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Split Null Point Problems and Fixed Point Problems for Demicontractive Multivalued Mappings

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Abstract. In this paper, we consider the split null point problem and the fixed point problem for multivalued mappings in Hilbert spaces. We introduce a Halpern-type algorithm for solving the problem for maximal monotone operators and demicontractive multivalued mappings, and establish a strong convergence result under some suitable conditions. Also, we apply our problem of main result to other split problems, that is, the split feasibility problem, the split equilibrium problem, and the split minimization problem. Finally, a numerical result for supporting our main result is also supplied.

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Keywords. Split null point problems, fixed point problems, demicontractive multivalued mappings, maximal monotone operators, strong convergence.

1. Introduction

Throughout this paper, we shall assume that $\mathcal{H}$, $\mathcal{H}_1$ and $\mathcal{H}_2$ are real Hilbert spaces with inner products $\langle \cdot, \cdot \rangle$ and norms $\| \cdot \|$, and let I be the identity operator on a Hilbert space. Let $\mathbb{N}$ be the set of positive integers and $\mathbb{R}$ the set of real numbers.

Recently, the *split inverse problem* (SIP) was widely studied by many researchers [6,8–10,12,17,18,23] as its applications are desirable and can be used in real-world applications, for example, in image recovery, signal processing, the intensity-modulated radiation therapy, etc (see [4,7,8,11]). The SIP concerns a model which is to find a point

$$x^* \in \mathcal{H}_1 \quad \text{that solves} \quad \text{IP}_1 \tag{1.1}$$

such that

$$y^* := Ax^* \in \mathcal{H}_2 \quad \text{solves} \quad \text{IP}_2, \tag{1.2}$$

where IP_1 denotes an inverse problem formulated in $\mathcal{H}_1$ and IP_2 denotes an inverse problem formulated in $\mathcal{H}_2$, and $A : \mathcal{H}_1 \to \mathcal{H}_2$ is a bounded linear operator. In 1994, the first instance of the split inverse problem was introduced by Censor and Elfving, that is, the split feasibility problem [8]. After that, other split problems were introduced such as the split variational inequality problem [12], the split common null point problem [6], the split common fixed point problem [9], the split equilibrium problem [23], the split minimization problem, etc (see Sect. 4).

In this work, we focus our attention on the following *split null point problem* (SNPP) which was introduced by Byrne et al. [6]: given two multivalued mappings $B_1 : \mathcal{H}_1 \to 2^{\mathcal{H}_1}$ and $B_2 : \mathcal{H}_2 \to 2^{\mathcal{H}_2}$, the problem is formulated as finding a point

$$x^* \in \mathcal{H}_1 \text{ such that } 0 \in B_1(x^*) \text{ and } 0 \in B_2(Ax^*). \tag{1.3}$$

Let B be a multivalued mappings of $\mathcal{H}$ into $2^{\mathcal{H}}$, then the null point set of B is denoted by $B^{-1}0 := \{x \in \mathcal{H} : 0 \in Bx\}$. In other words of the SNPP, the problem of finding a point of the null point set of a multivalued mapping such that its image under a given bounded linear operator belongs to the null point set of another multivalued mapping in the image space. Then, the SNPP (1.3) can be rewritten as follows: Find a point $x^* \in \mathcal{H}_1$ such that

$$x^* \in B_1^{-1}0 \quad \text{and} \quad Ax^* \in B_2^{-1}0.$$

The SNPP for maximal monotone operators was studied by many researches in both Hilbert spaces and Banach spaces; see, for instance, [1,2,6,30,31]. The subdifferential of a lower semicontinuous and convex function is an important example of maximal monotone operators and its resolvent operators are often used to construct algorithms for solving the minimization problem of the function. In [15], Combettes and Pesquet considered proximal splitting methods constructed by the resolvent operators of the subdifferential of functions to study signal processing.

To solve the SNPP (1.3) for two maximal monotone operators B_1 and B_2, Byrne et al. [6] proposed the following two algorithms:

$$x_{n+1} = J_\lambda^{B_1}\left(x_n + \gamma A^*\left(J_\lambda^{B_2} - I\right)Ax_n\right), \quad n \in \mathbb{N}, \tag{1.4}$$

and

$$\begin{cases} u \in \mathcal{H}_1, \\ x_{n+1} = \alpha_n u + (1-\alpha_n)J_\lambda^{B_1}\left(x_n + \gamma A^*\left(J_\lambda^{B_2} - I\right)Ax_n\right), \quad n \in \mathbb{N}, \end{cases} \tag{1.5}$$

where $J_\lambda^{B_1}$ and $J_\lambda^{B_2}$ are resolvents of B_1 and B_2, respectively, A^* is the adjoint operator of A. They obtained a weak convergence result of Algorithm (1.4) and a strong convergence result of Algorithm (1.5) under some control conditions.

In many areas, stability or equilibrium is a fundamental concept that can be explained in terms of fixed points, and the fixed point theory is very significant in nonlinear analysis and has been widely studied. In 2015, the problem of finding a common solution of the null point and fixed point problem was first studied by Takahashi et al. [33]. It is well known that the class of demicontractive mappings [20, 24] includes several common types of classes of mappings occurring in nonlinear analysis and optimization problems. In 2017, Eslamain [16] considered the problem of finding a common solution of the split null point problem and the fixed point problem for maximal monotone operators and demicontractive mappings, respectively (see also [22]). Furthermore, fixed point theory was also studied in the case of multivalued mappings and it can be utilized in various areas such as game theory, control theory, mathematical economics, etc.

In this article, inspired and motivated by these works, we are interested to study the split null point problem and the fixed point problem for multivalued mappings in Hilbert spaces. In Sect. 3, we introduce a Halpern-type algorithm [19] for finding a common solution of the split null point problem and the fixed point problem for maximal monotone operators and demicontractive multivalued mappings, respectively, and prove a strong convergence theorem of the proposed algorithm under some suitable conditions. In Sect. 4, we reduce our problem to other split problems, i.e., the split feasibility problem, the split equilibrium problem and the split minimization problem. In Sect. 5, we also present the numerical example to demonstrate the convergence of our algorithm.

2. Preliminaries

We denote the strong and weak convergence of a sequence $\{x_n\} \subset \mathcal{H}$ to $z \in \mathcal{H}$ by $x_n \to z$ and $x_n \rightharpoonup z$, respectively. Let C be a nonempty closed convex subset of $\mathcal{H}$.

Recall that the (metric) projection from $\mathcal{H}$ onto C, denoted by P_C is defined for each $x \in \mathcal{H}$, $P_C x$ is the unique element in C such that

$$\|x - P_C x\| = d(x, C) := \inf\{\|x - z\| : z \in C\}.$$

It is known that $P_C x \in C$ is characterized by the following property:

$$\langle x - P_C x, z - P_C x\rangle \le 0 \quad \text{for all} \quad z \in C.$$

Let $T : C \to 2^C$ be a multivalued mapping. An element $p \in C$ is called a fixed point of T if $p \in Tp$. The set of all fixed points of T is denoted by $F(T)$. We say that T satisfies the endpoint condition if $Tp = \{p\}$ for all $p \in F(T)$.

A subset D of C is said to be proximal if for each $x \in C$, there exists $y \in D$ such that

$$\|x - y\| = d(x, D).$$

We denote by $CB(C)$, $K(C)$, and $P(C)$ the families of all nonempty closed bounded subsets of C, nonempty compact subsets of C, and nonempty proximal bounded subsets of C, respectively. The Pompeiu–Hausdorff metric on $CB(C)$ is defined by

$$H(A, B) := \max\left\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\right\}$$

for all $A, B \in CB(C)$.

We now recall the definitions of some multivalued mappings in Hilbert spaces.

Definition 2.1. Let C be a nonempty closed convex subset of $\mathcal{H}$. A multivalued mapping $T : C \to CB(C)$ is said to be

(i) *nonexpansive* if
$$H(Tx, Ty) \leq \|x - y\| \quad \text{for all} \quad x, y \in C,$$

(ii) *quasi-nonexpansive* if $F(T) \neq \emptyset$ and
$$H(Tx, Tp) \leq \|x - p\| \quad \text{for all} \quad x \in C, p \in F(T),$$

(iii) *demicontractive* [13,21] if $F(T) \neq \emptyset$ and there exists $k \in [0, 1)$ such that
$$H(Tx, Tp)^2 \leq \|x - p\|^2 + kd(x, Tx)^2 \quad \text{for all} \quad x \in C, p \in F(T).$$

It is noticed in Definition 2.1 that the class of demicontractive mappings includes classes of nonexpansive and quasi-nonexpansive mappings.

We provide an example of a demicontractive multivalued mapping which is not quasi-nonexpansive.

Example 2.2. Let $\mathcal{H} = \mathbb{R}$. Define a multivalued mapping $T : \mathbb{R} \to 2^{\mathbb{R}}$ by
$$Tx := \begin{cases} \left[-\frac{9x}{2}, -5x\right], & \text{if } \ x \leq 0, \\ \left[-5x, -\frac{9x}{2}\right], & \text{if } \ x > 0. \end{cases}$$
Then $F(T) = \{0\}$. For each $0 \neq x \in \mathbb{R}$,
$$H(Tx, T0)^2 = |-5x - 0|^2 = 25|x - 0|^2 = |x - 0|^2 + 24|x|^2.$$
Clearly, T is not quasi-nonexpansive. We also have
$$d(x, Tx)^2 = \left|x - \left(-\frac{9x}{2}\right)\right|^2 = \left|\frac{11x}{2}\right|^2 = \frac{121}{4}|x|^2.$$
Thus,
$$H(Tx, T0)^2 = |x - 0|^2 + \frac{96}{121}\left(\frac{121}{4}|x|^2\right) = \frac{96}{121}d(x, Tx)^2.$$
Hence, T is demicontractive with a constant $k = \frac{96}{121} \in (0, 1)$.

For a multivalued mapping $T : C \to P(C)$, the best approximation operator P_T is defined by
$$P_T(x) := \{w \in Tx : \|x - w\| = d(x, Tx)\}.$$
Note that $F(T) = F(P_T)$ and P_T satisfies the endpoint condition. In [27], they gave an example for the best approximation operator P_T which is nonexpansive, but T is not necessary to be nonexpansive.

Definition 2.3. Let C be a nonempty closed convex subset of $\mathcal{H}$ and let $T : C \to CB(C)$ be a multivalued mapping. The multivalued mapping $I - T$ is said to be *demiclosed at zero* if for any sequence $\{x_n\}$ in C which converges weakly to $x \in C$ and the sequence $\{\|x_n - y_n\|\}$ converges strongly to 0, where $y_n \in Tx_n$, then $x \in F(T)$.

Let us recall the maximal monotone operator. A multivalued mapping B of $\mathcal{H}$ into $2^{\mathcal{H}}$ is called a maximal monotone operator if B is monotone, i.e.,

$$\langle x-y, z-w\rangle \geq 0 \quad \text{for all} \quad x, y \in \operatorname{dom}(B), z \in Bx, w \in By,$$

where $\operatorname{dom}(B) := \{x \in \mathcal{H} : Bx \neq \emptyset\}$, and the graph $G(B)$ of B,

$$G(B) := \{(x, z) \in \mathcal{H} \times \mathcal{H} : z \in Bx\},$$

is not properly contained in the graph of any other monotone operator, i.e.,

$$(x, z) \in G(B) \;\Leftrightarrow\; \langle x-y, z-w\rangle \geq 0 \quad \text{for all} \quad (y, w) \in G(B).$$

For a maximal monotone operator $B : \mathcal{H} \to 2^{\mathcal{H}}$ and $\lambda > 0$, we define the resolvent of B with parameter λ by

$$J_\lambda^B := (I + \lambda B)^{-1}.$$

It is known [5] that $J_\lambda^B : \mathcal{H} \to \operatorname{dom}(B)$ is single-valued, firmly nonexpansive, i.e., for any $x, y \in \mathcal{H}$,

$$\left\|J_\lambda^B x - J_\lambda^B y\right\|^2 \leq \left\langle J_\lambda^B x - J_\lambda^B y, x - y\right\rangle,$$

this is equivalent to

$$\left\langle J_\lambda^B x - J_\lambda^B y, (J_\lambda^B x - x) - (J_\lambda^B y - y)\right\rangle \leq 0,$$

and $F(J_\lambda^B) = B^{-1}0 = \{x \in \mathcal{H} : 0 \in Bx\}$. Moreover, $I - J_\lambda^B$ is demiclosed at zero.

Let f be a convex function of $\mathcal{H}$ into $(-\infty, \infty]$, then a subdifferential ∂f of f at $x \in \mathcal{H}$ is defined by

$$\partial f(x) := \{y \in \mathcal{H} : f(x) + \langle y, z - x\rangle \leq f(z), \quad \forall z \in \mathcal{H}\}.$$

It was shown [26] that if f is a proper, lower semicontinuous and convex function, then ∂f is a maximal monotone operator.

Example 2.4. (Indicator Function). Let C be a nonempty closed convex subset of $\mathcal{H}$. Define a function $i_C : \mathcal{H} \to (-\infty, \infty]$ of C by

$$i_C(x) = \begin{cases} 0, & \text{if } x \in C, \\ \infty, & \text{if } x \notin C. \end{cases}$$

Thus, i_C is a proper, lower semicontinuous and convex function, and hence ∂i_C is maximal monotone. It is not difficult to show that

$$\partial i_C(x) = \begin{cases} \{y \in \mathcal{H} : \langle y, z - x\rangle \leq 0, \ \forall z \in C\}, & \text{if } x \in C, \\ \emptyset, & \text{if } x \notin C, \end{cases}$$

and $J_\lambda^{\partial i_C} = (I + \lambda \partial i_C)^{-1} = P_C$ for all $\lambda > 0$.

Next, we give some significant tools and facts for proving our main result.

Lemma 2.5. *Let $x, y \in \mathcal{H}, \alpha \in \mathbb{R}$. Then the following inequalities hold on $\mathcal{H}$:*

(i) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle$;
(ii) $\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x - y\|^2$.

Lemma 2.6 [34]. *Let C be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$. Let $T : C \to CB(C)$ be a k-demicontractive multivalued mapping. Then, we have*

(i) *$F(T)$ is closed;*
(ii) *If T satisfies the endpoint condition, then $F(T)$ is convex.*

Lemma 2.7 [35]. *Suppose that $\{t_n\}$ is a sequence of nonnegative real numbers such that*

$$t_{n+1} \leq (1-\alpha_n)t_n + \alpha_n\beta_n + \delta_n, \quad n \in \mathbb{N},$$

where $\{\alpha_n\}, \{\beta_n\}$ and $\{\delta_n\}$ satisfy the following conditions:

(i) $\{\alpha_n\} \subset [0,1]$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(ii) $\limsup_n \beta_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\alpha_n\beta_n| < \infty$;
(iii) $\delta_n \geq 0$ *for all* $n \in \mathbb{N}$, $\sum_{n=1}^{\infty} \delta_n < \infty$.

Then, $\lim_{n\to\infty} t_n = 0$.

Lemma 2.8 [25] *Let $\{\Gamma_n\}$ be a sequence of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ which satisfies $\Gamma_{n_i} < \Gamma_{n_i+1}$ for all $i \in \mathbb{N}$. Also consider the sequence of positive integers $\{\rho(n)\}$ defined by*

$$\rho(n) := \max\{m \leq n : \Gamma_m < \Gamma_{m+1}\}$$

for all $n \geq n_0$ (for some n_0 large enough). Then, $\{\rho(n)\}$ is a nondecreasing sequence such that $\rho(n) \to \infty$ as $n \to \infty$, and it holds that

$$\Gamma_{\rho(n)} \leq \Gamma_{\rho(n)+1}, \quad \Gamma_n \leq \Gamma_{\rho(n)+1}.$$

3. Main Results

In this section, we present an algorithm for finding a common solution of the split null point problem and the fixed point problem for maximal monotone operators and demicontractive multivalued mappings, respectively, and prove a strong convergence result.

We now prove our main theorem.

Theorem 3.1. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two real Hilbert spaces and C be a nonempty closed convex subset of $\mathcal{H}_1$. Let $A : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator. Let $B_1 : \mathcal{H}_1 \to 2^{\mathcal{H}_1}$ and $B_2 : \mathcal{H}_2 \to 2^{\mathcal{H}_2}$ be maximal monotone operators such that $\mathrm{dom}(B_1)$ is included in C, and let $J_\lambda^{B_1}$ and $J_\lambda^{B_2}$ be resolvents of B_1 and B_2, respectively, for $\lambda > 0$. Let $T : C \to CB(C)$ be a k-demicontractive multivalued mapping. Suppose that $I - T$ is demiclosed at zero and T satisfies the endpoint condition. Assume that $\Theta := F(T) \cap \Omega \neq \emptyset$, where $\Omega = \left\{x \in B_1^{-1}0 : Ax \in B_2^{-1}0\right\}$. Suppose that $u \in C$ and $\{x_n\}$ is a sequence generated by $x_1 \in C$ and*

$$\begin{cases} y_n = J_{\lambda_n}^{B_1}(x_n + \gamma A^*(J_{\lambda_n}^{B_2} - I)Ax_n), \\ u_n = (1-\delta)y_n + \delta z_n, \\ x_{n+1} = \alpha_n u + (1-\alpha_n)u_n, \quad n \in \mathbb{N}, \end{cases} \tag{3.1}$$

where $z_n \in Ty_n$, the parameters γ, δ and the sequences $\{\alpha_n\}$, $\{\lambda_n\}$ satisfy the following conditions:

(i) $\gamma \in \left(0, \frac{2}{\|A\|^2}\right)$ *and* $\delta \in (0, 1-k)$*;*
(ii) $\alpha_n \in (0,1)$ *such that* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$*;*
(iii) $\lambda_n \in (0,\infty)$ *such that* $\liminf_{n\to\infty} \lambda_n > 0$*.*

Then, the sequence $\{x_n\}$ *converges strongly to a point* $x^* \in \Theta$*, where* $x^* = P_\Theta u$*.*

Proof. By Lemma 2.6, we have $F(T)$ is closed and convex, and hence Θ is also closed and convex. Let $x^* = P_\Theta u$. By characterization of the metric projection, we get

$$\langle u - x^*, p - x^* \rangle \le 0 \quad \text{for all} \quad p \in \Theta. \tag{3.2}$$

Since $x^* \in \Theta$, we obtain $Tx^* = \{x^*\}$, $J_{\lambda_n}^{B_1} x^* = x^*$ and $J_{\lambda_n}^{B_2}(Ax^*) = Ax^*$. We first show that $\{x_n\}$ is bounded. Since $J_{\lambda_n}^{B_1}$ is nonexpansive and A is a bounded linear operator, we have

$$\begin{aligned}
\|y_n - x^*\|^2 &= \left\| J_{\lambda_n}^{B_1}\left(x_n + \gamma A^*\left(J_{\lambda_n}^{B_2} - I\right)Ax_n\right) - J_{\lambda_n}^{B_1} x^* \right\|^2 \\
&\le \left\| x_n + \gamma A^*\left(J_{\lambda_n}^{B_2} - I\right)Ax_n - x^* \right\|^2 \\
&= \|x_n - x^*\|^2 + \gamma^2 \left\| A^*(J_{\lambda_n}^{B_2} - I)Ax_n \right\|^2 \\
&\qquad + 2\gamma \left\langle x_n - x^*, A^*\left(J_{\lambda_n}^{B_2} - I\right)Ax_n \right\rangle \\
&\le \|x_n - x^*\|^2 + \gamma^2 \|A\|^2 \left\| J_{\lambda_n}^{B_2}(Ax_n) - Ax_n \right\|^2 \\
&\quad\ + 2\gamma \left\langle Ax_n - Ax^*, J_{\lambda_n}^{B_2}(Ax_n) - Ax_n \right\rangle.
\end{aligned} \tag{3.3}$$

Now we take

$$\mathcal{E}_n := 2\gamma \left\langle Ax_n - Ax^*, J_{\lambda_n}^{B_2}(Ax_n) - Ax_n \right\rangle.$$

Since $J_{\lambda_n}^{B_2}$ is firmly nonexpansive, we have

$$\begin{aligned}
\mathcal{E}_n &= 2\gamma \Big\langle Ax_n - Ax^* + \left(J_{\lambda_n}^{B_2}(Ax_n) - Ax_n\right) \\
&\qquad - \left(J_{\lambda_n}^{B_2}(Ax_n) - Ax_{n4}\right), J_{\lambda_n}^{B_2}(Ax_n) - Ax_n \Big\rangle \\
&= 2\gamma \left(\left\langle J_{\lambda_n}^{B_2}(Ax_n) - Ax^*, J_{\lambda_n}^{B_2}(Ax_n) - Ax_n \right\rangle - \left\| J_{\lambda_n}^{B_2}(Ax_n) - Ax_n \right\|^2 \right) \\
&\le -2\gamma \left\| J_{\lambda_n}^{B_2}(Ax_n) - Ax_n \right\|^2.
\end{aligned} \tag{3.4}$$

By (3.3) and (3.4), we obtain that

$$\|y_n - x^*\|^2 \le \|x_n - x^*\|^2 - \gamma(2 - \gamma\|A\|^2) \left\| J_{\lambda_n}^{B_2}(Ax_n) - Ax_n \right\|^2.$$

By Lemma 2.5 (ii) and the demicontractivity of T with the constant k, we have

$$\begin{aligned}
\|u_n - x^*\|^2 &= \|(1-\delta)(y_n - x^*) + \delta(z_n - x^*)\|^2 \\
&= (1-\delta)\|y_n - x^*\|^2 + \delta\|z_n - x^*\|^2 - \delta(1-\delta)\|y_n - z_n\|^2
\end{aligned}$$

$$
\begin{aligned}
&= (1-\delta)\|y_n - x^*\|^2 + \delta d(z_n, Tx^*)^2 - \delta(1-\delta)\|y_n - z_n\|^2 \\
&\leq (1-\delta)\|y_n - x^*\|^2 + \delta H(Ty_n, Tx^*)^2 - \delta(1-\delta)\|y_n - z_n\|^2 \\
&\leq (1-\delta)\|y_n - x^*\|^2 + \delta\left(\|y_n - x^*\|^2 + k\, d(y_n, Ty_n)^2\right) \\
&\quad - \delta(1-\delta)\|y_n - z_n\|^2 \\
&\leq (1-\delta)\|y_n - x^*\|^2 + \delta\|y_n - x^*\|^2 + \delta k\|y_n - z_n\|^2 \\
&\quad - \delta(1-\delta)\|y_n - z_n\|^2 \\
&= \|y_n - x^*\|^2 - \delta(1-k-\delta)\|y_n - z_n\|^2 \\
&\leq \|x_n - x^*\|^2 - \gamma(2-\gamma\|A\|^2)\|J_{\lambda_n}^{B_2}(Ax_n) - Ax_n\|^2 \\
&\quad - \delta(1-k-\delta)\|y_n - z_n\|^2. \qquad (3.5)
\end{aligned}
$$

It follows that

$$\|u_n - x^*\| \leq \|x_n - x^*\|.$$

Thus, we have

$$
\begin{aligned}
\|x_{n+1} - x^*\| &= \|\alpha_n(u - x^*) + (1-\alpha_n)(u_n - x^*)\| \\
&\leq \alpha_n\|u - x^*\| + (1-\alpha_n)\|u_n - x^*\| \\
&\leq \alpha_n\|u - x^*\| + (1-\alpha_n)\|x_n - x^*\| \\
&\leq \max\left\{\|x_n - x^*\|, \|u - x^*\|\right\}.
\end{aligned}
$$

By continuously taking this process, we obtain

$$\|x_n - x^*\| \leq \max\left\{\|x_1 - x^*\|, \|u - x^*\|\right\}$$

for all $n \in \mathbb{N}$. Therefore, $\{x_n\}$ is bounded. This implies that $\{y_n\}$ is also bounded. It follows from (3.5) that

$$
\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|\alpha_n(u - x^*) + (1-\alpha_n)(u_n - x^*)\|^2 \\
&\leq \alpha_n\|u - x^*\|^2 + (1-\alpha_n)\|u_n - x^*\|^2 \\
&\leq \alpha_n\|u - x^*\|^2 + \|x_n - x^*\|^2 \\
&\quad - \gamma(2-\gamma\|A\|^2)\|J_{\lambda_n}^{B_2}(Ax_n) - Ax_n\|^2 \\
&\quad - \delta(1-k-\delta)\|y_n - z_n\|^2. \qquad (3.6)
\end{aligned}
$$

Thus, by (3.6), we get the following two inequalities

$$\gamma(2-\gamma\|A\|^2)\|J_{\lambda_n}^{B_2}(Ax_n) - Ax_n\|^2 \leq \alpha_n\|u - x^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \qquad (3.7)$$

and

$$\delta(1-k-\delta)\|y_n - z_n\|^2 \leq \alpha_n\|u - x^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2. \qquad (3.8)$$

Now, we divide the rest of the proof into two cases.

Case 1 Assume that there exists $n_0 \in \mathbb{N}$ such that $\{\|x_n - x^*\|\}_{n \geq n_0}$ is either nonincreasing or nondecreasing. Since $\{\|x_n - x^*\|\}$ is bounded, then it converges and $\|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \to 0$ as $n \to \infty$. Since $\gamma \in \left(0, \frac{2}{\|A\|^2}\right)$ and $\alpha_n \to 0$ as $n \to \infty$, then by (3.7) we deduce that

$$\lim_{n\to\infty} \left\|J_{\lambda_n}^{B_2}(Ax_n) - Ax_n\right\| = 0. \qquad (3.9)$$

Similarly, in view of (3.8), since $\delta \in (0, 1-k)$ and $\alpha_n \to 0$ as $n \to \infty$, we have

$$\lim_{n\to\infty} \|y_n - z_n\| = 0. \tag{3.10}$$

By (3.3) and (3.4), we also have

$$\left\|x_n + \gamma A^*\left(J_{\lambda_n}^{B_2} - I\right)Ax_n - x^*\right\|^2 \leq \|x_n - x^*\|^2. \tag{3.11}$$

By the firmly nonexpansivity of $J_{\lambda_n}^{B_1}$ and (3.11), we have

$$\begin{aligned}
\|y_n - x^*\|^2 &= \left\|J_{\lambda_n}^{B_1}\left(x_n + \gamma A^*\left(J_{\lambda_n}^{B_2} - I\right)Ax_n\right) - J_{\lambda_n}^{B_1}x^*\right\|^2 \\
&\leq \Big\langle J_{\lambda_n}^{B_1}\left(x_n + \gamma A^*\left(J_{\lambda_n}^{B_2} - I\right)Ax_n\right) \\
&\quad - J_{\lambda_n}^{B_1}x^*, x_n + \gamma A^*\left(J_{\lambda_n}^{B_2} - I\right)Ax_n - x^*\Big\rangle \\
&= \left\langle y_n - x^*, x_n + \gamma A^*\left(J_{\lambda_n}^{B_2} - I\right)Ax_n - x^*\right\rangle \\
&= \frac{1}{2}\Big(\|y_n - x^*\|^2 + \left\|x_n + \gamma A^*\left(J_{\lambda_n}^{B_2} - I\right)Ax_n - x^*\right\|^2 \\
&\quad - \left\|y_n - x_n - \gamma A^*\left(J_{\lambda_n}^{B_2} - I\right)Ax_n\right\|^2\Big) \\
&\leq \frac{1}{2}\left(\|y_n - x^*\|^2 + \|x_n - x^*\|^2 - \left\|y_n - x_n - \gamma A^*\left(J_{\lambda_n}^{B_2} - I\right)Ax_n\right\|^2\right) \\
&= \frac{1}{2}\Big(\|y_n - x^*\|^2 + \|x_n - x^*\|^2 - \|y_n - x_n\|^2 - \gamma^2\|A^*\left(J_{\lambda_n}^{B_2} - I\right)Ax_n\|^2 \\
&\quad + 2\gamma\left\langle y_n - x_n, A^*\left(J_{\lambda_n}^{B_2} - I\right)Ax_n\right\rangle\Big) \\
&\leq \frac{1}{2}\Big(\|y_n - x^*\|^2 + \|x_n - x^*\|^2 - \|y_n - x_n\|^2 - \gamma^2\left\|A^*\left(J_{\lambda_n}^{B_2} - I\right)Ax_n\right\|^2 \\
&\quad + 2\gamma\|y_n - x_n\|\left\|A^*\left(J_{\lambda_n}^{B_2} - I\right)Ax_n\right\|\Big),
\end{aligned}$$

which implies that

$$\|y_n - x^*\|^2 \leq \|x_n - x^*\|^2 - \|y_n - x_n\|^2 + 2\gamma\|y_n - x_n\|\left\|A^*\left(J_{\lambda_n}^{B_2} - I\right)Ax_n\right\|. \tag{3.12}$$

Since T is k-demicontractive, then it follows from (3.12) that

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leq \alpha_n\|u - x^*\|^2 + (1-\alpha_n)\|u_n - x^*\|^2 \\
&\leq \alpha_n\|u - x^*\|^2 + (1-\alpha_n)\left((1-\delta)\|y_n - x^*\|^2 + \delta\|z_n - x^*\|^2\right) \\
&\leq \alpha_n\|u - x^*\|^2 + (1-\delta)\|y_n - x^*\|^2 + \delta\, d(z_n, Tx^*)^2 \\
&\leq \alpha_n\|u - x^*\|^2 + (1-\delta)\|y_n - x^*\|^2 + \delta H(Ty_n, Tx^*)^2 \\
&\leq \alpha_n\|u - x^*\|^2 + (1-\delta)\|y_n - x^*\|^2 + \delta\|y_n - x^*\|^2 \\
&\quad + \delta k\, d(y_n, Ty_n)^2 \\
&\leq \alpha_n\|u - x^*\|^2 + \|y_n - x^*\|^2 + \delta k\|y_n - z_n\|^2 \\
&\leq \alpha_n\|u - x^*\|^2 + \|x_n - x^*\|^2 - \|y_n - x_n\|^2 \\
&\quad + 2\gamma\|y_n - x_n\|\left\|A^*\left(J_{\lambda_n}^{B_2} - I\right)Ax_n\right\| + \delta k\|y_n - z_n\|^2.
\end{aligned} \tag{3.13}$$

By (3.9), (3.10) and (3.13), we deduce that

$$\begin{aligned}\|y_n - x_n\|^2 \le \alpha_n \|u - x^*\|^2 + 2\gamma \|y_n - x_n\| \left\| A^* \left(J_{\lambda_n}^{B_2} - I \right) A x_n \right\| \\ + \delta k \|y_n - z_n\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \to 0\end{aligned}$$

as $n \to \infty$, which implies that $\|y_n - x_n\| \to 0$ as $n \to \infty$. We next show that

$$\limsup_{n\to\infty} \langle u - x^*, x_n - x^* \rangle \le 0.$$

To show this, let $\{x_{n_j}\}$ be a subsequence of $\{x_n\}$ such that

$$\lim_{j\to\infty} \langle u - x^*, x_{n_j} - x^* \rangle = \limsup_{n\to\infty} \langle u - x^*, x_n - x^* \rangle.$$

Since $\{x_{n_j}\}$ is bounded, there exists a subsequence $\{x_{n_{j_k}}\}$ of $\{x_{n_j}\}$ and $p \in \mathcal{H}_1$ such that $x_{n_{j_k}} \rightharpoonup p$. Without loss of generality, we can assume that $x_{n_j} \rightharpoonup p$. Since A is a bounded linear operator, we have $\langle z, Ax_{n_j} - Ap \rangle = \langle A^* z, x_{n_j} - p \rangle \to 0$ as $j \to \infty$, for all $z \in \mathcal{H}_2$, this implies that $Ax_{n_j} \rightharpoonup Ap$. From (3.9) and by the demiclosedness of $I - J_{\lambda_n}^{B_2}$ at zero, we get $Ap \in F(J_{\lambda_n}^{B_2}) = B_2^{-1}0$. Since $x_{n_j} \rightharpoonup p$ and $\|y_n - x_n\| \to 0$ as $n \to \infty$, we have $y_{n_j} \rightharpoonup p$. From (3.10) and by the demiclosedness of $I - T$ at zero, we obtain $p \in F(T)$. Now let us show that $p \in B_1^{-1}0$. From $y_n = J_{\lambda_n}^{B_1}(x_n + \gamma A^*(J_{\lambda_n}^{B_2} - I)Ax_n)$, then we can easily prove that

$$\frac{1}{\lambda_n} \left(x_n - y_n + \gamma A^* \left(J_{\lambda_n}^{B_2} - I \right) A x_n \right) \in B_1 y_n.$$

By the monotonicity of B_1, we have

$$\left\langle y_n - v, \frac{1}{\lambda_n} \left(x_n - y_n + \gamma A^* \left(J_{\lambda_n}^{B_2} - I \right) A x_n \right) - w \right\rangle \ge 0$$

for all $(v, w) \in G(B_1)$. Thus, we also have

$$\left\langle y_{n_j} - v, \frac{1}{\lambda_{n_j}} \left(x_{n_j} - y_{n_j} + \gamma A^* \left(J_{\lambda_{n_j}}^{B_2} - I \right) A x_{n_j} \right) - w \right\rangle \ge 0 \tag{3.14}$$

for all $(v, w) \in G(B_1)$. Since $y_{n_j} \rightharpoonup p$, $\|x_{n_j} - y_{n_j}\| \to 0$ and $\|(J_{\lambda_{n_j}}^{B_2} - I)Ax_{n_j}\| \to 0$ as $j \to \infty$, then by taking the limit as $j \to \infty$ in (3.14) yields

$$\langle p - v, -w \rangle \ge 0$$

for all $(v, w) \in G(B_1)$. By the maximal monotonicity of B_1, we get $0 \in B_1 p$, i.e., $p \in B_1^{-1}0$. Therefore, $p \in \Theta$. Since x^* satisfies the inequality (3.2), we have

$$\limsup_{n\to\infty} \langle u - x^*, x_n - x^* \rangle = \lim_{j\to\infty} \langle u - x^*, x_{n_j} - x^* \rangle = \langle u - x^*, p - x^* \rangle \le 0.$$

By using Lemma 2.5 (i), we deduce that

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &= \|(1 - \alpha_n)(u_n - x^*) + \alpha_n (u - x^*)\|^2 \\ &\le (1 - \alpha_n)^2 \|u_n - x^*\|^2 + 2\alpha_n \langle u - x^*, x_{n+1} - x^* \rangle \\ &\le (1 - \alpha_n) \|x_n - x^*\|^2 + 2\alpha_n \langle u - x^*, x_{n+1} - x^* \rangle.\end{aligned}$$

Hence, by Lemma 2.7, we can conclude that $x_n \to x^*$ as $n \to \infty$.

Case 2 Suppose that $\{\|x_n - x^*\|\}$ is not a monotone sequence. Then, there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $\|x_{n_i} - x^*\| < \|x_{n_i+1} - x^*\|$ for all $i \in \mathbb{N}$. We now define a positive integer sequence $\{\rho(n)\}$ by

$$\rho(n) := \max\{m \leq n : \|x_m - x^*\| < \|x_{m+1} - x^*\|\}$$

for all $n \geq n_0$ (for some n_0 large enough). By Lemma 2.8, we have $\{\rho(n)\}$ is a nondecreasing sequence such that $\rho(n) \to \infty$ as $n \to \infty$ and

$$\|x_{\rho(n)} - x^*\|^2 - \|x_{\rho(n)+1} - x^*\|^2 \leq 0$$

for all $n \geq n_0$. From (3.7), we obtain that

$$\lim_{n\to\infty} \left\| \left(J^{B_2}_{\lambda_{\rho(n)}} - I \right) Ax_{\rho(n)} \right\| = 0. \tag{3.15}$$

From (3.8), we have

$$\lim_{n\to\infty} \|y_{\rho(n)} - z_{\rho(n)}\| = 0. \tag{3.16}$$

By (3.15), (3.16) and by the same proof as in case 1, we obtain that

$$\limsup_{n\to\infty} \langle u - x^*, x_{\rho(n)} - x^* \rangle \leq 0.$$

By the same computation as in case 1, we deduce that

$$\|x_{\rho(n)+1} - x^*\|^2 \leq (1 - \alpha_{\rho(n)})\|x_{\rho(n)} - x^*\|^2 + 2\alpha_{\rho(n)}\langle u - x^*, x_{\rho(n)+1} - x^* \rangle.$$

By applying Lemma 2.7 again, we obtain that $\|x_{\rho(n)} - x^*\| \to 0$ as $n \to \infty$. It follows from Lemma 2.8 that

$$0 \leq \|x_n - x^*\| \leq \|x_{\rho(n)+1} - x^*\| \to 0$$

as $n \to \infty$. Hence $\{x_n\}$ converges strongly to x^*. This completes the proof. □

By properties of the best approximation operator, we have the following corollary.

Corollary 3.2. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two real Hilbert spaces and C be a nonempty closed convex subset of $\mathcal{H}_1$. Let $A : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator. Let $B_1 : \mathcal{H}_1 \to 2^{\mathcal{H}_1}$ and $B_2 : \mathcal{H}_2 \to 2^{\mathcal{H}_2}$ be maximal monotone operators such that* $\mathrm{dom}(B_1)$ *is included in C, and let $J^{B_1}_\lambda$ and $J^{B_2}_\lambda$ be resolvents of B_1 and B_2, respectively, for $\lambda > 0$. Let $T : C \to P(C)$ be a multivalued mapping such that P_T is k-demicontractive. Assume that $I - P_T$ is demiclosed at zero and $\Theta := F(T) \cap \Omega \neq \emptyset$, where $\Omega = \left\{x \in B_1^{-1}0 : Ax \in B_2^{-1}0\right\}$. Suppose that $u \in C$ and $\{x_n\}$ is a sequence generated by $x_1 \in C$ and*

$$\begin{cases} y_n = J^{B_1}_{\lambda_n}(x_n + \gamma A^*(J^{B_2}_{\lambda_n} - I)Ax_n), \\ u_n = (1-\delta)y_n + \delta z_n, \\ x_{n+1} = \alpha_n u + (1-\alpha_n)u_n, \quad n \in \mathbb{N}, \end{cases} \tag{3.17}$$

where $z_n \in P_T(y_n)$, the parameters γ, δ and the sequences $\{\alpha_n\}$, $\{\lambda_n\}$ satisfy conditions (i)–(iii) *in Theorem 3.1. Then, the sequence $\{x_n\}$ defined by* (3.17) *converges strongly to $x^* \in \Theta$, where $x^* = P_\Theta u$.*

Proof. Since P_T satisfies the end point condition and $F(T) = F(P_T)$, then the result is obtained directly by Theorem 3.1. □

We also obtain a result for solving the fixed point problem for demicontractive multivalued mappings as follows.

Theorem 3.3. *Let $\mathcal{H}$ be a real Hilbert space and C be a nonempty closed convex subset of $\mathcal{H}$. Let $T : C \to CB(C)$ be a k-demicontractive multivalued mapping. Assume that $I - T$ is demiclosed at zero and T satisfies the endpoint condition. Let $u \in C$ and $\{x_n\}$ be a sequence generated by $x_1 \in C$, and*

$$\begin{cases} u_n = (1-\delta)x_n + \delta z_n, \\ x_{n+1} = \alpha_n u + (1-\alpha_n)u_n, \quad n \in \mathbb{N}, \end{cases} \tag{3.18}$$

where $z_n \in Tx_n$, $\delta \in (0, 1-k)$ and $\alpha_n \in (0,1)$ such that $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then, the sequence $\{x_n\}$ defined by (3.18) converges strongly to $x^* \in F(T)$, where $x^* = P_{F(T)}u$.

Proof. Set $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$, $A := I$, $B_1 := \partial i_C$ and $B_2 := 0$, where i_C is the indicator function of C and 0 is a zero operator. Then B_1 and B_2 are maximal monotone such that $\operatorname{dom}(B_1) = C$, $J_\lambda^{B_1} = P_C$ and $J_\lambda^{B_2} = I$ for $\lambda > 0$. We also have $\Omega := \left\{x \in B_1^{-1}0 : Ax \in B_2^{-1}0\right\} = C$, then $\Theta := F(T) \cap \Omega = F(T)$. So the result is obtained directly by Theorem 3.1. □

When we take $C = \mathcal{H}_1$ and $T = I$ is a single-valued mapping in Theorem 3.1, then the result of Byrne et al. [6] for solving the SNPP (1.3) for maximal monotone operators is a consequence of our main result.

Corollary 3.4 [6] *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two real Hilbert spaces, and let $A : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator. Let $B_1 : \mathcal{H}_1 \to 2^{\mathcal{H}_1}$ and $B_2 : \mathcal{H}_2 \to 2^{\mathcal{H}_2}$ be two maximal monotone operators, and let $J_\lambda^{B_1}$ and $J_\lambda^{B_2}$ be resolvents of B_1 and B_2, respectively, for $\lambda > 0$. Assume that $\Omega = \left\{x \in B_1^{-1}0 : Ax \in B_2^{-1}0\right\} \neq \emptyset$. Suppose that $u \in \mathcal{H}_1$ and $\{x_n\}$ is a sequence generated by $x_1 \in \mathcal{H}_1$ and*

$$x_{n+1} = \alpha_n u + (1-\alpha_n) J_\lambda^{B_1}\left(x_n + \gamma A^*\left(J_\lambda^{B_2} - I\right)Ax_n\right), \quad n \in \mathbb{N}, \tag{3.19}$$

where $\gamma \in \left(0, \frac{2}{\|A\|^2}\right)$ and $\alpha_n \in (0,1)$ such that $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$ Then, the sequence $\{x_n\}$ defined by (3.19) converges strongly to a point $x^* \in \Omega$.

4. Other Split Problems Deduced from Main Problem

In this section, we reduce our main problem to the following split problems:

4.1. The Split Feasibility Problem

Let C and Q are nonempty closed convex subsets of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, and let $A : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator. The *split feasibility problem* (SFP) which was introduced by Censor and Elfving [8] for modeling inverse problems is formulated as finding a point

$$x^* \in C \quad \text{such that} \quad Ax^* \in Q. \tag{4.1}$$

Byrne [3] was the first who introduced the so-called CQ algorithm which does not involve matrix inverses in finite-dimensional spaces for solving the SFP (4.1).

Now, we obtain a result for finding a common solution of the split feasibility problem and the fixed point problem for demicontractive multivalued mappings as follows.

Theorem 4.1. *Let C and Q be nonempty closed convex subsets of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, and let $A : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator. Let $T : C \to CB(C)$ be a k-demicontractive multivalued mapping. Suppose that $I - T$ is demiclosed at zero and T satisfies the endpoint condition. Assume that $\Theta := F(T) \cap A^{-1}(Q) \neq \emptyset$. Suppose that $u \in C$ and $\{x_n\}$ is a sequence generated by $x_1 \in C$ and*

$$\begin{cases} y_n = P_C(x_n + \gamma A^*(P_Q - I)Ax_n), \\ u_n = (1-\delta)y_n + \delta z_n, \\ x_{n+1} = \alpha_n u + (1-\alpha_n)u_n, \quad n \in \mathbb{N}, \end{cases} \tag{4.2}$$

where $z_n \in Ty_n$, the parameters γ, δ and the sequence $\{\alpha_n\}$ satisfy the following conditions:

(i) $\gamma \in \left(0, \frac{2}{\|A\|^2}\right)$ *and* $\delta \in (0, 1-k)$*;*
(ii) $\alpha_n \in (0,1)$ *such that* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$*.*

Then $\{x_n\}$ defined by (4.2) converges strongly to a point $x^* \in \Theta$, where $x^* = P_\Theta u$.

Proof. Set $B_1 := \partial i_C$ and $B_2 := \partial i_Q$. Then B_1 and B_2 are maximal monotone such that $J_\lambda^{B_1} = P_C$ and $J_\lambda^{B_2} = P_Q$ for $\lambda > 0$. We also have $B_1^{-1}0 = C$ and $B_2^{-1}0 = Q$. Hence the result is obtained directly by Theorem 3.1. □

4.2. The Split Equilibrium Problem

Let K be a nonempty closed convex subset of $\mathcal{H}$, and let $h : K \times K \to \mathbb{R}$ be a bifunction. Recall that the *equilibrium problem* is to find a point $x^* \in K$ such that

$$h(x^*, y) \geq 0 \quad \text{for all} \quad y \in K. \tag{4.3}$$

The solution set of Problem (4.3) is denoted by $EP(h)$. To study the equilibrium problem, we assume that the bifunction $h : K \times K \to \mathbb{R}$ satisfies the following assumptions:

(A1) $h(x,x) = 0$ for all $x \in K$;
(A2) h is monotone, i.e., $h(x,y) + h(y,x) \leq 0$ for all $x, y \in K$;
(A3) For each $x, y, z \in K$, $\limsup_{t\to o} h(tz + (1-t)x, y) \leq h(x,y)$;
(A4) For each $x \in K$, the function $y \mapsto h(x,y)$ is convex and lower semicontinuous.

Kazmi and Rizvi [23] introduced and studied the *split equilibrium problem* (SEP): Let K_1 and K_2 be nonempty closed convex subsets of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Let $h_1 : K_1 \times K_1 \to \mathbb{R}$ and $h_2 : K_2 \times K_2 \to \mathbb{R}$ be two bifunctions, and $A:\mathcal{H}_1\to\mathcal{H}_2$ a bounded linear operator, the problem is to find a point $x^*\in K_1$ such that

$$x^*\in EP(h_1)\quad\text{and}\quad Ax^*\in EP(h_2). \tag{4.4}$$

We also need the following lemma and theorem.

Lemma 4.2. [14] *Let C be a nonempty closed convex subset of $\mathcal{H}$ and let h be a bifunction from $C\times C$ to $\mathbb{R}$ satisfying (A1)-(A4). For any $r>0$ and $x\in\mathcal{H}$, define $T_r^h:\mathcal{H}\to C$ by*

$$T_r^h(x)=\left\{y\in C: h(y,z)+\frac{1}{r}\langle z-y,y-x\rangle\geq 0,\ \forall z\in C\right\}.$$

Then, the following hold:

(i) *T_r^h is nonempty and single-valued;*
(ii) *T_r^h is firmly nonexpansive;*
(iii) *$F(T_r^h)=EP(h)$;*
(iv) *$EP(h)$ is closed and convex.*

Theorem 4.3 [32]. *Let C be a nonempty closed convex subset of $\mathcal{H}$ and let h be a bifunction from $C\times C$ to $\mathbb{R}$ satisfying (A1)-(A4). Define a multivalued mapping $A_h:\mathcal{H}\to 2^{\mathcal{H}}$ by*

$$A_h(x)=\begin{cases}\{y\in\mathcal{H}: h(x,z)\geq\langle z-x,y\rangle,\ \forall z\in C\}, & \text{if } x\in C,\\ \emptyset, & \text{if } x\notin C.\end{cases}$$

Then, the following hold:

(i) *A_h is maximal monotone;*
(ii) *$EP(h)=A_h^{-1}0$;*
(iii) *$T_r^h=(I+rA_h)^{-1}$ for $r>0$, i.e., T_r^h is the resolvent of A_h.*

Recently, Suantai et al. [28] studied a problem of finding a common solution of the split equilibrium problem and the fixed point problem for $\frac{1}{2}$-nonspreading multivalued mappings, and proved a weak convergence theorem.

Now, we obtain a strong convergence result for solving the split equilibrium problem and the fixed point problem for demicontractive multivalued mappings as follows.

Theorem 4.4. *Let C and Q be nonempty closed convex subsets of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Let $A:\mathcal{H}_1\to\mathcal{H}_2$ be a bounded linear operator. Let $h_1:C\times C\to\mathbb{R}$ and $h_2:Q\times Q\to\mathbb{R}$ be bifunctions satisfying (A1)-(A4), and let $T_r^{h_1}$ and $T_r^{h_2}$ be resolvents of A_{h_1} and A_{h_2} in Theorem 4.3, respectively, for $r>0$. Let $T:C\to CB(C)$ be a k-demicontractive multivalued mapping. Suppose that $I-T$ is demiclosed at zero and T satisfies the endpoint condition. Assume that $\Theta:=F(T)\cap\Omega\neq\emptyset$, where $\Omega=\{x\in EP(h_1): Ax\in EP(h_2)\}$. Suppose that $u\in C$ and $\{x_n\}$ is a sequence generated by $x_1\in C$ and*

$$\begin{cases}y_n=T_{r_n}^{h_1}\left(x_n+\gamma A^*(T_{r_n}^{h_2}-I)Ax_n\right),\\ u_n=(1-\delta)y_n+\delta z_n,\\ x_{n+1}=\alpha_n u+(1-\alpha_n)u_n,\quad n\in\mathbb{N},\end{cases} \tag{4.5}$$

where $z_n \in Ty_n$*, the parameters* γ*,* δ *and the sequences* $\{\alpha_n\}$*,* $\{r_n\}$ *satisfy the following conditions:*

(i) $\gamma \in \left(0, \frac{2}{\|A\|^2}\right)$ *and* $\delta \in (0, 1-k)$*;*
(ii) $\alpha_n \in (0,1)$ *such that* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$*;*
(iii) $r_n \in (0,\infty)$ *such that* $\liminf_{n\to\infty} r_n > 0$*.*

Then, $\{x_n\}$ *defined by* (4.5) *converges strongly to a point* $x^* \in \Theta$*, where* $x^* = P_\Theta u$*.*

Proof. Set $B_1 := A_{h_1}$ and $B_2 := A_{h_2}$. By Theorem 4.3, we know that B_1 and B_2 are maximal monotone, $EP(h_1) = B_1^{-1}0, EP(h_2) = B_2^{-1}0, T_{r_n}^{h_1} = J_{r_n}^{B_1}$ and $T_{r_n}^{h_2} = J_{r_n}^{B_2}$. Then the result is obtained directly by Theorem 3.2. □

4.3. The Split Minimization Problem

Let $f : \mathcal{H} \to (-\infty, \infty]$ be a function, we define the set of minimizer of f by

$$\operatorname{Argmin} f := \{x \in \mathcal{H} : f(x) \le f(z), \quad \forall z \in \mathcal{H}\}.$$

If f is a proper, lower semicontinuous and convex function, then ∂f is a maximal monotone operator. Moreover,

$$x \in (\partial f)^{-1}0 \ \Leftrightarrow\ 0 \in \partial f(x) \ \Leftrightarrow\ f(x) \le f(z),\ \forall z \in \mathcal{H} \ \Leftrightarrow\ x \in \operatorname{Argmin} f,$$

i.e., $\operatorname{Argmin} f = (\partial f)^{-1}0$. In this case, the resolvent of ∂f is called the proximity operator of f (see [15]).

Let $f_1 : \mathcal{H}_1 \to (-\infty, \infty]$ and $f_2 : \mathcal{H}_2 \to (-\infty, \infty]$ be two proper, lower semicontinuous and convex functions, and let $A : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator. The *split minimization problem* (SMP) is to find a point $x^* \in \mathcal{H}_1$ such that

$$x^* \in \operatorname{Argmin} f_1 \quad \text{and} \quad Ax^* \in \operatorname{Argmin} f_2. \tag{4.6}$$

The following result is immediately obtained when we take $B_1 = \partial f_1$ and $B_2 = \partial f_2$ in Theorem 3.2.

Theorem 4.5. *Let* $f_1 : \mathcal{H}_1 \to (-\infty, \infty]$ *and* $f_2 : \mathcal{H}_2 \to (-\infty, \infty]$ *be two proper, lower semicontinuous and convex functions, and let* $A : \mathcal{H}_1 \to \mathcal{H}_2$ *be a bounded linear operator. Let* $T : \mathcal{H}_1 \to CB(\mathcal{H}_1)$ *be a* k*-demicontractive multivalued mapping. Suppose that* $I - T$ *is demiclosed at zero and* T *satisfies the endpoint condition. Assume that* $\Theta := F(T) \cap \Omega \ne \emptyset$*, where* $\Omega = \{x \in \operatorname{Argmin} f_1 : Ax \in \operatorname{Argmin} f_2\}$*. Suppose that* $u \in \mathcal{H}_1$ *and* $\{x_n\}$ *is a sequence generated by* $x_1 \in \mathcal{H}_1$ *and*

$$\begin{cases} y_n = J_{\lambda_n}^{\partial f_1}\left(x_n + \gamma A^*\left(J_{\lambda_n}^{\partial f_2} - I\right)Ax_n\right), \\ u_n = (1-\delta)y_n + \delta z_n, \\ x_{n+1} = \alpha_n u + (1-\alpha_n)u_n, \quad n \in \mathbb{N}, \end{cases} \tag{4.7}$$

where $z_n \in Ty_n$*, the parameters* γ*,* δ *and the sequences* $\{\alpha_n\}$*,* $\{\lambda_n\}$ *satisfy the following conditions:*

(i) $\gamma \in \left(0, \frac{2}{\|A\|^2}\right)$ *and* $\delta \in (0, 1-k)$*;*
(ii) $\alpha_n \in (0,1)$ *such that* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$*;*
(iii) $\lambda_n \in (0,\infty)$ *such that* $\liminf_{n\to\infty} \lambda_n > 0$*.*

Then, $\{x_n\}$ *defined by* (4.7) *converges strongly to a point* $x^* \in \Theta$*, where* $x^* = P_\Theta u$*.*

5. A Numerical Example

In this section, we provide a numerical result for supporting our main.

Example 5.1. Let $\mathcal{H}_1 = \mathbb{R}$ and $\mathcal{H}_2 = \mathbb{R}^3$ with the usual norms. Define a multivalued mapping $T : \mathbb{R} \to CB(\mathbb{R})$ by

$$Tx := \begin{cases} \left[-\frac{9x}{2}, -5x\right], & \text{if } x \leq 0, \\ \left[-5x, -\frac{9x}{2}\right], & \text{if } x > 0. \end{cases}$$

By Example 2.2, it was shown that T is demicontractive with a constant $k = \frac{96}{121}$. Let $h : [-9, 3] \times [-9, 3] \to \mathbb{R}$ be a bifunction defined by $h(x, y) = y^2 + xy - 2x^2$ and let $f : \mathbb{R}^3 \to (-\infty, \infty]$ be a function defined by $f(z) = \frac{1}{2}\|Pz\|^2$, where $P = \begin{bmatrix} -4 & 2 & 7 \\ 1 & -5 & 8 \end{bmatrix}$. We define two maximal monotone operators $B_1 : \mathbb{R} \to 2^{\mathbb{R}}$ and $B_2 : \mathbb{R}^3 \to 2^{\mathbb{R}^3}$ by $B_1 := A_h$ (see Theorem 4.3) and $B_2 := \partial f$. By [29] and [15], we can write the explicit resolvents of B_1 and B_2 in the following forms:

$$J_1^{B_1} x = \frac{x}{4} \quad \text{and} \quad J_1^{B_2} z = \left(P^T P + I\right)^{-1} z$$

for all $x \in \mathbb{R}$ and $z \in \mathbb{R}^3$. Define a bounded linear operator $A : \mathbb{R} \to \mathbb{R}^3$ by $Ax := (2x, -5x, 3x)$. Let $\Theta := F(T) \cap \Omega$, where $\Omega = \left\{x \in B_1^{-1}0 : Ax \in B_2^{-1}0\right\}$. Take $\alpha_n = \frac{1}{9500n}$, $\lambda_n = 1$, $\delta = \frac{1}{8}$, $\gamma = \frac{1}{\|A\|^2}$, $u = \frac{1}{2}$ and choose $z_n = -5y_n$. Thus, Algorithm (3.1) in our main result becomes

$$x_{n+1} = \frac{1}{19000n} + \frac{1}{16}\left(1 - \frac{1}{9500n}\right) \times \left\{x_n + \frac{1}{\|A\|^2} A^* \left[\left(P^T P + I\right)^{-1} (Ax_n) - Ax_n\right]\right\}. \tag{5.1}$$

We first start with the initial point $x_1 = 14$ and the stopping criterion for our testing process is set as: $|x_n - x_{n-1}| < 10^{-7}$. Now, a convergence of Algorithm (5.1) is shown by Table 1 and it converges to $0 \in \Theta$.

6. Concluding Remarks

Our problem is the problem of finding a common solution of the split null point problem and the fixed point problem for multivalued mappings in Hilbert spaces. We focus on the class of maximal monotone operators for the split null point problem and the class of demicontractive multivalued mappings for the fixed point problem. We present an algorithm based on Halpern's method for solving the problem and also obtain some sufficient conditions for the strong convergence of the proposed algorithm. The result of Byrne et al. [6, Theorem 4.5] for solving the split null point problem and

Table 1. Numerical experiment of Algorithm (5.1)

n	x_n	$\|x_n - x_{n-1}\|$
2	0.0250010	13.974999
3	0.0000709	0.0249301
4	0.0000177	0.0000532
5	0.0000132	0.0000045
⋮	⋮	⋮
15	0.0000038	0.0000003
⋮	⋮	⋮
24	0.0000023	0.0000001
25	0.0000022	0.00000009

the result for solving the fixed point problems for demicontractive multivalued mappings are consequences of our main result. Moreover, our problem of main result can be applied to other split problems, i.e., the split feasibility problem, the split equilibrium problem, and the split minimization problem.

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