutions of a Second-Order Functional Differential Equation with State-Dependent Argument

Hou Yu Zhao and Jia Liu

Abstract. In this paper, we use Schauder and Banach fixed point theorem to study the existence, uniqueness and stability of periodic solutions of a class of iterative differential equation

$$c_0x''(t) + c_1x'(t) + c_2x(t) = x(p(t) + bx(t)) + h(t).$$

Mathematics Subject Classification. 39B12, 39B82.

Keywords. Iterative differential equation, periodic solutions, fixed point theorem.

1. Introduction

Delay differential equation of the form

$$x'(t) = f(t, x(t - \tau(t)))$$

has been discussed in [1,5]. In particular, the delay function $\tau(t)$ depends not only on unknown function, but also state $\tau(t, x(t))$ have been studied in many literatures in the past few years ([6,7,12,13]). Cooke [2] points out that it is highly desirable to establish the existence and stability properties of periodic solutions for equations of the form

$$x'(t) + ax(t - h(t, x(t))) = F(t),$$

in which the lag $h(t, x(t))$ implicitly involves $x(t)$. Eder [3] considers the iterative functional differential equation

$$x'(t) = x^{[2]}(t)$$

This work was partially supported by the National Natural Science Foundation of China (Grant no. 11501069), Science and Technology Research Program of Chongqing Municipal Education Commission (Grant no. KJQN201800502), Foundation of youth talent of Chongqing Normal University (02030307-00039).

Birkhäuser

and obtains that every solution either vanishes identically or is strictly monotonic. Fečkan [4] studies the equation

$$x'(t) = f(x^{[2]}(t))$$

by obtaining an existence theorem for solutions satisfying $x(0) = 0$. Si and Cheng [8] consider the analytic solutions of the form

$$x'(t) = x(at + bx(t)).$$

Further discussion is made in [9–11] for existence of analytic solutions of several iterative functional differential equations with state or state derivative dependent. Recently, Liu and Li [6] consider the analytic solutions of the form

$$c_0x''(t) + c_1x'(t) + c_2x(t) = x(p(t) + bx(t)) + h(t), \tag{1.1}$$

in a neighborhood of the origin.

In this note, we will study the existence of periodic solutions of Eq. (1.1). Reducing (1.1) with $y(t) = p(t) + bx(t)$ to the auxiliary equation

$$\begin{aligned} c_0y''(t) + c_1y'(t) + c_2y(t) = y(y(t)) - p(y(t)) + c_0p''(t) \\ + c_1p'(t) + c_2p(t) + bh(t). \end{aligned} \tag{1.2}$$

The periodic solutions of (1.1) will be considered by (1.2). For convenience, we will make use $C(\mathbb{R}, \mathbb{R})$ to denote the set of all real-valued continuous functions map $\mathbb{R}$ into $\mathbb{R}$.

For $T > 0$, define

$$\mathcal{P}_T = \left\{ x \in C(\mathbb{R}, \mathbb{R}) : x(t + T) = x(t),\ \forall t \in \mathbb{R} \right\}.$$

Then $\mathcal{P}_T$ is a Banach space with the norm

$$\|x\| = \max_{t\in[0,T]} |x(t)| = \max_{t\in\mathbb{R}} |x(t)|.$$

For $P > 0, L \geq 0$, define the sets

$$\mathcal{P}_T(P, L) = \left\{ x \in \mathcal{P}_T : \|x\| \leq P,\ |x(t_2) - x(t_1)| \leq L|t_2 - t_1|,\ \forall t_1, t_2 \in \mathbb{R} \right\},$$

which is a closed convex and bounded subset of $\mathcal{P}_T$, and we wish to find T-periodic functions $x \in \mathcal{P}_T(P, L)$ satisfies (1.1).

2. Periodic Solutions of (1.2)

In this section, the existence of periodic solutions of Eq. (1.2) will be proved. Let us state the Schauder fixed point theorem, which will be used to prove our main theorem.

Theorem 2.1 (Schauder). *Let Ω be a closed convex nonempty subset of a Banach space $(\mathbb{B}, \|\cdot\|)$. Suppose that A maps Ω into Ω and is compact and continuous. Then there exists $z \in \Omega$ with $z = Az$.*

Throughout this paper, we assume that all functions are continuous with respect to their arguments and the following condition holds:

(H) $b \neq 0,\ c_i \neq 0, i = 0, 1, 2,\ p \in \mathcal{P}_T(P_p, L_p)$ and $h \in \mathcal{P}_T(P_h, L_h)$ are given.

We begin with the following lemma:

Lemma 2.1 ([14]). *It holds*

$$\mathcal{P}_T(P,L) = \Bigg\{ x \in \mathcal{P}_T : \|x\| \le P, |x(t_2) - x(t_1)| \le L|t_2 - t_1|, \forall t_1, t_2 \in [0,T] \Bigg\}. \tag{2.1}$$

Lemma 2.2 *For any $\varphi, \psi \in \mathcal{P}_T(P, L)$,*

$$||\varphi^{[n]} - \psi^{[n]}|| \le \sum_{j=0}^{n-1} L^j \|\varphi - \psi\|,\ n = 1, 2, \dots \tag{2.2}$$

Proof. The result follows from the definition of $\mathcal{P}_T(P, L)$. □

Now we rewrite (1.2) as a fixed point equation.

Lemma 2.3. *$y \in \mathcal{P}_T$ is a solution of Eq. (1.2) if and only if*

$$y(t) = \frac{1}{c_0} E(c_0, \widetilde{c}_1) E(c_0, \overline{c}_1) \int_t^{t+T} \int_u^{u+T} \Phi_y(s) \mathrm{e}^{\frac{\widetilde{c}_1}{c_0}(s-u)} \mathrm{e}^{\frac{\overline{c}_1}{c_0}(u-t)} \mathrm{d}s \mathrm{d}u, \tag{2.3}$$

where

$$E(c_0, \widetilde{c}_1) = \frac{1}{\mathrm{e}^{\frac{\widetilde{c}_1}{c_0}T} - 1},\ E(c_0, \overline{c}_1) = \frac{1}{\mathrm{e}^{\frac{\overline{c}_1}{c_0}T} - 1} \tag{2.4}$$

and

$$\Phi_y(t) = y(y(t)) - p(y(t)) + c_0 p''(t) + c_1 p'(t) + c_2 p(t) + bh(t), \tag{2.5}$$

$\widetilde{c}_1 = \frac{c_1 + \sqrt{c_1^2 - 4c_0c_2}}{2}, \overline{c}_1 = \frac{c_1 - \sqrt{c_1^2 - 4c_0c_2}}{2}$ *or* $\widetilde{c}_1 = \frac{c_1 - \sqrt{c_1^2 - 4c_0c_2}}{2}, \overline{c}_1 = \frac{c_1 + \sqrt{c_1^2 - 4c_0c_2}}{2}$ *and we see* $\widetilde{c}_1 + \overline{c}_1 = c_1, \widetilde{c}_1 \overline{c}_1 = c_0 c_2$.

Proof. By direct calculation, we can see that (2.3) is a T-periodic solution of (1.2).

Suppose $y(t)$ is a T-periodic solution of (1.2); then it is easy to find Eq. (1.2) can be written in the form of

$$\begin{aligned} &y''(t)\mathrm{e}^{\frac{\widetilde{c}_1}{c_0}t} + \frac{\widetilde{c}_1}{c_0} y'(t) \mathrm{e}^{\frac{\widetilde{c}_1}{c_0}t} + \frac{\overline{c}_1}{c_0} y'(t) \mathrm{e}^{\frac{\widetilde{c}_1}{c_0}t} + \frac{c_2}{c_0} y(t) \mathrm{e}^{\frac{\widetilde{c}_1}{c_0}t} \\ &\quad = \frac{1}{c_0} \Bigg(y(y(t)) - p(y(t)) + c_0 p''(t) + c_1 p'(t) + c_2 p(t) + bh(t) \Bigg) \mathrm{e}^{\frac{\widetilde{c}_1}{c_0}t}, \end{aligned}$$

or

$$\left(y'(t)\mathrm{e}^{\frac{\widetilde{c}_1}{c_0}t}\right)' + \frac{\overline{c}_1}{c_0}\left(y(t)\mathrm{e}^{\frac{\widetilde{c}_1}{c_0}t}\right)' \\ = \frac{1}{c_0}\left(y(y(t)) - p(y(t)) + c_0 p''(t) + c_1 p'(t) + c_2 p(t) + bh(t)\right)\mathrm{e}^{\frac{\widetilde{c}_1}{c_0}t}. \tag{2.6}$$

Integrating (2.6) from t to $t+T$ and using the fact $y(t+T) = y(t)$ obtain

$$y'(t) + \frac{\overline{c}_1}{c_0}y(t) = \frac{1}{c_0}\int_t^{t+T}\left(y(y(s)) - p(y(s)) \right. \\ \left. + c_0 p''(s) + c_1 p'(s) + c_2 p(s) + bh(s)\right)\frac{\mathrm{e}^{\frac{\widetilde{c}_1}{c_0}(s-t)}}{\mathrm{e}^{\frac{\widetilde{c}_1}{c_0}T} - 1}\mathrm{d}s,$$

Therefore,

$$y(t) = \frac{1}{c_0}\int_t^{t+T}\int_u^{u+T}\left(y(y(s)) - p(y(s)) + c_0 p''(s) + c_1 p'(s) + c_2 p(s) + bh(s)\right) \\ \times \frac{\mathrm{e}^{\frac{\widetilde{c}_1}{c_0}(s-u)}}{\mathrm{e}^{\frac{\widetilde{c}_1}{c_0}T} - 1}\frac{\mathrm{e}^{\frac{\overline{c}_1}{c_0}(u-t)}}{\mathrm{e}^{\frac{\overline{c}_1}{c_0}T} - 1}\mathrm{d}s\mathrm{d}u.$$

This completes the proof. □

Now we will need to construct a mapping that satisfies the hypotheses of Theorem 2.1. To this aim, consider the map $A : \mathcal{P}_T(P, L) \to \mathcal{P}_T$ defined as follows:

$$(Ay)(t) = \frac{1}{c_0}E(c_0, \widetilde{c}_1)E(c_0, \overline{c}_1)\int_t^{t+T}\int_u^{u+T}\Phi_y(s)\mathrm{e}^{\frac{\widetilde{c}_1}{c_0}(s-u)+\frac{\overline{c}_1}{c_0}(u-t)}\mathrm{d}s\mathrm{d}u, \tag{2.7}$$

where $E(c_0, \widetilde{c}_1), E(c_0, \overline{c}_1)$ and $\Phi_y(t)$ are defined as in Lemma 2.3.

Lemma 2.4. *Suppose* (H) *holds and* $p \in \mathcal{P}_T(P_p, L_p)$*; then operator* A *is continuous and compact on* $\mathcal{P}_T(P, L)$*.*

Proof. Take $\varphi, \psi \in \mathcal{P}_T(P, L), t \in \mathbb{R}$; then by (2.2),

$$|(A\varphi)(t) - (A\psi)(t)| \\ \le \frac{1}{|c_0|}|E(c_0, \widetilde{c}_1)||E(c_0, \overline{c}_1)|\left|\int_t^{t+T}\int_u^{u+T}|\Phi_\varphi(s) - \Phi_\psi(s)|\mathrm{e}^{\frac{\widetilde{c}_1}{c_0}(s-u)+\frac{\overline{c}_1}{c_0}(u-t)}\mathrm{d}s\mathrm{d}u\right| \\ \le \frac{(1 + L + L_p)}{|c_2|}\|\varphi - \psi\|; \tag{2.8}$$

thus A is continuous.

Now we will show that A is a compact map. It is easy to see that $\mathcal{P}_T(P, L)$ is a uniformly bounded and equicontinuous on $\mathbb{R}$; then using Arzela-Ascoli theorem we know $\mathcal{P}_T(P, L)$ is a compact set. Since A is continuous, it maps compact sets into compact sets; therefore A is compact. This completes the proof. □

Theorem 2.2. *Suppose* (H) *holds; furthermore, the following inequalities hold*

$$|b|P_h + |c_2|P_p\left(5 + 4\frac{|\widetilde{c}_1|}{|\overline{c}_1|} + \frac{1}{|c_2|}\right) \leq (|c_2| - 1)P, \tag{2.9}$$

$$\begin{aligned}
&\mathrm{e}^{\frac{|\overline{c}_1|}{|c_0|}T}|E(c_0,\overline{c}_1)|\left(\frac{|\overline{c}_1|}{|c_2|}(P + P_p + |b|P_h)\left(1 + \mathrm{e}^{\frac{|\overline{c}_1|}{|c_0|}T} + \frac{1}{|E(c_0,\overline{c}_1)|}\right)\right.\\
&\quad + \frac{|\overline{c}_1|P_p}{|E(c_0,\overline{c}_1)|}(3 + \frac{|\widetilde{c}_1|}{|c_0|}) + 2\left(|c_0|L_p + 2|c_0|P_p + P_p|\overline{c}_1|\mathrm{e}^{\frac{|\overline{c}_1|}{|c_0|}T} + P_p|\widetilde{c}_1|\mathrm{e}^{\frac{|\overline{c}_1|}{|c_0|}T}\right)\\
&\quad + |c_1|P_p\left(1 + \mathrm{e}^{\frac{|\overline{c}_1|}{|c_0|}T} + \frac{2}{|E(c_0,\overline{c}_1)|}\right)\\
&\quad \left. + |\overline{c}_1|P_p\left(1 + \mathrm{e}^{\frac{|\overline{c}_1|}{|c_0|}T} + \frac{1}{|E(c_0,\overline{c}_1)|}\right)\right) < |c_0|L,
\end{aligned} \tag{2.10}$$

then Eq. (1.2) *has a periodic solution in* $\mathcal{P}_T(P,L)$.

Proof. For any $\varphi \in \mathcal{P}_T(P,L)$. let

$$\phi_y(t) = y(y(t)) - p(y(t)) + bh(t), \tag{2.11}$$

then

$$\begin{aligned}
&\left|\int_t^{t+T}\int_u^{u+T}\phi_y(s)\mathrm{e}^{\frac{\widetilde{c}_1}{c_0}(s-u)+\frac{\overline{c}_1}{c_0}(u-t)}\mathrm{d}s\mathrm{d}u\right|\\
&\quad \leq \mathrm{e}^{-\frac{\overline{c}_1}{c_0}t}(P + P_p + |b|P_h)\left|\int_t^{t+T}\int_u^{u+T}\mathrm{e}^{\frac{\widetilde{c}_1}{c_0}s+\frac{\overline{c}_1-\widetilde{c}_1}{c_0}u}\mathrm{d}s\mathrm{d}u\right|\\
&\quad \leq \frac{|c_0|}{|c_2|}\frac{P + P_p + |b|P_h}{|E(c_0,\widetilde{c}_1)||E(c_0,\overline{c}_1)|},
\end{aligned} \tag{2.12}$$

$$\begin{aligned}
&\left|\int_t^{t+T}\int_u^{u+T}p(s)\mathrm{e}^{\frac{\widetilde{c}_1}{c_0}(s-u)+\frac{\overline{c}_1}{c_0}(u-t)}\mathrm{d}s\mathrm{d}u\right|\\
&\quad \leq P_p\mathrm{e}^{-\frac{\overline{c}_1}{c_0}t}\left|\int_t^{t+T}\int_u^{u+T}\mathrm{e}^{\frac{\widetilde{c}_1}{c_0}s+\frac{\overline{c}_1-\widetilde{c}_1}{c_0}u}\mathrm{d}s\mathrm{d}u\right|\\
&\quad \leq \frac{|c_0|}{|c_2|}\frac{P_p}{|E(c_0,\widetilde{c}_1)||E(c_0,\overline{c}_1)|},
\end{aligned} \tag{2.13}$$

$$\begin{aligned}
&\left|\int_t^{t+T}\int_u^{u+T}p'(s)\mathrm{e}^{\frac{\widetilde{c}_1}{c_0}(s-u)+\frac{\overline{c}_1}{c_0}(u-t)}\mathrm{d}s\mathrm{d}u\right|\\
&\quad = \mathrm{e}^{-\frac{\overline{c}_1}{c_0}t}\left|\int_t^{t+T}\mathrm{e}^{\frac{\overline{c}_1-\widetilde{c}_1}{c_0}u}\int_u^{u+T}\mathrm{e}^{\frac{\widetilde{c}_1}{c_0}s}\mathrm{d}p(s)\mathrm{d}u\right|
\end{aligned}$$

$$= \mathrm{e}^{-\frac{\bar{c}_1}{c_0}t}\left|\int_t^{t+T}\mathrm{e}^{\frac{\bar{c}_1-\tilde{c}_1}{c_0}u}\left(p(u)\mathrm{e}^{\frac{\tilde{c}_1}{c_0}u}(\mathrm{e}^{\frac{\tilde{c}_1}{c_0}T}-1)-\frac{\tilde{c}_1}{c_0}\int_u^{u+T}p(s)\mathrm{e}^{\frac{\tilde{c}_1}{c_0}s}\mathrm{d}s\right)\mathrm{d}u\right|$$
$$\le \frac{2|c_0|P_p}{|\bar{c}_1||E(c_0,\tilde{c}_1)||E(c_0,\bar{c}_1)|} \tag{2.14}$$

and

$$\left|\int_t^{t+T}\int_u^{u+T}p''(s)\mathrm{e}^{\frac{\tilde{c}_1}{c_0}(s-u)+\frac{\bar{c}_1}{c_0}(u-t)}\mathrm{d}s\mathrm{d}u\right|$$
$$= \mathrm{e}^{-\frac{\bar{c}_1}{c_0}t}\left|\int_t^{t+T}\int_u^{u+T}p''(s)\mathrm{e}^{\frac{\tilde{c}_1}{c_0}s+\frac{\bar{c}_1-\tilde{c}_1}{c_0}u}\mathrm{d}s\mathrm{d}u\right|$$
$$= \mathrm{e}^{-\frac{\bar{c}_1}{c_0}t}\left|\int_t^{t+T}\mathrm{e}^{\frac{\bar{c}_1-\tilde{c}_1}{c_0}u}\int_u^{u+T}\mathrm{e}^{\frac{\tilde{c}_1}{c_0}s}\mathrm{d}p'(s)\mathrm{d}u\right|$$
$$= \mathrm{e}^{-\frac{\bar{c}_1}{c_0}t}\left|\int_t^{t+T}\mathrm{e}^{\frac{\bar{c}_1-\tilde{c}_1}{c_0}u}\left(p'(u)\mathrm{e}^{\frac{\tilde{c}_1}{c_0}u}(\mathrm{e}^{\frac{\tilde{c}_1}{c_0}T}-1)-\frac{\tilde{c}_1}{c_0}\int_u^{u+T}\mathrm{e}^{\frac{\tilde{c}_1}{c_0}s}\mathrm{d}p(s)\right)\mathrm{d}u\right|$$
$$\le \mathrm{e}^{-\frac{\bar{c}_1}{c_0}t}\left(|\mathrm{e}^{\frac{\tilde{c}_1}{c_0}T}-1|\left|\int_t^{t+T}\mathrm{e}^{\frac{\bar{c}_1}{c_0}u}\mathrm{d}p(u)\right|\right.$$
$$\left.+\left|\frac{\tilde{c}_1}{c_0}\int_t^{t+T}\mathrm{e}^{\frac{\bar{c}_1-\tilde{c}_1}{c_0}u}\left(p(u)\mathrm{e}^{\frac{\tilde{c}_1}{c_0}u}(\mathrm{e}^{\frac{\tilde{c}_1}{c_0}T}-1)-\frac{\tilde{c}_1}{c_0}\int_u^{u+T}p(s)\mathrm{e}^{\frac{\tilde{c}_1}{c_0}s}\mathrm{d}s\right)\mathrm{d}u\right|\right)$$
$$\le \frac{2P_p}{|E(c_0,\tilde{c}_1)||E(c_0,\bar{c}_1)|}\left(1+\frac{|\tilde{c}_1|}{|\bar{c}_1|}\right). \tag{2.15}$$

By (2.12)–(2.15) and (2.9), we have

$$|(A\varphi)(t)| \le \frac{1}{|c_0|}|E(c_0,\tilde{c}_1)||E(c_0,\bar{c}_1)|\left|\int_t^{t+T}\int_u^{u+T}\Phi_y(s)\mathrm{e}^{\frac{\tilde{c}_1}{c_0}(s-u)+\frac{\bar{c}_1}{c_0}(u-t)}\mathrm{d}s\mathrm{d}u\right|$$
$$\le \frac{1}{|c_0|}|E(c_0,\tilde{c}_1)||E(c_0,\bar{c}_1)|\left(\left|\int_t^{t+T}\int_u^{u+T}\phi_y(s)\mathrm{e}^{\frac{\tilde{c}_1}{c_0}(s-u)+\frac{\bar{c}_1}{c_0}(u-t)}\mathrm{d}s\mathrm{d}u\right|\right.$$
$$+|c_0|\left|\int_t^{t+T}\int_u^{u+T}p''(s)\mathrm{e}^{\frac{\tilde{c}_1}{c_0}(s-u)+\frac{\bar{c}_1}{c_0}(u-t)}\mathrm{d}s\mathrm{d}u\right|$$
$$+|c_1|\left|\int_t^{t+T}\int_u^{u+T}p'(s)\mathrm{e}^{\frac{\tilde{c}_1}{c_0}(s-u)+\frac{\bar{c}_1}{c_0}(u-t)}\mathrm{d}s\mathrm{d}u\right|$$
$$\left.+|c_2|\left|\int_t^{t+T}\int_u^{u+T}p(s)\mathrm{e}^{\frac{\tilde{c}_1}{c_0}(s-u)+\frac{\bar{c}_1}{c_0}(u-t)}\mathrm{d}s\mathrm{d}u\right|\right)$$
$$\le \frac{1}{|c_0|}|E(c_0,\tilde{c}_1)||E(c_0,\bar{c}_1)|\left(\frac{|c_0|}{|c_2|}\frac{P+P_p+|b|P_h}{|E(c_0,\tilde{c}_1)||E(c_0,\bar{c}_1)|}\right.$$
$$+\frac{2|c_0|P_p}{|E(c_0,\tilde{c}_1)||E(c_0,\bar{c}_1)|}\left(1+\frac{|\tilde{c}_1|}{|\bar{c}_1|}\right)$$
$$\left.+\frac{2|c_0||c_1|P_p}{|\bar{c}_1||E(c_0,\tilde{c}_1)||E(c_0,\bar{c}_1)|}+\frac{|c_0|P_p}{|E(c_0,\tilde{c}_1)||E(c_0,\bar{c}_1)|}\right)$$

$$
\begin{aligned}
&= \frac{1}{|c_2|}(P+P_p+|b|P_h)+2P_p\left(1+\frac{|\widetilde{c}_1|}{|\overline{c}_1|}\right)+2\frac{|c_1|}{|\overline{c}_1|}P_p+P_p\\
&\le \frac{1}{|c_2|}(P+P_p+|b|P_h)+P_p\left(5+4\frac{|\widetilde{c}_1|}{|\overline{c}_1|}\right)\\
&\le P.
\end{aligned}
$$

Without loss of generality, assume $t_1, t_2 \in [0,T]$; we obtain

$$
\begin{aligned}
&\Bigg|\int_{t_2}^{t_2+T}\int_u^{u+T} p(s)\mathrm{e}^{\frac{\widetilde{c}_1}{c_0}(s-u)+\frac{\overline{c}_1}{c_0}(u-t_2)}\mathrm{d}s\mathrm{d}u\\
&\quad -\int_{t_1}^{t_1+T}\int_u^{u+T} p(s)\mathrm{e}^{\frac{\widetilde{c}_1}{c_0}(s-u)+\frac{\overline{c}_1}{c_0}(u-t_1)}\mathrm{d}s\mathrm{d}u\Bigg|\\
&\quad\le \left|\int_{t_2}^{t_2+T}\int_u^{u+T} p(s)\mathrm{e}^{\frac{\widetilde{c}_1}{c_0}(s-u)+\frac{\overline{c}_1}{c_0}u}\mathrm{d}s\mathrm{d}u\right|\left|\mathrm{e}^{-\frac{\overline{c}_1}{c_0}t_2}-\mathrm{e}^{-\frac{\overline{c}_1}{c_0}t_1}\right|\\
&\quad +\mathrm{e}^{-\frac{\overline{c}_1}{c_0}t_1}\Bigg|\int_{t_2}^{t_2+T}\int_u^{u+T} p(s)\mathrm{e}^{\frac{\widetilde{c}_1}{c_0}(s-u)+\frac{\overline{c}_1}{c_0}u}\mathrm{d}s\mathrm{d}u\\
&\quad -\int_{t_1}^{t_1+T}\int_u^{u+T} p(s)\mathrm{e}^{\frac{\widetilde{c}_1}{c_0}(s-u)+\frac{\overline{c}_1}{c_0}u}\mathrm{d}s\mathrm{d}u\Bigg|\\
&\quad\le P_p\frac{|\overline{c}_1|}{|c_2|}\frac{1}{|E(c_0,\widetilde{c}_1)||E(c_0,\overline{c}_1)|}\mathrm{e}^{\frac{|\overline{c}_1|}{|c_0|}T}|t_2-t_1|\\
&\quad +\mathrm{e}^{-\frac{\overline{c}_1}{c_0}t_1}\left|\int_{t_2+T}^{t_1+T}\int_u^{u+T} p(s)\mathrm{e}^{\frac{\widetilde{c}_1}{c_0}(s-u)+\frac{\overline{c}_1}{c_0}u}\mathrm{d}s\mathrm{d}u\right|\\
&\quad +\mathrm{e}^{-\frac{\overline{c}_1}{c_0}t_1}\left|\int_{t_1}^{t_2}\int_u^{u+T} p(s)\mathrm{e}^{\frac{\widetilde{c}_1}{c_0}(s-u)+\frac{\overline{c}_1}{c_0}u}\mathrm{d}s\mathrm{d}u\right|\\
&\le P_p\frac{|\overline{c}_1|}{|c_2|}\frac{1}{|E(c_0,\widetilde{c}_1)||E(c_0,\overline{c}_1)|}\mathrm{e}^{\frac{|\overline{c}_1|}{|c_0|}T}|t_2-t_1|\\
&\quad +P_p\frac{|\overline{c}_1|}{|c_2|}\frac{1}{|E(c_0,\widetilde{c}_1)|}\mathrm{e}^{2\frac{|\overline{c}_1|}{|c_0|}T}|t_2-t_1|\\
&\quad +P_p\frac{|\overline{c}_1|}{|c_2|}\frac{1}{|E(c_0,\widetilde{c}_1)|}\mathrm{e}^{\frac{|\overline{c}_1|}{|c_0|}T}|t_2-t_1|\\
&= \frac{|\overline{c}_1|}{|c_2|}\frac{P_p\mathrm{e}^{\frac{|\overline{c}_1|}{|c_0|}T}}{|E(c_0,\widetilde{c}_1)|}\left(1+\mathrm{e}^{\frac{|\overline{c}_1|}{|c_0|}T}+\frac{1}{|E(c_0,\overline{c}_1)|}\right)|t_2-t_1|, \qquad (2.16)
\end{aligned}
$$

$$
\begin{aligned}
&\Bigg|\int_{t_2}^{t_2+T}\int_u^{u+T} p'(s)\mathrm{e}^{\frac{\widetilde{c}_1}{c_0}(s-u)+\frac{\overline{c}_1}{c_0}(u-t_2)}\mathrm{d}s\mathrm{d}u\\
&\quad -\int_{t_1}^{t_1+T}\int_u^{u+T} p'(s)\mathrm{e}^{\frac{\widetilde{c}_1}{c_0}(s-u)+\frac{\overline{c}_1}{c_0}(u-t_1)}\mathrm{d}s\mathrm{d}u\Bigg|\\
&\le \left|\int_{t_2}^{t_2+T}\int_u^{u+T} p'(s)\mathrm{e}^{\frac{\widetilde{c}_1}{c_0}(s-u)+\frac{\overline{c}_1}{c_0}u}\mathrm{d}s\mathrm{d}u\right|\left|\mathrm{e}^{-\frac{\overline{c}_1}{c_0}t_2}-\mathrm{e}^{-\frac{\overline{c}_1}{c_0}t_1}\right|\\
&\quad +\mathrm{e}^{-\frac{\overline{c}_1}{c_0}t_1}\Bigg|\int_{t_2}^{t_2+T}\int_u^{u+T} p'(s)\mathrm{e}^{\frac{\widetilde{c}_1}{c_0}(s-u)+\frac{\overline{c}_1}{c_0}u}\mathrm{d}s\mathrm{d}u
\end{aligned}
$$

$$
\begin{aligned}
&-\int_{t_1}^{t_1+T}\int_u^{u+T} p'(s)\mathrm{e}^{\frac{\tilde{c}_1}{c_0}(s-u)+\frac{\bar{c}_1}{c_0}u}\mathrm{d}s\mathrm{d}u\Bigg|\\
&\le 2P_p\frac{1}{|E(c_0,\tilde{c}_1)||E(c_0,\bar{c}_1)|}\mathrm{e}^{\frac{|\bar{c}_1|}{|c_0|}T}|t_2-t_1|\\
&\quad+\mathrm{e}^{-\frac{\bar{c}_1}{c_0}t_1}\left|\int_{t_2+T}^{t_1+T}\int_u^{u+T} p'(s)\mathrm{e}^{\frac{\tilde{c}_1}{c_0}(s-u)+\frac{\bar{c}_1}{c_0}u}\mathrm{d}s\mathrm{d}u\right|\\
&\quad+\mathrm{e}^{-\frac{\bar{c}_1}{c_0}t_1}\left|\int_{t_1}^{t_2}\int_u^{u+T} p'(s)\mathrm{e}^{\frac{\tilde{c}_1}{c_0}(s-u)+\frac{\bar{c}_1}{c_0}u}\mathrm{d}s\mathrm{d}u\right|\\
&\le 2P_p\frac{1}{|E(c_0,\tilde{c}_1)||E(c_0,\bar{c}_1)|}\mathrm{e}^{\frac{|\bar{c}_1|}{|c_0|}T}|t_2-t_1|\\
&\quad+\frac{(1+P_p)}{|E(c_0,\tilde{c}_1)|}\mathrm{e}^{2\frac{|\bar{c}_1|}{|c_0|}T}|t_2-t_1|\\
&\quad+\frac{(1+P_p)}{|E(c_0,\tilde{c}_1)|}\mathrm{e}^{\frac{|\bar{c}_1|}{|c_0|}T}|t_2-t_1|\\
&=\frac{P_p\mathrm{e}^{\frac{|\bar{c}_1|}{|c_0|}T}}{|E(c_0,\tilde{c}_1)|}\left(1+\mathrm{e}^{\frac{|\bar{c}_1|}{|c_0|}T}+\frac{2}{|E(c_0,\bar{c}_1)|}\right)|t_2-t_1|,
\end{aligned}
\tag{2.17}
$$

$$
\begin{aligned}
&\Bigg|\int_{t_2}^{t_2+T}\int_u^{u+T} p''(s)\mathrm{e}^{\frac{\tilde{c}_1}{c_0}(s-u)+\frac{\bar{c}_1}{c_0}(u-t_2)}\mathrm{d}s\mathrm{d}u\\
&-\int_{t_1}^{t_1+T}\int_u^{u+T} p''(s)\mathrm{e}^{\frac{\tilde{c}_1}{c_0}(s-u)+\frac{\bar{c}_1}{c_0}(u-t_1)}\mathrm{d}s\mathrm{d}u\Bigg|\\
\le&\left|\int_{t_2}^{t_2+T}\int_u^{u+T} p''(s)\mathrm{e}^{\frac{\tilde{c}_1}{c_0}(s-u)+\frac{\bar{c}_1}{c_0}u}\mathrm{d}s\mathrm{d}u\right|\left|\mathrm{e}^{-\frac{\bar{c}_1}{c_0}t_2}-\mathrm{e}^{-\frac{\bar{c}_1}{c_0}t_1}\right|\\
&\quad+\mathrm{e}^{-\frac{\bar{c}_1}{c_0}t_1}\Bigg|\int_{t_2}^{t_2+T}\int_u^{u+T} p''(s)\mathrm{e}^{\frac{\tilde{c}_1}{c_0}(s-u)+\frac{\bar{c}_1}{c_0}u}\mathrm{d}s\mathrm{d}u\\
&\quad-\int_{t_1}^{t_1+T}\int_u^{u+T} p''(s)\mathrm{e}^{\frac{\tilde{c}_1}{c_0}(s-u)+\frac{\bar{c}_1}{c_0}u}\mathrm{d}s\mathrm{d}u\Bigg|\\
\le&\frac{P_p\mathrm{e}^{\frac{|\bar{c}_1|}{|c_0|}T}}{|E(c_0,\tilde{c}_1)||E(c_0,\bar{c}_1)|}\frac{|\bar{c}_1|}{|c_0|}\left(3+\frac{|\tilde{c}_1|}{|c_0|}\right)|t_2-t_1|\\
&\quad+\mathrm{e}^{-\frac{\bar{c}_1}{c_0}t_1}\left|\int_{t_2+T}^{t_1+T}\int_u^{u+T} p''(s)\mathrm{e}^{\frac{\tilde{c}_1}{c_0}(s-u)+\frac{\bar{c}_1}{c_0}u}\mathrm{d}s\mathrm{d}u\right|\\
&\quad+\mathrm{e}^{-\frac{\bar{c}_1}{c_0}t_1}\left|\int_{t_1}^{t_2}\int_u^{u+T} p''(s)\mathrm{e}^{\frac{\tilde{c}_1}{c_0}(s-u)+\frac{\bar{c}_1}{c_0}u}\mathrm{d}s\mathrm{d}u\right|\\
\le&\frac{P_p\mathrm{e}^{\frac{|\bar{c}_1|}{|c_0|}T}}{|E(c_0,\tilde{c}_1)||E(c_0,\bar{c}_1)|}\frac{|\bar{c}_1|}{|c_0|}\left(3+\frac{|\tilde{c}_1|}{|c_0|}\right)|t_2-t_1|
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\mathrm{e}^{\frac{|\bar{c}_1|}{|c_0|}T}}{|E(c_0,\widetilde{c}_1)|}\left(L_p+2P_p\frac{|\bar{c}_1|}{|c_0|}\mathrm{e}^{\frac{|\bar{c}_1|}{|c_0|}T}+2P_p\frac{|\widetilde{c}_1|}{|c_0|}\mathrm{e}^{\frac{|\bar{c}_1|}{|c_0|}T}\right)|t_2-t_1|\\
&+\frac{\mathrm{e}^{\frac{|\bar{c}_1|}{c_0}T}}{|E(c_0,\widetilde{c}_1)|}\left(L_p+4P_p\right)|t_2-t_1|\\
=&\frac{P_p\mathrm{e}^{\frac{|\bar{c}_1|}{|c_0|}T}}{|E(c_0,\widetilde{c}_1)||E(c_0,\bar{c}_1)|}\frac{|\bar{c}_1|}{|c_0|}\left(3+\frac{|\widetilde{c}_1|}{|c_0|}\right)|t_2-t_1|\\
&\quad+\frac{2\mathrm{e}^{\frac{|\bar{c}_1|}{|c_0|}T}}{|E(c_0,\widetilde{c}_1)|}\left(L_p+2P_p+P_p\frac{|\bar{c}_1|}{|c_0|}\mathrm{e}^{\frac{|\bar{c}_1|}{|c_0|}T}+P_p\frac{|\widetilde{c}_1|}{|c_0|}\mathrm{e}^{\frac{|\bar{c}_1|}{|c_0|}T}\right)|t_2-t_1|
\end{aligned}
\tag{2.18}
$$

and

$$
\begin{aligned}
&\left|\int_{t_2}^{t_2+T}\int_u^{u+T}\phi_\varphi(s)\mathrm{e}^{\frac{\tilde{c}_1}{c_0}(s-u)+\frac{\bar{c}_1}{c_0}(u-t_2)}\mathrm{d}s\mathrm{d}u\right.\\
&\qquad\left.-\int_{t_1}^{t_1+T}\int_u^{u+T}\phi_\varphi(s)\mathrm{e}^{\frac{\tilde{c}_1}{c_0}(s-u)+\frac{\bar{c}_1}{c_0}(u-t_1)}\mathrm{d}s\mathrm{d}u\right|\\
\le&\left|\int_{t_2}^{t_2+T}\int_u^{u+T}\phi_\varphi(s)\mathrm{e}^{\frac{\tilde{c}_1}{c_0}(s-u)+\frac{\bar{c}_1}{c_0}u}\mathrm{d}s\mathrm{d}u\right|\left|\mathrm{e}^{-\frac{\bar{c}_1}{c_0}t_2}-\mathrm{e}^{-\frac{\bar{c}_1}{c_0}t_1}\right|\\
&\qquad+\mathrm{e}^{-\frac{\bar{c}_1}{c_0}t_1}\left|\int_{t_2}^{t_2+T}\int_u^{u+T}\phi_\varphi(s)\mathrm{e}^{\frac{\tilde{c}_1}{c_0}(s-u)+\frac{\bar{c}_1}{c_0}u}\mathrm{d}s\mathrm{d}u\right.\\
&\qquad\left.-\int_{t_1}^{t_1+T}\int_u^{u+T}\phi_\varphi(s)\mathrm{e}^{\frac{\tilde{c}_1}{c_0}(s-u)+\frac{\bar{c}_1}{c_0}u}\mathrm{d}s\mathrm{d}u\right|\\
\le&\frac{|\bar{c}_1|}{|c_2|}\frac{P+P_p+|b|P_h}{|E(c_0,\widetilde{c}_1)||E(c_0,\bar{c}_1)|}\mathrm{e}^{\frac{|\bar{c}_1|}{|c_0|}T}|t_2-t_1|\\
&\qquad+\mathrm{e}^{-\frac{\bar{c}_1}{c_0}t_1}\left|\int_{t_2+T}^{t_1+T}\int_u^{u+T}\phi_\varphi(s)\mathrm{e}^{\frac{\tilde{c}_1}{c_0}(s-u)+\frac{\bar{c}_1}{c_0}u}\mathrm{d}s\mathrm{d}u\right|\\
&\qquad+\mathrm{e}^{-\frac{\bar{c}_1}{c_0}t_1}\left|\int_{t_1}^{t_2}\int_u^{u+T}\phi_\varphi(s)\mathrm{e}^{\frac{\tilde{c}_1}{c_0}(s-u)+\frac{\bar{c}_1}{c_0}u}\mathrm{d}s\mathrm{d}u\right|\\
\le&\frac{|\bar{c}_1|}{|c_2|}\frac{P+P_p+|b|P_h}{|E(c_0,\widetilde{c}_1)||E(c_0,\bar{c}_1)|}\mathrm{e}^{\frac{|\bar{c}_1|}{|c_0|}T}|t_2-t_1|\\
&+\frac{|\bar{c}_1|}{|c_2|}\frac{P+P_p+|b|P_h}{|E(c_0,\widetilde{c}_1)|}\mathrm{e}^{2\frac{|\bar{c}_1|}{|c_0|}T}|t_2-t_1|\\
&+\frac{|\bar{c}_1|}{|c_2|}\frac{P+P_p+|b|P_h}{|E(c_0,\widetilde{c}_1)|}\mathrm{e}^{\frac{|\bar{c}_1|}{|c_0|}T}|t_2-t_1|\\
=&\frac{|\bar{c}_1|}{|c_2|}\frac{P+P_p+|b|P_h}{|E(c_0,\widetilde{c}_1)|}\mathrm{e}^{\frac{|\bar{c}_1|}{|c_0|}T}\left(1+\mathrm{e}^{\frac{|\bar{c}_1|}{|c_0|}T}+\frac{1}{|E(c_0,\bar{c}_1)|}\right)|t_2-t_1|
\end{aligned}
\tag{2.19}
$$

By (2.16)–(2.19), using (2.10), we have

$$
\begin{aligned}
&\Big|(A\varphi)(t_2)-(A\varphi)(t_1)\Big| \\
&\le \frac{1}{|c_0|}|E(c_0,\widetilde{c}_1)||E(c_0,\overline{c}_1)|\Bigg(\Bigg|\int_{t_2}^{t_2+T}\int_u^{u+T}\phi_\varphi(s)\mathrm{e}^{\frac{\widetilde{c}_1}{c_0}(s-u)+\frac{\overline{c}_1}{c_0}(u-t_2)}\mathrm{d}s\mathrm{d}u \\
&\quad -\int_{t_1}^{t_1+T}\int_u^{u+T}\phi_\varphi(s)\mathrm{e}^{\frac{\widetilde{c}_1}{c_0}(s-u)+\frac{\overline{c}_1}{c_0}(u-t_1)}\mathrm{d}s\mathrm{d}u\Bigg| \\
&\quad +|c_0|\Bigg|\int_{t_2}^{t_2+T}\int_u^{u+T}p''(s)\mathrm{e}^{\frac{\widetilde{c}_1}{c_0}(s-u)+\frac{\overline{c}_1}{c_0}(u-t_2)}\mathrm{d}s\mathrm{d}u \\
&\quad -\int_{t_1}^{t_1+T}\int_u^{u+T}p''(s)\mathrm{e}^{\frac{\widetilde{c}_1}{c_0}(s-u)+\frac{\overline{c}_1}{c_0}(u-t_1)}\mathrm{d}s\mathrm{d}u\Bigg| \\
&\quad +|c_1|\Bigg|\int_{t_2}^{t_2+T}\int_u^{u+T}p'(s)\mathrm{e}^{\frac{\widetilde{c}_1}{c_0}(s-u)+\frac{\overline{c}_1}{c_0}(u-t_2)}\mathrm{d}s\mathrm{d}u \\
&\quad -\int_{t_1}^{t_1+T}\int_u^{u+T}p'(s)\mathrm{e}^{\frac{\widetilde{c}_1}{c_0}(s-u)+\frac{\overline{c}_1}{c_0}(u-t_1)}\mathrm{d}s\mathrm{d}u\Bigg| \\
&\quad +|c_2|\Bigg|\int_{t_2}^{t_2+T}\int_u^{u+T}p(s)\mathrm{e}^{\frac{\widetilde{c}_1}{c_0}(s-u)+\frac{\overline{c}_1}{c_0}(u-t_2)}\mathrm{d}s\mathrm{d}u \\
&\quad -\int_{t_1}^{t_1+T}\int_u^{u+T}p(s)\mathrm{e}^{\frac{\widetilde{c}_1}{c_0}(s-u)+\frac{\overline{c}_1}{c_0}(u-t_1)}\mathrm{d}s\mathrm{d}u\Bigg|\Bigg) \\
&\le \frac{1}{|c_0|}\mathrm{e}^{\frac{|\overline{c}_1|}{|c_0|}T}|E(c_0,\overline{c}_1)|\Bigg(\frac{|\overline{c}_1|}{|c_2|}(P+P_p+|b|P_h)\left(1+\mathrm{e}^{\frac{|\overline{c}_1|}{|c_0|}T}+\frac{1}{|E(c_0,\overline{c}_1)|}\right) \\
&\quad +\frac{|\overline{c}_1|P_p}{|E(c_0,\overline{c}_1)|}\left(3+\frac{|\widetilde{c}_1|}{|c_0|}\right)+2\left(|c_0|L_p+2|c_0|P_p+P_p|\overline{c}_1|\mathrm{e}^{\frac{|\overline{c}_1|}{|c_0|}T}+P_p|\widetilde{c}_1|\mathrm{e}^{\frac{|\overline{c}_1|}{|c_0|}T}\right) \\
&\quad +|c_1|P_p\left(1+\mathrm{e}^{\frac{|\overline{c}_1|}{|c_0|}T}+\frac{2}{|E(c_0,\overline{c}_1)|}\right)+|\overline{c}_1|P_p\left(1+\mathrm{e}^{\frac{|\overline{c}_1|}{|c_0|}T}+\frac{1}{|E(c_0,\overline{c}_1)|}\right)\Bigg)|t_2-t_1| \\
&\le L|t_2-t_1|
\end{aligned}
$$

Therefore, $(A\varphi)(t)\in\mathcal{P}_T(P,L)$. So by Lemma 2.4, we see that all the conditions of Schauder's theorem are satisfied on $\mathcal{P}_T(P,L)$. Thus there exists a fixed point y in $\mathcal{P}_T(P,L)$ such that $y=Ay$, from Lemma 2.3, y is a T-periodic solution of Eq. (1.2). This completes the proof. □

From the relation between x and y, we have the following result:

Theorem 2.3. *Assume the conditions in Theorem 2.2 hold; then (1.1) has a periodic solution in* $\mathcal{P}_T(\frac{1}{|b|}(P+P_p),\frac{1}{|b|}(L+L_p))$.

3. Uniqueness and Stability

In this section, uniqueness and stability of (1.2) will be proved.

Theorem 3.1. *In addition to the assumption of Theorem 2.2, suppose that*

$$1+L+L_p<|c_2|; \tag{3.1}$$

then (1.2) has a unique solution in $\mathcal{P}_T(P,L)$.

Proof. We know from the proof of Theorem 2.2 that $A : \mathcal{P}_T(P, L) \to \mathcal{P}_T(P, L)$, Moreover, by (2.8), we get

$$\|A\varphi - A\psi\| \le \frac{1 + L + L_p}{|c_2|}\|\varphi - \psi\|, \quad \varphi, \psi \in \mathcal{P}_T(P, L),$$

(3.1) means $\frac{1+L+L_p}{|c_2|} < 1$, so the fixed point must be unique by the Banach fixed point theorem. □

Theorem 3.2. *The unique solution obtained in Theorem 3.1 depends continuously on the given functions $p(t)$ and $h(t)$.*

Proof. Let functions $p(t)$, $\widetilde{p}(t)$ and $h(t)$, $\widetilde{h}(t)$ in $\mathcal{P}_T(P_p, L_p)$ and $\mathcal{P}_T(P_h, L_h)$ be given. Then we consider the corresponding operators $A, \widetilde{A}$ defined by (2.7). Assuming corresponding conditions (2.9), (2.10) and (3.1), there are two unique corresponding functions $y(t)$ and $\widetilde{y}(t)$ in $\mathcal{P}_T(P_p, L_p)$ such that

$$y = Ay, \quad \widetilde{y} = \widetilde{A}\widetilde{y}.$$

Then we have

$$\|y - \widetilde{y}\| \le \|Ay - A\widetilde{y}\| + \|A\widetilde{y} - \widetilde{A}\widetilde{y}\| \le \frac{1 + L + L_p}{|c_2|}\|y - \widetilde{y}\| + \|A\widetilde{y} - \widetilde{A}\widetilde{y}\|,$$

which implies

$$\|y - \widetilde{y}\| \le \frac{|c_2|}{|c_2| - 1 - L - L_p}\|A\widetilde{y} - \widetilde{A}\widetilde{y}\|.$$

Now, for $t \in [0, T]$, we note

$$\begin{aligned} &\left|\int_t^{t+T}\int_u^{u+T}(p(\widetilde{y}(s)) - \widetilde{p}(\widetilde{y}(s)))\mathrm{e}^{\frac{\widetilde{c}_1}{c_0}(s-u)+\frac{\overline{c}_1}{c_0}(u-t)}\mathrm{d}s\mathrm{d}u\right| \\ &\quad\le \mathrm{e}^{-\frac{\overline{c}_1}{c_0}t}\left|\int_t^{t+T}\int_u^{u+T}\mathrm{e}^{\frac{\widetilde{c}_1}{c_0}s+\frac{\overline{c}_1-\widetilde{c}_1}{c_0}u}\mathrm{d}s\mathrm{d}u\right|\|p - \widetilde{p}\| \\ &\quad\le \frac{|c_0|}{|c_2|}\frac{\|p - \widetilde{p}\|}{|E(c_0, \widetilde{c}_1)||E(c_0, \overline{c}_1)|}, \end{aligned} \tag{3.2}$$

$$\begin{aligned} &\left|\int_t^{t+T}\int_u^{u+T}(p(s) - \widetilde{p}(s))\mathrm{e}^{\frac{\widetilde{c}_1}{c_0}(s-u)+\frac{\overline{c}_1}{c_0}(u-t)}\mathrm{d}s\mathrm{d}u\right| \\ &\quad\le \mathrm{e}^{-\frac{\overline{c}_1}{c_0}t}\left|\int_t^{t+T}\int_u^{u+T}\mathrm{e}^{\frac{\widetilde{c}_1}{c_0}s+\frac{\overline{c}_1-\widetilde{c}_1}{c_0}u}\mathrm{d}s\mathrm{d}u\right|\|p - \widetilde{p}\| \\ &\quad\le \frac{|c_0|}{|c_2|}\frac{\|p - \widetilde{p}\|}{|E(c_0, \widetilde{c}_1)||E(c_0, \overline{c}_1)|} \end{aligned} \tag{3.3}$$

$$\begin{aligned} &\left|\int_t^{t+T}\int_u^{u+T}(p'(s) - \widetilde{p}'(s))\mathrm{e}^{\frac{\widetilde{c}_1}{c_0}(s-u)+\frac{\overline{c}_1}{c_0}(u-t)}\mathrm{d}s\mathrm{d}u\right| \\ &\quad= \mathrm{e}^{-\frac{\overline{c}_1}{c_0}t}\left|\int_t^{t+T}\int_u^{u+T}(p(s) - \widetilde{p}(s))'\mathrm{e}^{\frac{\widetilde{c}_1}{c_0}s+\frac{\overline{c}_1-\widetilde{c}_1}{c_0}u}\mathrm{d}s\mathrm{d}u\right| \end{aligned}$$

$$\leq \frac{|c_0|}{|\overline{c}_1|}\frac{2\|p-\widetilde{p}\|}{|E(c_0,\widetilde{c}_1)||E(c_0,\overline{c}_1)|} \tag{3.4}$$

$$\begin{aligned}
&\left|\int_t^{t+T}\int_u^{u+T}(p''(s)-\widetilde{p}''(s))\mathrm{e}^{\frac{\widetilde{c}_1}{c_0}(s-u)+\frac{\overline{c}_1}{c_0}(u-t)}\mathrm{d}s\mathrm{d}u\right|\\
&\quad = \mathrm{e}^{-\frac{\overline{c}_1}{c_0}t}\left|\int_t^{t+T}\int_u^{u+T}(p(s)-\widetilde{p}(s))''\mathrm{e}^{\frac{\widetilde{c}_1}{c_0}s+\frac{\overline{c}_1-\widetilde{c}_1}{c_0}u}\mathrm{d}s\mathrm{d}u\right|\\
&\quad \leq (2+\frac{|\widetilde{c}_1|}{|\overline{c}_1|}+\frac{|\widetilde{c}_1|}{|c_0|})\frac{\|p-\widetilde{p}\|}{E(c_0,\widetilde{c}_1)E(c_0,\overline{c}_1)}
\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}
&\left|\int_t^{t+T}\int_u^{u+T}(h(s)-\widetilde{h}(s))\mathrm{e}^{\frac{\widetilde{c}_1}{c_0}(s-u)+\frac{\overline{c}_1}{c_0}(u-t)}\mathrm{d}s\mathrm{d}u\right|\\
&\quad \leq \mathrm{e}^{-\frac{\overline{c}_1}{c_0}t}\left|\int_t^{t+T}\int_u^{u+T}\mathrm{e}^{\frac{\widetilde{c}_1}{c_0}s+\frac{\overline{c}_1-\widetilde{c}_1}{c_0}u}\mathrm{d}s\mathrm{d}u\right|\|h-\widetilde{h}\|\\
&\quad \leq \frac{|c_0|}{|c_2|}\frac{\|h-\widetilde{h}\|}{|E(c_0,\widetilde{c}_1)||E(c_0,\overline{c}_1)|}.
\end{aligned} \tag{3.6}$$

From (3.2)–(3.6), we arrive at

$$\begin{aligned}
&\|y-\widetilde{y}\|\\
&\quad\leq \frac{|c_2|}{|c_2|-1-L-L_p}\|A\widetilde{y}-\widetilde{A}\widetilde{y}\|\\
&\quad\leq \frac{|c_2||E(c_0,\widetilde{c}_1)||E(c_0,\overline{c}_1)|}{|c_0|(|c_2|-1-L-L_p)}\Bigg(\left|\int_t^{t+T}\int_u^{u+T}(p(\widetilde{y}(s))-\widetilde{p}(\widetilde{y}(s)))\mathrm{e}^{\frac{\widetilde{c}_1}{c_0}(s-u)+\frac{\overline{c}_1}{c_0}(u-t)}\mathrm{d}s\mathrm{d}u\right|\\
&\qquad+|c_0|\left|\int_t^{t+T}\int_u^{u+T}(p''(s)-\widetilde{p}''(s))\mathrm{e}^{\frac{\widetilde{c}_1}{c_0}(s-u)+\frac{\overline{c}_1}{c_0}(u-t)}\mathrm{d}s\mathrm{d}u\right|\\
&\qquad+|c_1|\left|\int_t^{t+T}\int_u^{u+T}(p'(s)-\widetilde{p}'(s))\mathrm{e}^{\frac{\widetilde{c}_1}{c_0}(s-u)+\frac{\overline{c}_1}{c_0}(u-t)}\mathrm{d}s\mathrm{d}u\right|\\
&\qquad+|c_2|\left|\int_t^{t+T}\int_u^{u+T}(p(s)-\widetilde{p}(s))\mathrm{e}^{\frac{\widetilde{c}_1}{c_0}(s-u)+\frac{\overline{c}_1}{c_0}(u-t)}\mathrm{d}s\mathrm{d}u\right|\\
&\qquad+|b|\left|\int_t^{t+T}\int_u^{u+T}(h(s)-\widetilde{h}(s))\mathrm{e}^{\frac{\widetilde{c}_1}{c_0}(s-u)+\frac{\overline{c}_1}{c_0}(u-t)}\mathrm{d}s\mathrm{d}u\right|\Bigg)\\
&\quad\leq \frac{|c_2|}{|c_2|-1-L-L_p}\left(5+\frac{1}{|c_2|}+2\frac{|\widetilde{c}_1|}{|\overline{c}_1|}+\frac{|\widetilde{c}_1|}{|c_0|}\right)\|p-\widetilde{p}\|\\
&\qquad+\frac{|b|}{|c_2|-1-L-L_p}\|h-\widetilde{h}\|.
\end{aligned}$$

This completes the proof. □

Now, we have the results for (1.1) by Theorems 3.1 and 3.2.

Theorem 3.3. *Assume the conditions in Theorem 3.1 hold; then (1.1) has a unique periodic solution in* $\mathcal{P}_T(\frac{1}{|b|}(P+P_p),\frac{1}{|b|}(L+L_p))$.

Theorem 3.4. *The unique solution obtained in Theorem 3.3 depends continuously on the given functions $p(t)$ and $h(t)$.*

4. Examples

Give examples to illustrate that the assumptions of Theorem 2.3 do not self-contradict.

Example 4.1. Consider the following equation:

$$2x''(t)+5x'(t)-7x(t)=x(\cos(t)+x(t))+\sin(t), \tag{4.1}$$

where $c_0=2, c_1=5, c_2=-7, b=1, p(t)=\cos(t), h(t)=\sin(t)$. $\widetilde{c}_1=\frac{c_1+\sqrt{c_1^2-4c_0c_2}}{2}=7, \overline{c}_1=\frac{c_1-\sqrt{c_1^2-4c_0c_2}}{2}=-2$, $P_p=P_h=L_p=L_h=1$,

$$E(c_0,\widetilde{c}_1)=\frac{1}{\mathrm{e}^{\frac{\widetilde{c}_1}{c_0}T}-1}=\frac{1}{\mathrm{e}^{\frac{5}{2}T}-1},\ E(c_0,\overline{c}_1)=\frac{1}{\mathrm{e}^{\frac{\overline{c}_1}{c_0}T}-1}=\frac{\mathrm{e}^T}{1-\mathrm{e}^T};$$

here $T=2\pi$. Take $P=23, L=1.35\times 10^{12}$, then

$$|b|P_h+|c_2|P_p\left(5+4\frac{|\widetilde{c}_1|}{|\overline{c}_1|}+\frac{1}{|c_2|}\right)=135\le 138=(|c_2|-1)P$$

and

$$\begin{aligned}
&\mathrm{e}^{\frac{|\overline{c}_1|}{|c_0|}T}|E(c_0,\overline{c}_1)|\Bigg(\frac{|\overline{c}_1|}{|c_2|}(P+P_p+|b|P_h)\left(1+\mathrm{e}^{\frac{|\overline{c}_1|}{|c_0|}T}+\frac{1}{|E(c_0,\overline{c}_1)|}\right)\\
&+\frac{|\overline{c}_1|P_p}{|E(c_0,\overline{c}_1)|}\left(3+\frac{|\widetilde{c}_1|}{|c_0|}\right)+2\left(|c_0|L_p+2|c_0|P_p+P_p|\overline{c}_1|\mathrm{e}^{\frac{|\overline{c}_1|}{|c_0|}T}+P_p|\widetilde{c}_1|\mathrm{e}^{\frac{|\overline{c}_1|}{|c_0|}T}\right)\\
&+|c_1|P_p\left(1+\mathrm{e}^{\frac{|\overline{c}_1|}{|c_0|}T}+\frac{2}{|E(c_0,\overline{c}_1)|}\right)+|\overline{c}_1|P_p\left(1+\mathrm{e}^{\frac{|\overline{c}_1|}{|c_0|}T}+\frac{1}{|E(c_0,\overline{c}_1)|}\right)\Bigg)\\
&=\frac{\mathrm{e}^{4\pi}}{7(\mathrm{e}^{2\pi}-1)}\left(380+225\mathrm{e}^{2\pi}-\frac{225}{\mathrm{e}^{2\pi}}\right)<2.7\times 10^{12}=|c_0|L,
\end{aligned}$$

By Theorem 2.3, Eq. (4.1) has a 2π-periodic solution x such that $|x(t)|\le 24$, and $|x(t_2)-x(t_1)|\le(1.35\times 10^{12}+1)|t_2-t_1|,\ \forall t_1,t_2\in\mathbb{R}$.

Example 4.2. Consider

$$2x''(t)+5x'(t)-7x(t)=x(\lambda\cos(t)+x(t))+\lambda\sin(t), \tag{4.2}$$

where $\lambda>0$ is a parameter. Noting $c_0=2, c_1=5, c_2=-7, b=1$, $\widetilde{c}_1=\frac{c_1+\sqrt{c_1^2-4c_0c_2}}{2}=7, \overline{c}_1=\frac{c_1-\sqrt{c_1^2-4c_0c_2}}{2}=-2$ as in Example 4.1, $p(t)=\lambda\cos(t), h(t)=\lambda\sin(t)$. $P_p=P_h=L_p=L_h=\lambda$.

$$E(c_0,\widetilde{c}_1)=\frac{1}{\mathrm{e}^{\frac{\widetilde{c}_1}{c_0}T}-1}=\frac{1}{\mathrm{e}^{\frac{7}{2}T}-1},\ E(c_0,\overline{c}_1)=\frac{1}{\mathrm{e}^{\frac{\overline{c}_1}{c_0}T}-1}=\frac{\mathrm{e}^T}{1-\mathrm{e}^T};$$

here $T=2\pi$. Next, we consider P and L as variables to be defined by λ. In fact, if we take $P(\lambda)\ge\frac{45}{2}\lambda, L(\lambda)>\frac{\lambda\mathrm{e}^{2T}}{\mathrm{e}^T-1}\left(29+16\mathrm{e}^T-\frac{16}{\mathrm{e}^T}\right)$, a simple calculation yields

$$|b|P_h + |c_2|P_p\left(5 + 4\frac{|\widetilde{c}_1|}{|\overline{c}_1|} + \frac{1}{|c_2|}\right) = 135\lambda \le (|c_2|-1)P(\lambda),$$

$$\begin{aligned}&\mathrm{e}^{\frac{|\overline{c}_1|}{|c_0|}T}|E(c_0,\overline{c}_1)|\Bigg(\frac{|\overline{c}_1|}{|c_2|}(P+P_p+|b|P_h)\left(1+\mathrm{e}^{\frac{|\overline{c}_1|}{|c_0|}T}+\frac{1}{|E(c_0,\overline{c}_1)|}\right)\\&+\frac{|\overline{c}_1|P_p}{|E(c_0,\overline{c}_1)|}\left(3+\frac{|\widetilde{c}_1|}{|c_0|}\right)+2\bigg(|c_0|L_p+2|c_0|P_p+P_p|\overline{c}_1|\mathrm{e}^{\frac{|\overline{c}_1|}{|c_0|}T}+P_p|\widetilde{c}_1|\mathrm{e}^{\frac{|\overline{c}_1|}{|c_0|}T}\bigg)\\&+|c_1|P_p\left(1+\mathrm{e}^{\frac{|\overline{c}_1|}{|c_0|}T}+\frac{2}{|E(c_0,\overline{c}_1)|}\right)+|\overline{c}_1|P_p\left(1+\mathrm{e}^{\frac{|\overline{c}_1|}{|c_0|}T}+\frac{1}{|E(c_0,\overline{c}_1)|}\right)\Bigg)\\&\le\frac{\lambda\mathrm{e}^{2T}}{\mathrm{e}^T-1}\left(58+32\mathrm{e}^T-\frac{32}{\mathrm{e}^T}\right)<2L(\lambda)=|c_0|L.\end{aligned}$$

Then (2.9) and (2.10) hold. So the conditions for Theorem 2.3 are satisfied and thus Eq. (4.2) has a 2π-periodic solution such that $|x(t)| \le \lambda + P(\lambda)$, and $|x(t_2) - x(t_1)| \le (\lambda + L(\lambda))|t_2 - t_1|$, $\forall t_1, t_2 \in \mathbb{R}$. where $P(\lambda) \ge \frac{45\lambda}{2}$ and $L(\lambda) > \frac{\lambda \mathrm{e}^{4\pi}}{\mathrm{e}^{2\pi}-1}\left(29 + 16\mathrm{e}^{2\pi} - \frac{16}{\mathrm{e}^{2\pi}}\right)$. Furthermore, if we take $L(\lambda) < 6 - \lambda$, then

$$1 + L(\lambda) + L_p < |c_2|;$$

by Theorem 3.1, Eq. (4.2) has a unique 2π-periodic solution.

Remark 4.1. In fact, if $\lambda > \lambda_0 \approx 1.300912 \times 10^{-6}$, Eq. (4.2) has a solution in $\mathcal{P}_{2\pi}(\lambda + P(\lambda), \lambda + L(\lambda))$. If $0 < \lambda < \lambda_0 \approx 1.300912 \times 10^{-6}$, Eq. (4.2) has a unique solution in $\mathcal{P}_{2\pi}(\lambda + P(\lambda), \lambda + L(\lambda))$. Obviously, $\lambda = 1$ in Eq. (4.1) satisfies the first case.

References

[1] Bellman, R., Cooke, K.: Differential-Difference Equations. Academic Press, Cambridge (1963)

[2] Cooke, K. L.: Functional differential systems: some models and perturbation problems. In: Proceedings of the international symposium on differential equations and dynamical systems, Puerto Rico, 1965. Academic Press, New York (1967)

[3] Eder, E.: The functional differential equation $x'(t) = x(x(t))$. J. Differ. Equ. **54**, 390–400 (1984)

[4] Fečkan, M.: On a certain type of functional differential equations. Math. Slovaca **43**, 39–43 (1993)

[5] Hale, J.: Theory of functional differential equations. Springer, Berlin (1977)

[6] Liu, H.Z., Li, W.R.: Discussion on the analytic solutions of the second-order iterated differential equation. Bull. Korean Math. Soc. **43**(4), 791–804 (2006)

[7] Shi, B., Li, Z.X.: Existence of solutions and bifurcation of a class of first-order functional differential equations. Acta Math. Appl. Sin. **18**, 83–89 (1995)

[8] Si, J.G., Cheng, S.S.: Analytic solutions of a functional differential equation with state dependent argument. Taiwan. J. Math. **1**(4), 471–480 (1997)

[9] Si, J.G., Ma, M.H.: Local invertible analytic solution of a functional differential equation with deviating arguments depending on the state derivative. J. Math. Anal. Appl. **327**, 723–734 (2007)

[10] Si, J.G., Wang, X.P., Cheng, S.S.: Analytic solutions of a functional differential equation with a state derivative dependent delay. Aequ. Math. **57**, 75–86 (1999)

[11] Si, J.G., Zhang, W.N., Kim, G.H.: Analytic solutions of an iterative functional differential equation. Appl. Math. Comput. **150**, 647–659 (2004)

[12] Stephen, B.H.: On the existence of periodic solutions of $z'(t) = -az(t - r + \mu k(t, z(t))) + F(t)$. J. Differ. Equ. **6**, 408–419 (1969)

[13] Wu, H.Z.: A class of functional differential equations with deviating arguments depending on the state. Acta Math. Sin. **38**(6), 803–809 (1995)

[14] Zhao, H.Y., Fečkan, M.: Periodic solutions for a class of differential equations with delays depending on state. Math. Commun. **23**, 29–42 (2018)

Hou Yu Zhao
School of Mathematics
Chongqing Normal University
Chongqing 401331
People's Republic of China
e-mail: houyu19@gmail.com

Jia Liu
Department of Architectural Economic Management
Shandong Urban Construction Vocational College
Jinan 250103 Shandong
People's Republic of China

Received: April 12, 2018.
Accepted: October 29, 2018.