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An Elastic Frictional Contact Problem with Unilateral Constraint

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Abstract. We consider a mathematical model which describes the equilibrium of an elastic body in contact with two obstacles. We derive its weak formulation which is in a form of an elliptic quasi-variational inequality for the displacement field. Then, under a smallness assumption, we establish the existence of a unique weak solution to the problem. We also study the dependence of the solution with respect to the data and prove a convergence result. Finally, we consider an optimization problem associated with the contact model for which we prove the existence of a minimizer and a convergence result, as well.

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1. Introduction

Processes of contact between deformable bodies arise in industry and everyday life. They mathematical modeling leads to strongly elliptic or evolutionary nonlinear boundary value problems, and therefore, their study is carried out using arguments of nonsmooth analysis. Reference in the field is the books [6–10,16,17] and, more recently, [5,14,18]. There, various models of contact with elastic, viscoelastic, and viscoplastic materials have been considered, associated with a large number of contact boundary condition. Existence and uniqueness results have been proved, using arguments of variational and hemivariational inequalities. In part of these references, the numerical analysis of the models was also provided, together with error estimates and convergence results. Moreover, numerical simulations which represent an evidence of the theoretical results have been presented, together with their mechanical interpretations. Results on optimal control for various contact problems with elastic materials could be found in [1,5,11–13,20,21] and the references therein.

In [3], we considered a mathematical model which describes the frictionless contact between an elastic body and a foundation made of a rigid material, covered by a rigid-plastic layer of thickness g and yield limit F. The variational formulation of the model led to an elliptic variational inequality with unilateral constraints for the displacement field, governed by the parameter g. We provided the unique solvability of the model, the continuous dependence of the solution with respect to the data, and we discussed related optimal control problems.

The current paper represents a continuation of [3]. The first novelty arises in the model we use, which is frictional. It describes the equilibrium of an elastic body acted upon by body forces of density $\boldsymbol{f}_0$ and tractions of density $\boldsymbol{f}_2$, in contact with two obstacles. The first obstacle is made of a rigid material, covered by a rigid-plastic layer of thickness g and yield limit F. The second one is made of a rigid material, covered by a deformable layer of thickness k. The contact between the body and the first obstacle is frictionless, while the contact with the second one is frictional and is modeled with a version of Coulomb's law of dry friction. Our aim is to provide a clear and rigourous statement of the problem, to provide its variational analysis and to study-related optimization problems, in which the control could be $\boldsymbol{f}_0$, $\boldsymbol{f}_2$, F, g, and k. Note that, in contrast with the contact model in [3], the model considered in this current paper leads to an elliptic quasi-variational inequality for the displacement field in which the set of constraints is governed by two parameters, g and k. This gives rise to additional mathematical difficulties in its analysis. The approach used to overcome these difficulties represents the second trait of novelty of this paper.

This paper is structured as follows. In Sect. 2, we introduce the preliminary material we need. It includes a survey of the basic properties on the functional spaces we use as well as an existence and uniqueness result for elliptic quasi-variational inequalities. In Sect. 3, we introduce the contact model, and then, we list the assumptions on the data and derive its variational formulation. In Sect. 4, we state and prove the unique weak solvability of the problem, Theorem 4.1. In Sect. 5, we study the dependence of the solution with respect to the data and prove a convergence result, Theorem 5.1. The proof is based on arguments of compactness, monotonicity, and lower semicontinuity. Finally, in Sect. 6, we deal with an optimization problem related to the contact model. We start with an abstract result, Theorem 6.1; then, we show its applicability in two relevant particular cases, for which we provide the corresponding mechanical interpretations.

2. Preliminaries

Basic notation Everywhere in this paper, $d \in \{1, 2, 3\}$ and $\mathbb{S}^d$ represent the space of second-order symmetric tensors on $\mathbb{R}^d$ or, equivalently, the space of symmetric matrices of order d. The zero element of the spaces $\mathbb{R}^d$ and $\mathbb{S}^d$ will be denoted $\mathbf{0}$. The inner product and norm on $\mathbb{R}^d$ and $\mathbb{S}^d$ are defined by

$$\boldsymbol{u}\cdot\boldsymbol{v}=u_iv_i,\quad \|\boldsymbol{v}\|=(\boldsymbol{v}\cdot\boldsymbol{v})^{\frac{1}{2}}\qquad \forall\,\boldsymbol{u},\boldsymbol{v}\in\mathbb{R}^d,$$
$$\boldsymbol{\sigma}\cdot\boldsymbol{\tau}=\sigma_{ij}\tau_{ij},\quad \|\boldsymbol{\tau}\|=(\boldsymbol{\tau}\cdot\boldsymbol{\tau})^{\frac{1}{2}}\qquad \forall\,\boldsymbol{\sigma},\boldsymbol{\tau}\in\mathbb{S}^d,$$

where the indices i, j run between 1 and d, and unless stated otherwise, the summation convention over repeated indices is used.
Function spaces Let Ω be a bounded domain of $\mathbb{R}^d$ with a Lipschitz continuous boundary Γ and let Γ_1, Γ_1, Γ_3, Γ_4 be a partition of Γ into four measurable parts, such that meas$\,(\Gamma_1)>0$. We use $\boldsymbol{x}=(x_i)$ for the generic point in $\Omega\cup\Gamma$. An index that follows a comma will represent the partial derivative with respect to the corresponding component of the spatial variable $\boldsymbol{x}\in\Omega\cup\Gamma$, i.e., $f_i=\partial f/\partial x_i$. Moreover, $\boldsymbol{\nu}=(\nu_i)$ denotes the outward unit normal at Γ.

We use the standard notation for Sobolev and Lebesgue spaces associated with Ω and Γ. In particular, we use the spaces $L^2(\Omega)^d$, $L^2(\Gamma_2)^d$, $L^2(\Gamma_3)$, and $H^1(\Omega)^d$ endowed with their canonical inner products and associated norms. Moreover, we recall that for an element $\boldsymbol{v}\in H^1(\Omega)^d$, we sometimes write $\boldsymbol{v}$ for the trace $\gamma\boldsymbol{v}\in L^2(\Gamma)^d$ of $\boldsymbol{v}$ to Γ. In addition, we consider the following spaces:

$$V=\{\boldsymbol{v}\in H^1(\Omega)^d:\ \boldsymbol{v}=\boldsymbol{0}\text{ on }\Gamma_1\},\quad Q=\{\boldsymbol{\sigma}=(\sigma_{ij}):\ \sigma_{ij}=\sigma_{ji}\in L^2(\Omega)\}.$$

The spaces V and Q are real Hilbert spaces endowed with the canonical inner products given by

$$(\boldsymbol{u},\boldsymbol{v})_V=\int_\Omega\boldsymbol{\varepsilon}(\boldsymbol{u})\cdot\boldsymbol{\varepsilon}(\boldsymbol{v})\,\mathrm{d}x,\qquad (\boldsymbol{\sigma},\boldsymbol{\tau})_Q=\int_\Omega\boldsymbol{\sigma}\cdot\boldsymbol{\tau}\,\mathrm{d}x. \tag{2.1}$$

Here and below $\boldsymbol{\varepsilon}$ and Div will represent the deformation and the divergence operators, respectively, i.e., $\boldsymbol{\varepsilon}(\boldsymbol{u})=(\varepsilon_{ij}(\boldsymbol{u})),\ \varepsilon_{ij}(\boldsymbol{u})=\frac{1}{2}\,(u_{i,j}+u_{j,i}),\ \mathrm{Div}\,\boldsymbol{\sigma}=(\sigma_{ij,j})$. The associated norms on these spaces are denoted by $\|\cdot\|_V$ and $\|\cdot\|_Q$, respectively. In addition, recall that the completeness of the space V follows from the assumption meas$\,(\Gamma_1)>0$ which allows the use of Korn's inequality.

We denote by $\boldsymbol{0}_V$ the zero element of V, and for any element $\boldsymbol{v}\in V$, we denote by v_ν and $\boldsymbol{v}_\tau$ its normal and tangential components on Γ given by $v_\nu=\boldsymbol{v}\cdot\boldsymbol{\nu}$ and $\boldsymbol{v}_\tau=\boldsymbol{v}-v_\nu\boldsymbol{\nu}$, respectively. For a regular function $\boldsymbol{\sigma}:\Omega\to\mathbb{S}^d$, we denote by σ_ν and $\boldsymbol{\sigma}_\tau$ its normal and tangential components on Γ, that is $\sigma_\nu=(\boldsymbol{\sigma}\boldsymbol{\nu})\cdot\boldsymbol{\nu}$ and $\boldsymbol{\sigma}_\tau=\boldsymbol{\sigma}\boldsymbol{\nu}-\sigma_\nu\boldsymbol{\nu}$, and we recall that the following Green's formula holds

$$\int_\Omega\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}(\boldsymbol{v})\,\mathrm{d}x+\int_\Omega\mathrm{Div}\,\boldsymbol{\sigma}\cdot\boldsymbol{v}\,\mathrm{d}x=\int_\Gamma\boldsymbol{\sigma}\boldsymbol{\nu}\cdot\boldsymbol{v}\,\mathrm{d}a\quad\forall\,\boldsymbol{v}\in H^1(\Omega)^d. \tag{2.2}$$

We also recall that there exists $c_0>0$ which depends on Ω and Γ_1, such that

$$\|\boldsymbol{v}\|_{L^2(\Gamma)^d}\le c_0\|\boldsymbol{v}\|_V\quad\forall\,\boldsymbol{v}\in V. \tag{2.3}$$

Inequality (2.3) represents a consequence of the Sobolev trace theorem.

Let $Y=L^2(\Omega)^d\times L^2(\Gamma_2)^d$ be the product Hilbert space endowed with the canonical inner product $(\cdot,\cdot)_Y$ and the associated norm $\|\cdot\|_Y$. We denote by $\pi\colon V\to Y$ the operator defined by

$$\pi\boldsymbol{v}=(\boldsymbol{v},\gamma_2\boldsymbol{v})\qquad\forall\,\boldsymbol{v}\in V, \tag{2.4}$$

where $\gamma_2\boldsymbol{v}$ represents the trace of the function $\boldsymbol{v} \in V$ to the boundary Γ_2. Note that π is a linear continuous operator, and therefore, there exists a constant $d_0 > 0$, such that

$$\|\pi\boldsymbol{v}\|_Y \le d_0\|\boldsymbol{v}\|_V \quad \forall\, \boldsymbol{v} \in V. \tag{2.5}$$

Moreover, the compactness of the trace operator combined with the compactness of the embedding $H^1(\Omega)^d \subset L^2(\Omega)^d$ imply that π is a weakly–strongly continuous operator, that is

$$\boldsymbol{v}_n \rightharpoonup \boldsymbol{v} \quad \text{in } V \qquad \Longrightarrow \qquad \pi\boldsymbol{v}_n \to \pi\boldsymbol{v} \quad \text{in } Y. \tag{2.6}$$

Here and below, we use notation "$\rightharpoonup$" and "$\to$" for the weak and strong convergence in various Hilbert spaces. All the limits, upper and lower limits below are considered as $n \to \infty$, even if we do not mention it explicitly.

Quasi-variational inequalities Consider a real Hilbert space X endowed with the inner product $(\cdot,\cdot)_X$ and the associated norm $\|\cdot\|_X$. Let $K \subset X$, $A : X \to X$, $j\colon X \times X \to \mathbb{R}$, $f \in X$ and consider the following problem.

Problem 1. Find an element u such that

$$u \in K, \quad (Au, v-u)_X + j(u,v) - j(u,u) \ge (f, v-u)_X \quad \forall\, v \in K. \tag{2.7}$$

Note that the function j depends on the solution u, and for this reason, we refer to (2.7) as a quasi-variational inequality. Quasi-variational inequalities of the form (2.7) have been studied by many authors, using different functional methods, including fixed point and topological degree arguments. Existence and uniqueness results for such inequalities could be found in [4,15,18], for instance, under various assumption on the operator A and the function j. Here, we consider the following assumptions:

$$K \text{ is a nonempty, closed and convex subset of } X. \tag{2.8}$$

$$\left\{\begin{array}{l} A \text{ is a strongly monotone Lipschitz continuous operator, i.e.,} \\ \quad \text{there exists } m_A > 0 \text{ and } L_A > 0 \text{ such that} \\ \text{(a) } (Au - Av, u - v)_X \ge m_A\|u-v\|_X^2 \quad \forall\, u,\, v \in X, \\ \text{(b) } \|Au - Av\|_X \le L_A\,\|u-v\|_X \quad \forall\, u,\, v \in X. \end{array}\right. \tag{2.9}$$

$$\left\{\begin{array}{l} \text{(a) For all } \eta \in X,\ j(\eta,\cdot) : X \to \mathbb{R} \text{ is convex} \\ \quad \text{and lower semicontinuous.} \\ \text{(b)There exists } \alpha_j \ge 0 \text{ such that} \\ \quad j(\eta_1, v_2) - j(\eta_1, v_1) + j(\eta_2, v_1) - j(\eta_2, v_2) \\ \quad \le \alpha_j\,\|\eta_1 - \eta_2\|_X\|v_1 - v_2\|_X \quad \forall\, \eta_1,\, \eta_2,\, v_1,\, v_2 \in X. \end{array}\right. \tag{2.10}$$

$$m_A > \alpha_j. \tag{2.11}$$

Under these assumptions, we have the following existence and uniqueness result.

Theorem 2.1. *Assume* (2.8)–(2.11). *Then, Problem* 1 *has a unique solution.*

A proof of Theorem 2.1 can be found in [18, p. 49], based on the Banach fixed point argument.

3. The Model

The physical setting is depicted in Fig. 1 and is described as follows. An elastic body occupies, in its reference configuration, the domain $\Omega \subset \mathbb{R}^d$. Recall that the boundary of Ω, denoted Γ, is divided into four measurable disjoint parts Γ_1, Γ_2, Γ_3, and Γ_4, such that meas $(\Gamma_1) > 0$. The body is fixed on Γ_1, is acted upon by given surface tractions on Γ_2, and is in contact with two obstacles on Γ_3 and Γ_4, respectively. The mathematical model which corresponds to the equilibrium of the body in the physical setting above, based on specific interface boundary condition that will be described below, is the following.

Problem 2. Find a displacement field $\boldsymbol{u}\colon \Omega \to \mathbb{R}^d$ and a stress field $\boldsymbol{\sigma}\colon \Omega \to \mathbb{S}^d$ such that

$$\boldsymbol{\sigma} = \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}) \quad \text{in } \Omega, \tag{3.1}$$

$$\mathrm{Div}\,\boldsymbol{\sigma} + \boldsymbol{f}_0 = \boldsymbol{0} \quad \text{in } \Omega, \tag{3.2}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma_1, \tag{3.3}$$

$$\boldsymbol{\sigma\nu} = \boldsymbol{f}_2 \quad \text{on } \Gamma_2, \tag{3.4}$$

$$\left.\begin{array}{ll} u_\nu \le g, & \\ \sigma_\nu = 0 & \text{if } u_\nu < 0,\\ -F \le \sigma_\nu \le 0 & \text{if } u_\nu = 0,\\ \sigma_\nu = -F & \text{if } 0 < u_\nu < g,\\ \sigma_\nu \le -F & \text{if } u_\nu = g,\\ \boldsymbol{\sigma}_\tau = \boldsymbol{0} & \end{array}\right\} \quad \text{on } \Gamma_3, \tag{3.5}$$

$$\left.\begin{array}{l} u_\nu \le k,\\ \sigma_\nu + p(u_\nu) \le 0,\\ (u_\nu - k)(\sigma_\nu + p(u_\nu)) = 0,\\ \|\boldsymbol{\sigma}_\tau\| \le \mu\, p(u_\nu),\\ -\boldsymbol{\sigma}_\tau = \mu\, p(u_\nu)\,\frac{\boldsymbol{u}_\tau}{\|\boldsymbol{u}_\tau\|} \quad \text{if } \boldsymbol{u}_\tau \ne \boldsymbol{0} \end{array}\right\} \quad \text{on } \Gamma_4. \tag{3.6}$$

We now provide a description of the equations and boundary conditions in Problem 2. First, Eq. (3.1) represents the elastic constitutive law of the material in which $\mathcal{F}$ is assumed to be a nonlinear constitutive operator. Equation (3.2) is the equation of equilibrium. We use it here, since the contact process is assumed to be static, and therefore, the inertial term in the equation of motion is neglected. Conditions (3.3) and (3.4) represent the displacement and traction boundary conditions, respectively.

Condition (3.5) describes the frictionless contact with an obstacle made of a rigid body covered by a layer made of rigid-plastic material of thickness g. This condition was already used in [3]. There, its detailed description was provided, together with various mechanical interpretation. Here, we restrict ourselves to mention that the function F could be interpreted as the yield limit of the rigid-plastic layer. Indeed, this layer does not allow penetration (and therefore, it behaves like a rigid body) as far as the inequality $-F < \sigma_\nu \le 0$ holds. It could allow penetration only when $\sigma_\nu = -F$, and in this case, it offers no additional resistance.

Condition (3.6) describes the frictional contact with an obstacle made of a rigid body covered by a layer made of deformable material of thickness

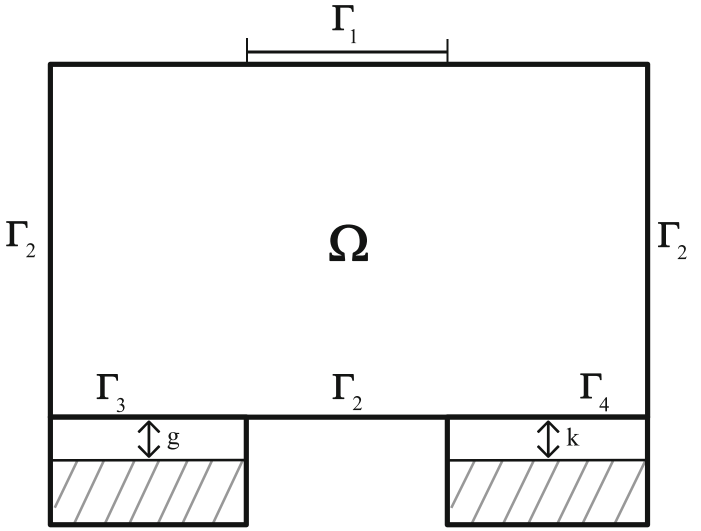

Figure 1. Physical setting

k. Here, μ represent the coefficient of friction and p is the normal compliance function which will be described in the next section. This condition was introduced in [2] and used in number of papers, see [19] and the references therein. It describes a contact with normal compliance, as far as the normal displacement satisfies the condition $u_\nu < k$, associated with the classical Coulomb's law of dry friction. When $u_\nu = k$, the contact is with a Signorini-type condition and is associated with the Tresca friction law with the bound $\mu p(k)$. It follows from here that condition (3.6) describes a natural transition from the Coulomb law of dry friction (which is valid as far as $0 \le u_\nu < k$) to the Tresca law (which is valid when $u_\nu = k$).

4. Existence and Uniqueness

In the study of the mechanical problem (3.1)–(3.6), we assume that the elasticity operator $\mathcal{F}$ satisfies the following conditions:

$$\left\{\begin{array}{l}
\text{(a) } \mathcal{F}\colon \Omega\times\mathbb{S}^d\to\mathbb{S}^d.\\
\text{(b) There exists } L_{\mathcal{F}}>0 \text{ such that}\\
\quad \|\mathcal{F}(\boldsymbol{x},\boldsymbol{\varepsilon}_1)-\mathcal{F}(\boldsymbol{x},\boldsymbol{\varepsilon}_2)\|\le L_{\mathcal{F}}\|\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2\|\\
\qquad \forall\,\boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2\in\mathbb{S}^d,\ \text{a.e. } \boldsymbol{x}\in\Omega.\\
\text{(c)There exists } m_{\mathcal{F}}>0 \text{ such that}\\
\quad (\mathcal{F}(\boldsymbol{x},\boldsymbol{\varepsilon}_1)-\mathcal{F}(\boldsymbol{x},\boldsymbol{\varepsilon}_2))\cdot(\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2)\ge m_{\mathcal{F}}\,\|\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2\|^2\\
\qquad \forall\,\boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2\in\mathbb{S}^d,\ \text{a.e. } \boldsymbol{x}\in\Omega.\\
\text{(d) The mapping } \boldsymbol{x}\mapsto\mathcal{F}(\boldsymbol{x},\boldsymbol{\varepsilon}) \text{ is measurable on}\Omega,\\
\qquad \text{for any } \boldsymbol{\varepsilon}\in\mathbb{S}^d.\\
\text{(e) The mapping } \boldsymbol{x}\mapsto\mathcal{F}(\boldsymbol{x},\boldsymbol{0}) \text{ belongs to } Q.
\end{array}\right. \tag{4.1}$$

The normal compliance function satisfies

$$\begin{cases} \text{(a) } p\colon \Gamma_4 \times \mathbb{R} \to \mathbb{R}_+. \\ \text{(b) There exists } L_p > 0 \text{such that} \\ \qquad |p(\boldsymbol{x}, r_1) - p(\boldsymbol{x}, r_2)| \le L_p |r_1 - r_2| \quad \forall\, r_1,\, r_2 \in \mathbb{R},\ \text{a.e. } \boldsymbol{x} \in \Gamma_4. \\ \text{(c) The mapping } \boldsymbol{x} \mapsto p(\boldsymbol{x}, r) \text{ is measurable on } \Gamma_4, \text{ for any } r \in \mathbb{R}. \\ \text{(d) } (p(\boldsymbol{x}, r_1) - p(\boldsymbol{x}, r_2))\,(r_1 - r_2) \ge 0 \quad \forall\, r_1,\, r_2 \in \mathbb{R},\ \text{a.e. } \boldsymbol{x} \in \Gamma_4. \\ \text{(e) } p(\boldsymbol{x}, r) = 0 \quad \text{for all } r \le 0,\ \text{a.e. } \boldsymbol{x} \in \Gamma_4. \end{cases} \tag{4.2}$$

Finally, we assume that the densities of body forces and tractions, the yield limit, the coefficient of friction and the given thicknesses are such that

$$\boldsymbol{f}_0 \in L^2(\Omega)^d, \tag{4.3}$$

$$\boldsymbol{f}_2 \in L^2(\Gamma_2)^d, \tag{4.4}$$

$$F \in L^2(\Gamma_3), \qquad F(\boldsymbol{x}) \ge 0 \quad \text{a.e. } \boldsymbol{x} \in \Gamma_3, \tag{4.5}$$

$$\mu \in L^\infty(\Gamma_4), \qquad \mu(\boldsymbol{x}) \ge 0 \quad \text{a.e. } \boldsymbol{x} \in \Gamma_4, \tag{4.6}$$

$$g > 0, \quad k > 0. \tag{4.7}$$

Under these assumptions we introduce the set $K \subset V$, the operator $A\colon V \to V$, the function $j\colon V \times V \to \mathbb{R}$, and the element $\boldsymbol{f} \in Y$ defined by

$$K = \{\, \boldsymbol{v} \in V : v_\nu \le g \text{ a.e. on } \Gamma_3, \quad v_\nu \le k \ \text{ a.e. on } \Gamma_4\}, \tag{4.8}$$

$$(A\boldsymbol{u}, \boldsymbol{v})_V = \int_\Omega \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v})\, \mathrm{d}x + \int_{\Gamma_4} p(u_\nu) v_\nu \, \mathrm{d}a \quad \forall\, \boldsymbol{u},\, \boldsymbol{v} \in V, \tag{4.9}$$

$$j(\boldsymbol{u}, \boldsymbol{v}) = \int_{\Gamma_3} F v_\nu^+ \, \mathrm{d}a + \int_{\Gamma_4} \mu\, p(u_\nu) \|\boldsymbol{v}_\tau\| \, \mathrm{d}a \quad \forall\, \boldsymbol{u},\ \boldsymbol{v} \in V, \tag{4.10}$$

$$\boldsymbol{f} = (\boldsymbol{f}_0, \boldsymbol{f}_2). \tag{4.11}$$

Here and below, r^+ denotes the positive part of r, i.e., $r = \max\{0, r\}$. Moreover, note that the definition (2.4) implies that

$$(\boldsymbol{f}, \pi\boldsymbol{v})_Y = \int_\Omega \boldsymbol{f}_0 \cdot \boldsymbol{v} \, \mathrm{d}x + \int_{\Gamma_2} \boldsymbol{f}_2 \cdot \boldsymbol{v} \, \mathrm{d}a \qquad \forall\, \boldsymbol{v} \in V. \tag{4.12}$$

We are now in a position to derive the variational formulation of Problem $\mathcal{P}$, and to this end, we assume that $(\boldsymbol{u}, \boldsymbol{\sigma})$ are sufficiently regular functions which satisfy (3.1)–(3.6). Then, using (3.3), (3.5), (3.6), and (4.8), it follows that

$$\boldsymbol{u} \in K. \tag{4.13}$$

Let $\boldsymbol{v} \in K$. We use Green's formula (2.2) and equalities (3.2)–(3.4) to see that

$$\begin{aligned} \int_\Omega \boldsymbol{\sigma} \cdot (\boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}))\, \mathrm{d}x &= \int_\Omega \boldsymbol{f}_0 \cdot (\boldsymbol{v} - \boldsymbol{u})\, \mathrm{d}x + \int_{\Gamma_2} \boldsymbol{f}_2 \cdot (\boldsymbol{v} - \boldsymbol{u})\, \mathrm{d}a \\ &\quad + \int_{\Gamma_3} \boldsymbol{\sigma\nu} \cdot (\boldsymbol{v} - \boldsymbol{u})\, \mathrm{d}a + \int_{\Gamma_4} \boldsymbol{\sigma\nu} \cdot (\boldsymbol{v} - \boldsymbol{u})\, \mathrm{d}a. \end{aligned} \tag{4.14}$$

Moreover, using standard arguments, we see that the contact condition (3.5) implies that

$$\int_{\Gamma_3} \boldsymbol{\sigma\nu} \cdot (\boldsymbol{v} - \boldsymbol{u})\, \mathrm{d}a \ge \int_{\Gamma_3} F(u_\nu^+ - v_\nu^+)\, \mathrm{d}a. \tag{4.15}$$

In addition, the contact condition (3.6) yields

$$\int_{\Gamma_4} \boldsymbol{\sigma\nu}\cdot(\boldsymbol{v}-\boldsymbol{u})\,\mathrm{d}a \\ \geq \int_{\Gamma_4} p(u_\nu)(u_\nu - v_\nu)\,\mathrm{d}a + \int_{\Gamma_4} \mu\, p(u_\nu)(\|\boldsymbol{u}_\tau\| - \|\boldsymbol{v}_\tau\|)\,\mathrm{d}a. \quad (4.16)$$

Next, we combine equality (4.14) with inequalities (4.15), (4.16), then we use the constitutive law (3.1), the definitions (4.9), (4.10), equality (4.12) and the regularity (4.13). As a result, we find the following variational formulation of Problem $\mathcal{P}$.

Problem 3. Find a displacement field $\boldsymbol{u}$, such that

$$\boldsymbol{u}\in K, \quad (A\boldsymbol{u},\boldsymbol{v}-\boldsymbol{u})_V + j(\boldsymbol{u},\boldsymbol{v}) - j(\boldsymbol{u},\boldsymbol{u}) \geq (\boldsymbol{f},\pi\boldsymbol{v}-\pi\boldsymbol{u})_Y \quad \forall\,\boldsymbol{v}\in K. \quad (4.17)$$

Note that Problem 3 is formulated in terms of the displacement field. Once the displacement field is known, the stress field can be easily obtained using the constitutive law (3.1). A couple $(\boldsymbol{u},\boldsymbol{\sigma})$ which satisfies (3.1) and (4.17) is called a *weak solution* to the contact problem $\mathcal{P}$.

The unique solvability of Problem 3 is provided by the following theorem.

Theorem 4.1. *Assume* (4.1)–(4.7). *Then, there exists a constant* μ_0*, which depends on* Ω, Γ_1 *and* $\mathcal{F}$, *such that Problem* 3 *has a unique solution, if*

$$L_p\|\mu\|_{L^\infty(\Gamma_4)} < \mu_0. \quad (4.18)$$

Proof. We use Theorem 2.1 with the choice $X = V$. To this end, we note that the set (4.8) is a nonempty closed convex subset of V and, moreover, $\mathbf{0}_V \in K$. Therefore, condition (2.8) is satisfied. Next, we use the definition (4.9) and the properties (4.1)(c) and (4.2)(d) to see that

$$(A\boldsymbol{u}-A\boldsymbol{v},\boldsymbol{u}-\boldsymbol{v})_V \geq m_{\mathcal{F}}\,\|\boldsymbol{u}-\boldsymbol{v}\|_V^2 \quad \forall\,\boldsymbol{u},\,\boldsymbol{v}\in V. \quad (4.19)$$

On the other hand, using (4.9), (4.1)(b), (4.2)(b) and the trace inequality (2.3) yields

$$\|A\boldsymbol{u}-A\boldsymbol{v}\|_V \leq (L_{\mathcal{F}} + c_0^2 L_p)\,\|\boldsymbol{u}-\boldsymbol{v}\|_V \qquad \forall\,\boldsymbol{u},\,\boldsymbol{v}\in V. \quad (4.20)$$

We conclude by (4.19) and (4.20) that A is a strongly monotone Lipschitz continuous operator on the space V which shows that (2.9) holds. Moreover, it is easy to see that the functional j defined by (4.10) satisfies condition (2.10)(a).

Assume now that $\boldsymbol{\eta}_1$, $\boldsymbol{\eta}_2$, $\boldsymbol{v}_1$, $\boldsymbol{v}_2\in V$. Then, using the definition (2.10), after some elementary computation, we find that

$$j(\boldsymbol{\eta}_1,\boldsymbol{v}_2) - j(\boldsymbol{\eta}_1,\boldsymbol{v}_1) + j(\boldsymbol{\eta}_2,\boldsymbol{v}_1) - j(\boldsymbol{\eta}_2,\boldsymbol{v}_2) \\ \leq \int_{\Gamma_4} \mu\big|p(\eta_{1\nu}) - (p(\eta_{2\nu})\big|\big|\|\boldsymbol{v}_{1\tau}\| - \|\boldsymbol{v}_{2\tau}\|\big|\,\mathrm{d}a$$

and using the properties (4.2), (4.6), of the function p and μ, together with the trace inequality (2.3), we deduce that

$$j(\boldsymbol{\eta}_1,\boldsymbol{v}_2) - j(\boldsymbol{\eta}_1,\boldsymbol{v}_1) + j(\boldsymbol{\eta}_2,\boldsymbol{v}_1) - j(\boldsymbol{\eta}_2,\boldsymbol{v}_2) \\ \leq c_0^2 L_p\|\mu\|_{L^\infty(\Gamma_4)}\|\boldsymbol{\eta}_1-\boldsymbol{\eta}_2\|_V\|\boldsymbol{v}_1-\boldsymbol{v}_2\|_V. \quad (4.21)$$

It follows from here that j satisfies condition (2.10)(b) with $\alpha_j = c_0^2 L_p\| \mu\|_{L^\infty(\Gamma_4)}$.

Let

$$\mu_0 = \frac{m_{\mathcal{F}}}{c_0^2}. \tag{4.22}$$

and note that, clearly, μ_0 depends on Ω, Γ_1 and $\mathcal{F}$. Assume that (4.18) holds. Then, since $m_A = m_{\mathcal{F}}$, it follows that condition (2.11) holds, too. Finally, using (2.5) and the Riesz representation theorem, we deduce that there exists a unique element $\widetilde{\boldsymbol{f}} \in V$, such that

$$(\widetilde{\boldsymbol{f}}, \boldsymbol{v})_V = (\boldsymbol{f}, \pi\boldsymbol{v})_Y \quad \forall\, \boldsymbol{v} \in V. \tag{4.23}$$

Theorem 4.1 now is a direct consequence of Theorem 2.1. □

5. A Continuous Dependence Result

The solution $\boldsymbol{u}$ of Problem 3 depends on the data $\boldsymbol{f}_0$, $\boldsymbol{f}_2$, F, μ, g, k and, therefore, we sometimes denote it by $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{f}_0, \boldsymbol{f}_2, F, \mu, g, k)$. In what follows, we study its dependence with respect to these data. To this end, in the rest of this section, we assume that (4.1)–(4.7), (4.18) hold and, in addition, we consider a perturbation $\boldsymbol{f}_{0n}$, $\boldsymbol{f}_{2n}$, F_n, μ_n, g_n, and k_n of $\boldsymbol{f}_0$, $\boldsymbol{f}_2$, F, μ, g, and k, respectively, which satisfy conditions (4.3)–(4.7), (4.18). For each $n \in \mathbb{N}$, we consider the set $K_n \subset V$, the functional $j_n\colon V \to \mathbb{R}$ and the element $\boldsymbol{f}_n \in Y$ defined by

$$K_n = \{\boldsymbol{v} \in V : v_\nu \le g_n \ \text{ a.e. on } \Gamma_3, \quad v_\nu \le k_n \ \text{ a.e. on } \Gamma_4\}, \tag{5.1}$$

$$j_n(\boldsymbol{u}, \boldsymbol{v}) = \int_{\Gamma_3} F_n v_\nu^+ \, da + \int_{\Gamma_4} \mu_n\, p(u_\nu)\|\boldsymbol{v}_\tau\| \, da \quad \forall\, \boldsymbol{u},\ \boldsymbol{v} \in V, \tag{5.2}$$

$$\boldsymbol{f}_n = (\boldsymbol{f}_{0n}, \boldsymbol{f}_{2n}). \tag{5.3}$$

Moreover, we consider the following perturbation of Problem 3.

Problem 4. Find a displacement field $\boldsymbol{u}_n$, such that

$$\begin{aligned} \boldsymbol{u}_n \in K_n, \quad & (A\boldsymbol{u}_n, \boldsymbol{v} - \boldsymbol{u}_n)_V + j_n(\boldsymbol{u}_n, \boldsymbol{v}) - j_n(\boldsymbol{u}_n, \boldsymbol{u}_n) \\ & \ge (\boldsymbol{f}_n, \pi\boldsymbol{v} - \pi\boldsymbol{u}_n)_Y \quad \forall\, \boldsymbol{v} \in K_n. \end{aligned} \tag{5.4}$$

It follows from Theorem 4.1 that, for each $n \in \mathbb{N}$, Problem 4 has a unique solution $\boldsymbol{u}_n = \boldsymbol{u}_n(\boldsymbol{f}_{0n}, \boldsymbol{f}_{2n}, F_n \mu_n, g_n, k_n)$. Our main result in this section is the following.

Theorem 5.1. *Assume that*

$$\boldsymbol{f}_{0n} \rightharpoonup \boldsymbol{f}_0 \quad \text{in } L^2(\Omega)^d, \tag{5.5}$$

$$\boldsymbol{f}_{2n} \rightharpoonup \boldsymbol{f}_2 \quad \text{in } L^2(\Gamma_2)^d, \tag{5.6}$$

$$F_n \rightharpoonup F \quad \text{in } L^2(\Gamma_3), \tag{5.7}$$

$$\mu_n \to \mu \quad \text{in } L^\infty(\Gamma_4), \tag{5.8}$$

$$g_n \to g, \quad k_n \to k. \tag{5.9}$$

Then, the solution $\boldsymbol{u}_n$ *of Problem* 4 *converges to the solution* $\boldsymbol{u}$ *of Problem* 3*, that is*

$$\boldsymbol{u}_n \to \boldsymbol{u} \quad \text{in } V. \tag{5.10}$$

The proof of Theorem 5.1 will be carried out in several steps that we present in what follows. We start by considering the following intermediate problem.

Problem 5. Find a displacement field $\widetilde{\boldsymbol{u}}_n$, such that

$$\begin{aligned} &\widetilde{\boldsymbol{u}}_n \in K, \quad (A\widetilde{\boldsymbol{u}}_n, \boldsymbol{v} - \widetilde{\boldsymbol{u}}_n)_V + j_n(\widetilde{\boldsymbol{u}}_n, \boldsymbol{v}) - j_n(\widetilde{\boldsymbol{u}}_n, \widetilde{\boldsymbol{u}}_n) \\ &\quad \geq (\boldsymbol{f}_n, \pi\boldsymbol{v} - \pi\widetilde{\boldsymbol{u}}_n)_Y \quad \forall\, \boldsymbol{v} \in K. \end{aligned} \tag{5.11}$$

The unique solvability of this problem is a direct consequence of Theorem 2.1. We now proceed with the following result.

Lemma 5.2. *There exists a constant* $D > 0$*, such that*

$$\|\boldsymbol{u}_n\|_V \leq D, \qquad \|\widetilde{\boldsymbol{u}}_n\|_V \leq D, \quad \text{for all } n \in \mathbb{N}. \tag{5.12}$$

Proof. Let $n \in \mathbb{N}$. We take $\boldsymbol{v} = \boldsymbol{0}_V$ in (5.4) to obtain

$$(A\boldsymbol{u}_n, \boldsymbol{u}_n)_V + j_n(\boldsymbol{u}_n, \boldsymbol{u}_n) \leq (\boldsymbol{f}_n, \pi\boldsymbol{u}_n)_Y.$$

Next, using assumption (4.19), the positivity of the function j_n and the continuity of the operator π, (2.5), it follows that

$$\|\boldsymbol{u}_n\|_V \leq \frac{1}{m_{\mathcal{F}}} \max(1, d_0)\,(\|\boldsymbol{f}_n\|_Y + \|A\boldsymbol{0}_V\|_V). \tag{5.13}$$

Similar arguments show that

$$\|\widetilde{\boldsymbol{u}}_n\|_V \leq \frac{1}{m_{\mathcal{F}}} \max(1, d_0)\,(\|\boldsymbol{f}_n\|_Y + \|A\boldsymbol{0}_V\|_V). \tag{5.14}$$

Finally, assumptions (5.5) and (5.6) imply that

$$\boldsymbol{f}_n \rightharpoonup \boldsymbol{f} \quad \text{in } Y \tag{5.15}$$

and, therefore, the sequence $\{\boldsymbol{f}_n\} \subset Y$ is bounded, i.e., there exists $E > 0$ which does not depend on n, such that

$$\|\boldsymbol{f}_n\|_Y \leq E. \tag{5.16}$$

Lemma 5.2 is now a direct consequence of inequalities (5.13), (5.14) and (5.16). □

Lemma 5.3. *The sequence* $\{\widetilde{\boldsymbol{u}}_n\}$ *converge weakly in* V *to* $\boldsymbol{u}$*, that is*

$$\widetilde{\boldsymbol{u}}_n \rightharpoonup \boldsymbol{u} \quad \text{in } V. \tag{5.17}$$

Proof. Lemma 5.2 combined with a standard compactness argument implies that there exists $\widetilde{\boldsymbol{u}} \in V$, such that passing to a subsequence, still denoted $\{\widetilde{\boldsymbol{u}}_n\}$, we have

$$\widetilde{\boldsymbol{u}}_n \rightharpoonup \widetilde{\boldsymbol{u}} \quad \text{in } V. \tag{5.18}$$

We now prove that

$$\widetilde{\boldsymbol{u}} = \boldsymbol{u}. \tag{5.19}$$

To this end, we first note that K is a closed convex subset of the space V and $\{\widetilde{\boldsymbol{u}}_n\} \subset K$. Therefore, the convergence (5.18) implies that $\widetilde{\boldsymbol{u}} \in K$. Let $n \in \mathbb{N}$. We take $\boldsymbol{v} = \widetilde{\boldsymbol{u}} \in K$ in (5.11) to obtain that

$$(A\widetilde{\boldsymbol{u}}_n, \widetilde{\boldsymbol{u}}_n - \widetilde{\boldsymbol{u}})_V \leq (\boldsymbol{f}_n, \pi\widetilde{\boldsymbol{u}}_n - \pi\widetilde{\boldsymbol{u}})_Y + j_n(\widetilde{\boldsymbol{u}}_n, \widetilde{\boldsymbol{u}}) - j_n(\widetilde{\boldsymbol{u}}_n, \widetilde{\boldsymbol{u}}_n). \tag{5.20}$$

We now use the convergences (5.7), (5.8), (5.15), (5.18), (2.6) and the compactness of the trace operator to see that

$$(\boldsymbol{f}_n, \pi\widetilde{\boldsymbol{u}}_n - \pi\widetilde{\boldsymbol{u}})_Y \to 0, \tag{5.21}$$

$$j_n(\widetilde{\boldsymbol{u}}_n, \widetilde{\boldsymbol{u}}) - j_n(\widetilde{\boldsymbol{u}}_n, \widetilde{\boldsymbol{u}}_n) \to 0. \tag{5.22}$$

We now pass to the upper limit in (5.20) and use (5.21), (5.22) to deduce that

$$\limsup\, (A\widetilde{\boldsymbol{u}}_n, \widetilde{\boldsymbol{u}}_n - \widetilde{\boldsymbol{u}})_V \leq 0.$$

Therefore, using (4.19) and (4.20), the convergence (5.18) and standard arguments on pseudomonotone operators, we deduce that

$$\liminf\, (A\widetilde{\boldsymbol{u}}_n, \widetilde{\boldsymbol{u}}_n - \boldsymbol{v})_V \geq (A\widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{u}} - \boldsymbol{v})_V \quad \forall\, \boldsymbol{v} \in K. \tag{5.23}$$

On the other hand, using again inequality (5.11), the convergences (5.7), (5.8), (5.15), (5.18), and (2.6), we obtain that

$$\limsup\, (A\widetilde{\boldsymbol{u}}_n, \widetilde{\boldsymbol{u}}_n - \boldsymbol{v})_V \leq (\boldsymbol{f}, \pi\widetilde{\boldsymbol{u}} - \pi\boldsymbol{v})_Y + j(\widetilde{\boldsymbol{u}}, \boldsymbol{v}) - j(\widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{u}}) \quad \forall\, \boldsymbol{v} \in K. \tag{5.24}$$

We combine now the inequalities (5.23) and (5.24) to see that

$$(A\widetilde{\boldsymbol{u}}, \boldsymbol{v} - \widetilde{\boldsymbol{u}})_V + j(\widetilde{\boldsymbol{u}}, \boldsymbol{v}) - j(\widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{u}}) \geq (\boldsymbol{f}, \pi\boldsymbol{v} - \pi\widetilde{\boldsymbol{u}})_Y \quad \forall\, \boldsymbol{v} \in K. \tag{5.25}$$

Next, we take $\boldsymbol{v} = \boldsymbol{u}$ in (5.25) and $\boldsymbol{v} = \widetilde{\boldsymbol{u}}$ in (4.17). Then, adding the resulting inequalities and using the strong monotonicity of the operator A, (4.19), we obtain that

$$m_{\mathcal{F}} \|\widetilde{\boldsymbol{u}} - \boldsymbol{u}\|_V^2 \leq j(\widetilde{\boldsymbol{u}}, \boldsymbol{u}) - j(\widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{u}}) + j(\boldsymbol{u}, \widetilde{\boldsymbol{u}}) - j(\boldsymbol{u}, \boldsymbol{u}).$$

Finally, we use inequality (4.21) to see that

$$(m_{\mathcal{F}} - c_0^2 L_p \|\mu\|_{L^\infty(\Gamma_4)}) \|\widetilde{\boldsymbol{u}} - \boldsymbol{u}\|_V^2 \leq 0.$$

Equality (5.19) is now a consequence of assumption (4.18) and definition (4.22).

A carefully examination of the arguments used above shows that any weakly convergent subsequence of the sequence $\{\widetilde{\boldsymbol{u}}_n\} \subset V$ converges weakly to $\boldsymbol{u}$, where recall, $\boldsymbol{u}$, is the unique solution of inequality (4.17). Therefore, using the bound (5.13), we find that the whole sequence $\{\widetilde{\boldsymbol{u}}_n\}$ converges weakly in V to $\boldsymbol{u}$, which concludes the proof. □

We proceed with the following strong convergence result.

Lemma 5.4. *The sequence* $\{\widetilde{\boldsymbol{u}}_n\}$ *converges strongly in* V *to* $\boldsymbol{u}$, *that is*

$$\|\widetilde{\boldsymbol{u}}_n - \boldsymbol{u}\|_V \to 0. \tag{5.26}$$

Proof. Let $n \in \mathbb{N}$. We take $\boldsymbol{v} = \boldsymbol{u} \in K$ in (5.11) to obtain that

$$(A\widetilde{\boldsymbol{u}}_n, \widetilde{\boldsymbol{u}}_n - \boldsymbol{u})_V \leq (\boldsymbol{f}_n, \pi\widetilde{\boldsymbol{u}}_n - \pi\boldsymbol{u})_Y + j_n(\widetilde{\boldsymbol{u}}_n, \boldsymbol{u}) - j_n(\widetilde{\boldsymbol{u}}_n, \widetilde{\boldsymbol{u}}_n). \quad (5.27)$$

Next, we use (4.19) to find that

$$\begin{aligned} m_{\mathcal{F}} \|\widetilde{\boldsymbol{u}}_n - \boldsymbol{u}\|_V^2 &\leq (A\widetilde{\boldsymbol{u}}_n - A\boldsymbol{u}, \widetilde{\boldsymbol{u}}_n - \boldsymbol{u})_V \\ &= (A\widetilde{\boldsymbol{u}}_n, \widetilde{\boldsymbol{u}}_n - \boldsymbol{u})_V - (A\boldsymbol{u}, \widetilde{\boldsymbol{u}}_n - \boldsymbol{u})_V \end{aligned}$$

and, therefore, (5.27) yields

$$\begin{aligned} &m_{\mathcal{F}} \|\widetilde{\boldsymbol{u}}_n - \boldsymbol{u}\|_V^2 \\ &\quad \leq (\boldsymbol{f}_n, \pi\widetilde{\boldsymbol{u}}_n - \pi\boldsymbol{u})_Y + j_n(\widetilde{\boldsymbol{u}}_n, \boldsymbol{u}) - j_n(\widetilde{\boldsymbol{u}}_n, \widetilde{\boldsymbol{u}}_n) - (A\boldsymbol{u}, \widetilde{\boldsymbol{u}}_n - \boldsymbol{u})_V. \end{aligned} \quad (5.28)$$

We now use (5.7), (5.8), (5.15), (5.17), (2.6) and the compactness of the trace operator to see that

$$(\boldsymbol{f}_n, \pi\widetilde{\boldsymbol{u}}_n - \pi\boldsymbol{u})_Y \to 0, \quad (5.29)$$

$$j_n(\widetilde{\boldsymbol{u}}_n, \boldsymbol{u}) - j_n(\widetilde{\boldsymbol{u}}_n, \widetilde{\boldsymbol{u}}_n) \to 0, \quad (5.30)$$

$$(A\boldsymbol{u}, \widetilde{\boldsymbol{u}}_n - \boldsymbol{u})_V \to 0. \quad (5.31)$$

Finally, we combine the inequality (5.28) with the convergences (5.29)–(5.31) to see that (5.26) holds. □

Lemma 5.5. *The following convergence holds*

$$\|\widetilde{\boldsymbol{u}}_n - \boldsymbol{u}_n\|_V \to 0. \quad (5.32)$$

Proof. Let $n \in \mathbb{N}$ and let $\alpha_n > 0$, $\beta_n > 0$ be given by

$$\alpha_n = \min\left\{\frac{g_n}{g}, \frac{k_n}{k}\right\}, \quad \beta_n = \min\left\{\frac{g}{g_n}, \frac{k}{k_n}\right\}. \quad (5.33)$$

Then, it is since $\widetilde{\boldsymbol{u}}_n \in K$ and $\boldsymbol{u}_n \in K_n$, it is easy to see that

$$\alpha_n\widetilde{\boldsymbol{u}}_n \in K_n. \quad (5.34)$$

and

$$\beta_n\boldsymbol{u}_n \in K. \quad (5.35)$$

The regularity (5.34) allows to test in (5.4) with $\boldsymbol{v} = \alpha_n\widetilde{\boldsymbol{u}}_n$ to obtain

$$\begin{aligned} &(A\boldsymbol{u}_n, \alpha_n\widetilde{\boldsymbol{u}}_n - \boldsymbol{u}_n)_V + j_n(\boldsymbol{u}_n, \alpha_n\widetilde{\boldsymbol{u}}_n) - j_n(\boldsymbol{u}_n, \boldsymbol{u}_n) \\ &\quad \geq (\boldsymbol{f}_n, \pi(\alpha_n\widetilde{\boldsymbol{u}}_n) - \pi\boldsymbol{u}_n)_Y. \end{aligned} \quad (5.36)$$

Moreover, the regularity (5.35) allows to test in (5.11) with $v = \beta_n\boldsymbol{u}_n$. As a result, we find that

$$\begin{aligned} &(A\widetilde{\boldsymbol{u}}_n, \beta_n\boldsymbol{u}_n - \widetilde{\boldsymbol{u}}_n)_V + j_n(\widetilde{\boldsymbol{u}}_n, \beta_n\boldsymbol{u}_n) - j_n(\widetilde{\boldsymbol{u}}_n, \widetilde{\boldsymbol{u}}_n) \\ &\quad \geq (\boldsymbol{f}_n, \pi(\beta_n\boldsymbol{u}_n) - \pi\widetilde{\boldsymbol{u}}_n)_Y. \end{aligned} \quad (5.37)$$

We now add inequalities (5.36), (5.37) to find that

$$\begin{aligned} &(A\widetilde{\boldsymbol{u}}_n, \widetilde{\boldsymbol{u}}_n - \beta_n\boldsymbol{u}_n)_V + (A\boldsymbol{u}_n, \boldsymbol{u}_n - \alpha_n\widetilde{\boldsymbol{u}}_n)_V \\ &\quad \leq (1 - \beta_n)(\boldsymbol{f}_n, \pi\boldsymbol{u}_n)_Y + (1 - \alpha_n)(\boldsymbol{f}_n, \pi\widetilde{\boldsymbol{u}}_n)_Y \\ &\qquad + j_n(\widetilde{\boldsymbol{u}}_n, \beta_n\boldsymbol{u}_n) - j_n(\widetilde{\boldsymbol{u}}_n, \widetilde{\boldsymbol{u}}_n) + j_n(\boldsymbol{u}_n, \alpha_n\widetilde{\boldsymbol{u}}_n) - j_n(\boldsymbol{u}_n, \boldsymbol{u}_n). \end{aligned}$$

Then, we use the positively homogeneity of the function j with respect to the second argument, and after some algebra, we deduce that

$$\begin{aligned}(A\widetilde{\boldsymbol{u}}_n - A\boldsymbol{u}_n, \widetilde{\boldsymbol{u}}_n - \boldsymbol{u}_n)_V &\\ \le (\alpha_n - 1)(A\boldsymbol{u}_n, \widetilde{\boldsymbol{u}}_n)_V + (\beta_n - 1)(A\widetilde{\boldsymbol{u}}_n, \boldsymbol{u}_n)_V &\\ \le (1-\beta_n)(\boldsymbol{f}_n, \pi\boldsymbol{u}_n)_Y + (1-\alpha_n)(\boldsymbol{f}_n, \pi\widetilde{\boldsymbol{u}}_n)_Y &\\ + (\beta_n - 1)j(\widetilde{\boldsymbol{u}}_n, \boldsymbol{u}_n) + (\alpha_n - 1)j(\boldsymbol{u}_n, \widetilde{\boldsymbol{u}}_n) &\\ + j_n(\widetilde{\boldsymbol{u}}_n, \boldsymbol{u}_n) - j_n(\widetilde{\boldsymbol{u}}_n, \widetilde{\boldsymbol{u}}_n) + j_n(\boldsymbol{u}_n, \widetilde{\boldsymbol{u}}_n) - j_n(\boldsymbol{u}_n, \boldsymbol{u}_n). &\quad (5.38)\end{aligned}$$

We now use inequalities (4.19), (4.20), (2.5) and the bounds (5.12), (5.16) to see that

$$(A\widetilde{\boldsymbol{u}}_n - A\boldsymbol{u}_n, \widetilde{\boldsymbol{u}}_n - \boldsymbol{u}_n)_V \ge m_{\mathcal{F}}\, \|\boldsymbol{u}_n - \widetilde{\boldsymbol{u}}_n\|_V^2 \quad (5.39)$$

$$(\alpha_n - 1)(A\boldsymbol{u}_n, \widetilde{\boldsymbol{u}}_n)_V \le |\alpha_n - 1|(L_{\mathcal{F}} + c_0^2 L_p + \|A\mathbf{0}_V\|_V)D, \quad (5.40)$$

$$(\beta_n - 1)(A\widetilde{\boldsymbol{u}}_n, \boldsymbol{u}_n)_V \le |\beta_n - 1|(L_{\mathcal{F}} + c_0^2 L_p + \|A\mathbf{0}_V\|_V)D, \quad (5.41)$$

$$(1-\beta_n)(\boldsymbol{f}_n, \pi\boldsymbol{u}_n)_Y \le |\beta_n - 1| E d_0 D, \quad (5.42)$$

$$(1-\alpha_n)(\boldsymbol{f}_n, \pi\widetilde{\boldsymbol{u}}_n)_Y \le |\alpha_n - 1| E d_0 D. \quad (5.43)$$

On the other hand, using inequality (4.21) with $\boldsymbol{\eta}_1 = \widetilde{\boldsymbol{u}}_n$, $\boldsymbol{\eta}_2 = \mathbf{0}_V$, $\boldsymbol{v}_1 = \mathbf{0}_V$, $\boldsymbol{v}_2 = \boldsymbol{u}_n$ combined with assumption (4.2)(e), it follows that

$$j(\widetilde{\boldsymbol{u}}_n, \boldsymbol{u}_n) \le c_0^2 L_p \|\mu\|_{L^\infty(\Gamma_4)} \|\widetilde{\boldsymbol{u}}_n\|_V \|\boldsymbol{u}_n\|_V.$$

Therefore, the bounds (5.12) imply that

$$(\beta_n - 1)j(\widetilde{\boldsymbol{u}}_n, \boldsymbol{u}_n) \le |\beta_n - 1| c_0^2 L_p D^2 \|\mu\|_{L^\infty(\Gamma_4)}. \quad (5.44)$$

A similar arguments yields

$$(\alpha_n - 1)j(\boldsymbol{u}_n, \widetilde{\boldsymbol{u}}_n) \le |\alpha_n - 1| c_0^2 L_p D^2 \|\mu\|_{L^\infty(\Gamma_4)}. \quad (5.45)$$

Finally, using again inequality (4.21), we deduce that

$$\begin{aligned}&j_n(\widetilde{\boldsymbol{u}}_n, \boldsymbol{u}_n) - j_n(\widetilde{\boldsymbol{u}}_n, \widetilde{\boldsymbol{u}}_n) + j_n(\boldsymbol{u}_n, \widetilde{\boldsymbol{u}}_n) - j_n(\boldsymbol{u}_n, \boldsymbol{u}_n)\\ &\quad \le c_0^2 L_p \|\mu_n\|_{L^\infty(\Gamma_4)} \|\boldsymbol{u}_n - \widetilde{\boldsymbol{u}}_n\|_V^2. \quad (5.46)\end{aligned}$$

We now combine inequalities (5.38)–(5.46) to see that there exists a positive constant G which does not depend on n, such that

$$(m_{\mathcal{F}} - c_0^2 L_p \|\mu_n\|_{L^\infty(\Gamma_4)}) \|\boldsymbol{u}_n - \widetilde{\boldsymbol{u}}_n\|_V^2 \le G(|\alpha_n - 1| + |\beta_n - 1|). \quad (5.47)$$

On the other hand, the convergence (5.8) implies that

$$m_{\mathcal{F}} - c_0^2 L_p \|\mu_n\|_{L^\infty(\Gamma_4)} \to m_{\mathcal{F}} - c_0^2 L_p \|\mu\|_{L^\infty(\Gamma_4)} \quad (5.48)$$

and, in addition, the smallness assumption (4.18) and the definition (4.22) yield $m_{\mathcal{F}} - c_0^2 L_p \|\mu\|_{L^\infty(\Gamma_4)} > 0$. We combine this inequality with the convergence (5.48) to deduce that for n large enough, we have

$$(m_{\mathcal{F}} - c_0^2 L_p \|\mu_n\|_{L^\infty(\Gamma_4)}) \ge \frac{1}{2}\,(m_{\mathcal{F}} - c_0^2 L_p \|\mu\|_{L^\infty(\Gamma_4)}). \quad (5.49)$$

The inequalities (5.47) and (5.49) show that there exists a positive constant C which does not depend on n, such that

$$\|\boldsymbol{u}_n - \widetilde{\boldsymbol{u}}_n\|_V^2 \le C(|\alpha_n - 1| + |\beta_n - 1|). \quad (5.50)$$

Finally, note that the definition (5.33) and assumption (5.9) yield

$$\alpha_n \to 1, \quad \beta_n \to 1. \tag{5.51}$$

The convergence (5.32) is now a consequence of (5.50) and (5.51). □

We now have all the ingredients to provide the proof of Theorem 5.1.

Proof. Let $n \in \mathbb{N}$. We write

$$\|\boldsymbol{u}_n - \boldsymbol{u}\|_V \leq \|\boldsymbol{u}_n - \widetilde{\boldsymbol{u}}_n\|_V + \|\widetilde{\boldsymbol{u}}_n - \boldsymbol{u}\|_V$$

then we use Lemmas 5.4 and 5.5 to see that the convergence (5.10) holds, which concludes the proof. □

In addition to the mathematical interest in the convergence result (5.10), it is important from mechanical point of view, since it shows that the weak solution of the contact Problem 2 depends continuously on the data.

6. An Optimization Problem

In this section, we study an optimization problem associated with Problem 3, under the assumption that μ is fixed. As already mentioned, the solution of this problem depends on the data $\boldsymbol{f}_0$, $\boldsymbol{f}_2$, F, g, and k, and therefore, each of these quantities could play the role of a control for the variational inequality (4.17). Various choices can be considered, and for this reason, to avoid repetition, we shall consider a generic problem which can be constructed as follows.

Denote by $\boldsymbol{\theta}$ one or part of the data $\boldsymbol{f}_0$, $\boldsymbol{f}_2$, F, g, and k and let $\boldsymbol{\eta}$ be the reminder ones. To guarantee the conditions of Theorem 4.1, we assume that $\boldsymbol{\theta} \in U$ and $\boldsymbol{\eta} \in S$, where U and S are subsets of two appropriate Hilbert spaces Z and W, respectively. For instance, if $\boldsymbol{\theta} = \boldsymbol{f}_2$, then $\boldsymbol{\eta} = (\boldsymbol{f}_0, F, g, k)$, $Z = L^2(\Gamma_2)^d$ and $W = L^2(\Omega)^d \times L^2(\Gamma_3) \times \mathbb{R}^2$. If $\boldsymbol{\theta} = (g, k)$, then $\boldsymbol{\eta} = (\boldsymbol{f}_0, \boldsymbol{f}_2, F)$, $Z = \mathbb{R}^2$ and $W = L^2(\Omega)^d \times L^2(\Gamma_2)^d \times L^2(\Gamma_3)$. The sets U and S will be specified in Examples 6.3 and 6.4. For each $(\boldsymbol{\theta}, \boldsymbol{\eta}) \in U \times S$, we denote in what follows by $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{\theta}, \boldsymbol{\eta})$ the solution of the variational inequality (4.17) and we consider the following optimization problem.

Problem 6. Given $\boldsymbol{\eta} \in S$, find $\boldsymbol{\theta}^* \in U$ such that

$$\mathcal{L}\big(\boldsymbol{u}(\boldsymbol{\theta}^*, \boldsymbol{\eta})\big) = \min_{\boldsymbol{\theta} \in U} \mathcal{L}\big(\boldsymbol{u}(\boldsymbol{\theta}, \boldsymbol{\eta})\big). \tag{6.1}$$

We also consider the following assumptions:

$$\left\{\begin{array}{l} U \text{ is a bounded weakly closed subset of } Z, \text{i.e.,} \\ \{\boldsymbol{\theta}_n\} \subset U, \quad \boldsymbol{\theta} \in Z, \quad \boldsymbol{\theta}_n \rightharpoonup \boldsymbol{\theta} \quad \text{in } Z \implies \boldsymbol{\theta} \in U. \end{array}\right. \tag{6.2}$$

$$\left\{\begin{array}{l} \mathcal{L}\colon V \to \mathbb{R} \text{ is a lower semicontinuous function, i.e.,} \\ \{\boldsymbol{u}_n\} \subset V, \quad \boldsymbol{u} \in V, \quad \boldsymbol{u}_n \to \boldsymbol{u} \quad \text{in } V \implies \liminf \mathcal{L}(\boldsymbol{u}_n) \geq \mathcal{L}(\boldsymbol{u}). \end{array}\right. \tag{6.3}$$

$$\left\{\begin{array}{l} S \text{ is a weakly closed subset of } W, \text{i.e.,} \\ \{\boldsymbol{\eta}_n\} \subset S, \quad \boldsymbol{\eta} \in W, \quad \boldsymbol{\eta}_n \rightharpoonup \boldsymbol{\eta} \quad \text{in } W \implies \boldsymbol{\eta} \in S. \end{array}\right. \tag{6.4}$$

$$\mathcal{L}\colon V \to \mathbb{R} \text{ is a continuous function.} \tag{6.5}$$

We have the following existence result.

Theorem 6.1. *Assume* (6.2) *and* (6.3). *Then, for each* $\boldsymbol{\eta} \in S$, *there exists at least one solution* $\boldsymbol{\theta}^* \in U$ *to Problem* 6.

Proof. Let

$$\omega = \inf_{\boldsymbol{\theta} \in U} \mathcal{L}\big(\boldsymbol{u}(\boldsymbol{\theta}, \boldsymbol{\eta})\big) \tag{6.6}$$

and let $\{\boldsymbol{\theta}_n\} \subset U$ be a minimizing sequence for the functional J, that is

$$\lim \mathcal{L}\big(\boldsymbol{u}(\boldsymbol{\theta}_n, \boldsymbol{\eta})\big) = \omega. \tag{6.7}$$

Since $U \subset Z$ is bounded, it follows that the sequence $\{\boldsymbol{\theta}_n\}$ is bounded in Z, and therefore, there exists $\boldsymbol{\theta}^* \in Z$, such that passing to a subsequence still denoted $\{\boldsymbol{\theta}_n\}$, we have

$$\boldsymbol{\theta}_n \rightharpoonup \boldsymbol{\theta}^* \quad \text{in } Z. \tag{6.8}$$

Moreover, since $U \subset Z$ is weakly closed, we deduce that

$$\boldsymbol{\theta}^* \in U. \tag{6.9}$$

In addition, the convergence (6.8) and Theorem 5.1 yield

$$\boldsymbol{u}(\boldsymbol{\theta}_n, \boldsymbol{\eta}) \to \boldsymbol{u}(\boldsymbol{\theta}^*, \boldsymbol{\eta}) \quad \text{in } V \tag{6.10}$$

and therefore, assumption (6.3) implies that

$$\liminf \mathcal{L}(\boldsymbol{u}(\boldsymbol{\theta}_n, \boldsymbol{\eta})) \geq \mathcal{L}(\boldsymbol{u}(\boldsymbol{\theta}^*, \boldsymbol{\eta})). \tag{6.11}$$

It follows now from (6.7) and (6.11) that

$$\omega \geq \mathcal{L}(\boldsymbol{u}(\boldsymbol{\theta}^*, \boldsymbol{\eta})). \tag{6.12}$$

In addition, (6.9) and (6.6) yield

$$\omega \leq \mathcal{L}(\boldsymbol{u}(\boldsymbol{\theta}^*, \boldsymbol{\eta})). \tag{6.13}$$

Finally, we combine (6.9), (6.12) and (6.13) to see that (6.1) holds, which concludes the proof. □

The solution of the optimization Problem 6 depends on $\boldsymbol{\eta}$. Its dependence is provided by the following result.

Theorem 6.2. *Assume* (6.2), (6.4), (6.5). *Let* $\{\boldsymbol{\eta}_n\} \subset S$, *and for each* $n \in \mathbb{N}$, *let* $\boldsymbol{\theta}_n^*$ *be a solution to Problem* 6 *for* $\boldsymbol{\eta} = \boldsymbol{\eta}_n$. *In addition, assume that*

$$\boldsymbol{\eta}_n \rightharpoonup \boldsymbol{\eta} \quad \text{in } W. \tag{6.14}$$

Then, there exists a subsequence of the sequence $\{\boldsymbol{\theta}_n^*\}$, *again denoted* $\{\boldsymbol{\theta}_n^*\}$, *and an element* $\boldsymbol{\theta}^* \in U$, *such that*

$$\boldsymbol{\theta}_n^* \rightharpoonup \boldsymbol{\theta}^* \quad \text{in } Z. \tag{6.15}$$

Moreover, $\boldsymbol{\theta}^*$ *is a solution to Problem* 6.

Proof. We start with the remark that assumption (6.5) implies (6.3), and therefore, the existence of the solution $\boldsymbol{\theta}_n^*$ of Problem 6 is guaranteed by Theorem 6.1, for each $n \in \mathbb{N}$. Next, we use assumption (6.2) to see that the sequence $\{\boldsymbol{\theta}_n\}$ is bounded, and therefore, there exists $\boldsymbol{\theta}^* \in Z$, such that passing to a subsequence still denoted $\{\boldsymbol{\theta}_n\}$, the convergence (6.15) holds. Moreover, (6.2) implies that

$$\boldsymbol{\theta}^* \in U \tag{6.16}$$

and (6.4) implies that $\boldsymbol{\eta} \in S$. In addition, the convergences (6.14), (6.15), and Theorem 5.1 guarantee that

$$\boldsymbol{u}(\boldsymbol{\theta}_n^*, \boldsymbol{\eta}_n) \to \boldsymbol{u}(\boldsymbol{\theta}^*, \boldsymbol{\eta}) \quad \text{in } V. \tag{6.17}$$

Assume now that $\boldsymbol{\theta}_0^*$ is a solution of Problem 6. Then

$$\mathcal{L}\big(\boldsymbol{u}(\boldsymbol{\theta}_0^*, \boldsymbol{\eta})\big) \le \mathcal{L}\big(\boldsymbol{u}(\boldsymbol{\theta}^*, \boldsymbol{\eta})\big). \tag{6.18}$$

Moreover, (6.14) and Theorem 5.1 imply that

$$\boldsymbol{u}(\boldsymbol{\theta}_0^*, \boldsymbol{\eta}_n) \to \boldsymbol{u}(\boldsymbol{\theta}_0^*, \boldsymbol{\eta}) \quad \text{in } V. \tag{6.19}$$

On the other hand, by the optimality of $\boldsymbol{\theta}_n^*$ it follows that

$$\mathcal{L}\big(\boldsymbol{u}(\boldsymbol{\theta}_n^*, \boldsymbol{\eta}_n)\big) \le \mathcal{L}\big(\boldsymbol{u}(\boldsymbol{\theta}_0^*, \boldsymbol{\eta}_n)\big) \quad \text{for all } n \in \mathbb{N}.$$

Therefore, the convergences (6.17), (6.19) and the continuity of $\mathcal{L}$, (6.5), imply that

$$\mathcal{L}(\boldsymbol{u}(\boldsymbol{\theta}^*, \boldsymbol{\eta})) \le \mathcal{L}(\boldsymbol{u}(\boldsymbol{\theta}_0^*, \boldsymbol{\eta})). \tag{6.20}$$

We now combine inequalities (6.18) and (6.20) and use regularity (6.16) to deduce that $\boldsymbol{\theta}^*$ is a solution to Problem 6, which concludes the proof. □

We end this section with two examples of optimization problems for which the results provided by Theorems 6.1 and 6.2 hold.

Example 6.3. We chose $\boldsymbol{\theta} = \boldsymbol{f}_2$, and therefore, $\boldsymbol{\eta} = (\boldsymbol{f}_0, F, g, k)$, $Z = L^2(\Gamma_2)^d$, $W = L^2(\Omega)^d \times L^2(\Gamma_3) \times \mathbb{R} \times \mathbb{R}$. Let

$$U = \{\boldsymbol{\theta} \in Z : \|\boldsymbol{\theta}\|_Z \le E\},$$
$$S = \{\boldsymbol{\eta} = (\boldsymbol{f}_0, F, g, k) \in S : F \ge 0 \text{ a.e. on } \Gamma_3,\ g_0 \le g \le g_1,\ k_0 \le k \le k_1\},$$
$$\mathcal{L}(\boldsymbol{v}) = \int_{\Gamma_3} \|v_\nu - \phi\|^2 \,\mathrm{d}a \quad \text{for all } \boldsymbol{v} \in V$$

where E, g_0, g_1, k_0, and k_1 are strictly positive constants, such that $g_0 \le g_1$ and $k_0 \le k_1$ and $\phi \in L^2(\Gamma_3)$ is given. With this choice, the mechanical interpretation of Problem 6 is the following: given a contact process of the form (3.1)–(3.6), with the data $(\boldsymbol{f}_0, F, g, k) \in S$, we are looking for a traction $\boldsymbol{\theta}^* = \boldsymbol{f}_2^* \in U$, such that the corresponding normal displacement of the solution on Γ_3 is as close as possible to the "desired displacement" ϕ. It is easy to see that in this case all the assumptions of Theorems 6.1 and 6.2 are satisfied. Theorem 6.1 guarantees the existence of at least one solution of this optimization problem.

Example 6.4. A second example of Problem 6 can be obtained by taking $\boldsymbol{\theta} = (g, k)$, and therefore, $\boldsymbol{\eta} = (\boldsymbol{f}_0, \boldsymbol{f}_2, F)$, $Z = \mathbb{R} \times \mathbb{R}$, $W = L^2(\Omega)^d \times L^2(\Gamma_2)^d \times L^2(\Gamma_3)$. Let

$$U = \{\boldsymbol{\theta} = (g, k) \in \mathbb{R} \times \mathbb{R} : \ g_0 \leq g \leq g_1, \ k_0 \leq k \leq k_1\},$$
$$S = \{\boldsymbol{\eta} = (\boldsymbol{f}_0, \boldsymbol{f}_2, F) \in S : F \geq 0 \text{ a.e. on } \Gamma_3\},$$
$$\mathcal{L}(\boldsymbol{v}) = \int_\Omega \|\boldsymbol{\varepsilon}(\boldsymbol{v})\|^2 \, \mathrm{d}x \quad \text{for all } \boldsymbol{v} \in V,$$

where g_0, g_1, k_0, and k_1 are strictly positive constants, such that $g_0 \leq g_1$ and $k_0 \leq k_1$. With this choice, the mechanical interpretation of Problem 6 is the following: given a contact process of the form (3.1)–(3.6), with the data $(\boldsymbol{f}_0, \boldsymbol{f}_2, F) \in S$, we are looking for a thicknesses $\boldsymbol{\theta}^* = (g^*, k^*) \in U$, such that the corresponding deformation in the body is as small as possible. Theorem 6.1 guarantees the existence of at least one solution of all this optimization problem.

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