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New Properties on Normalized Null Hypersurfaces

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Abstract. Rigging technique introduced in Gutiérrez and Olea (Math Nachr 289:1219–1236, 2016) is a convenient way to address the study of null hypersurfaces. It offers, in addition, the extra benefit of inducing a Riemannian structure on the null hypersurface which is used to study geometric and topological properties on it. In this paper, we develop this technique showing new properties and applications. We first discuss the very existence of the rigging fields under prescribed geometric and topological constraints. We consider the completeness of the induced rigged Riemannian structure. This is potentially important, because it allows to use most of the usual Riemannian techniques.

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1. Introduction

A null hypersurface in a spacetime is a smooth codimension one submanifold, such that the ambient metric degenerates when restricted to it. Null hypersurfaces play an important role in general relativity, as they represent horizons of various sorts (event horizon of a black hole, killing horizon, etc.) and include light cones. The main drawback to study them as part of standard submanifold theory is the degeneracy of the induced metric. Some attempts to overcome this difficulty have had remarkable success. In [11], the approach consists in fixing a geometric data formed by a null section and a screen distribution on the null hypersurface. This allows to induce some geometric objects such as a connection, a null second fundamental form, and Gauss–Codazzi-type equations. In [22], the author uses the quotient vector bundle $TM/TM^{\perp}$ to "get rid" of the degeneracy of the induced metric. Returning to the approach in [11], the basic question is how to reduce as much as possible

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the arbitrary choices and to have a reasonable coupling between the properties of the null hypersurface and the ambient space. In [20], the authors used the rigging technique to study null hypersurfaces. It is based on the arbitrary choice of a unique vector field in a neighborhood of the null hypersurface, called rigging vector field, from which is constructed both a null section defined on the null hypersurface, called rigged vector field, and a screen distribution. This rigging technique has also the advantage to induce, on the whole null hypersurface, a Riemannian structure coupled with the rigging, which is used as a bridge to study the null hypersurface. The null geometry of the hypersurface is related to the properties of the induced Riemannian structure on the hypersurface, allowing handle it using Riemannian geometry. The question now arises of knowing whether it is always possible to operate a choice of a rigging vector field with fixed geometric properties (closedness, conformality, causality conditions, etc.) but also with geometric prescribed properties for the induced rigged Riemannian structure (completeness, pinching constraints, geodesibility, etc.). This is our concern in the present paper. The fact that there is a positive answer to a reasonable amount of the above questions reinforces our opinion that the rigging technique can be a good tool in this theory.

In Sect. 2, we review some facts about null hypersurfaces, fix notations, and give two technical lemmas. Obstruction results involving both topology and prescribed geometric conditions on the rigging vector field are established in Sect. 3, e.g., Theorem 3.1. The completeness properties of the induced Riemannian metric are considered in Sect. 4. The first part is concerned with some splitting results on the hypersurface equipped with its rigged Riemannian structure. This allows us to get completeness sufficient conditions in Robertson–Walker spaces, e.g., Theorem 4.1 and Proposition 4.3. After this, we consider the case of generalized Robertson–Walker (GRW) spaces. We show that there is natural rigging using the warping function leading to a complete induced Riemannian structure, e.g., Proposition 4.4. Finally, we show using closedness argument on the rigging field and compactness of the screen leaves that the induced Riemannian structure is complete, e.g., Theorem 4.9. In Sect. 5, we establish some results on null hypersurfaces under completeness assumption of the induced Riemannian metric. In Sect. 5.1, we establish some estimates on mean curvature on null hypersurfaces with complete rigged Riemannian structure, e.g., Theorems 5.1, 5.5, 5.7. Finally, Sect. 5.2 deals with null hypersurfaces for which the screen shape operator is semi-definite. We prove some obstruction results on the existence of closed geodesics, e.g., Proposition 5.10, and show, under a completeness condition, that the manifold structure of the null hypersurface (say) M in a three-dimensional simply connected Lorentzian manifold with no closed null curve is diffeomorphic to the plane or the cylinder, e.g., Theorem 5.13. Finally, we investigate about the existence of topologically closed totally geodesic null hypersurfaces in Robertson–Walker spaces and prove non-existence of lightlike line in some cases, e.g., Theorem 5.16 and Corollary 5.17.

2. Normalization and Rigged Riemannian Structure

Let $(\overline{M}, \overline{g})$ be a $(n+2)$-dimensional Lorentzian manifold and M a null hypersurface in $\overline{M}$. This means that, at each $p \in M$, the restriction $\overline{g_p}_{|T_pM}$ is degenerate; that is, there exists a non-zero vector $U \in T_pM$, such that $\overline{g}(U, X) = 0$ for all $X \in T_pM$. Hence, in null setting, the normal bundle $TM^\perp$ of the null hypersurface M^{n+1} is a rank 1 vector subbundle of the tangent bundle TM, contrary to the classical theory of non-degenerate hypersurfaces for which the normal bundle has trivial intersection $\{0\}$ with the tangent one playing an important role in introducing the main induced geometric objects on M. Let us start with the usual tools involved in the study of such hypersurfaces according to [11]. They consist in fixing, on the null hypersurface, geometric data formed by a null section and a screen distribution. By *screen distribution* on M^{n+1}, we mean a complementary bundle of $TM^\perp$ in TM. It is then a rank n non-degenerate distribution over M. In fact, there are infinitely many possibilities of choices for such a distribution. Each of them is canonically isomorphic to the factor vector bundle $TM/TM^\perp$. For reasons that will become obvious in a few lines below, let us denote such a distribution by $\mathscr{S}(N)$. We then have

$$TM = \mathscr{S}(N) \oplus TM^\perp, \tag{2.1}$$

where $\oplus$ denotes the orthogonal direct sum. From [11], it is known that, for a null hypersurface equipped with a screen distribution, there exists a unique rank 1 vector subbundle $tr(TM)$ of $T\overline{M}$ over M, such that for any non-zero section ξ of $TM^\perp$ on a coordinate neighborhood $\mathscr{U} \subset M$, there exists a unique section N of $tr(TM)$ on $\mathscr{U}$ satisfying

$$\overline{g}(N, \xi) = 1, \quad \overline{g}(N, N) = \overline{g}(N, W) = 0, \quad \forall W \in \mathscr{S}(N)|_{\mathscr{U}}). \tag{2.2}$$

Then, $T\overline{M}$ admits the splitting:

$$T\overline{M}|_M = TM \oplus tr(TM) = \{TM^\perp \oplus tr(TM)\} \oplus \mathscr{S}(N). \tag{2.3}$$

We call $tr(TM)$ a *(null) transverse vector bundle* along M. In fact, from (2.2) and (2.3), one shows that, conversely, a choice of a transversal bundle $tr(TM)$ determines uniquely the screen distribution $\mathscr{S}(N)$. A vector field N as in (2.2) is called a *null transverse vector field* of M. It is then noteworthy that the choice of a null transversal vector field N along M determines both the null transverse vector bundle, the screen distribution $\mathscr{S}(N)$, and a unique radical vector field, say ξ, satisfying (2.2).

Before continuing our discussion, we need to clarify the (general) concept of rigging for our null hypersurface.

Definition 2.1. Let M be a null hypersurface in a Lorentzian manifold. A rigging for M is a vector field ζ defined on some open set containing M, such that $\zeta_p \notin T_pM$ for each $p \in M$.

Given a rigging ζ in a neighborhood of M in $(\overline{M}, \overline{g})$, let α denote the 1-form $\overline{g}$-metrically equivalent to ζ, i.e., $\alpha = \overline{g}(\zeta, \ .)$. Take $\omega = i^\star\alpha$, being $i : M \hookrightarrow \overline{M}$ the canonical inclusion. Next, consider the tensors

$$\breve{g} = \overline{g} + \alpha \otimes \alpha \quad \text{and} \quad \widetilde{g} = i^\star \breve{g}. \tag{2.4}$$

It is easy to show that $\widetilde{g}$ defines a Riemannian metric on the (whole) hypersurface M. The *rigged vector field* of ζ is the $\widetilde{g}$-metrically equivalent vector field to the 1-form ω and it is denoted by ξ. In fact, the rigged vector field ξ is the unique lightlike vector field in M, such that $\overline{g}(\zeta, \xi) = 1$. Moreover, ξ is $\widetilde{g}$-unitary. A screen distribution on M is given by $\mathscr{S}(\zeta) = TM \cap \zeta^{\perp}$. It is the $\widetilde{g}$-orthogonal subspace to ξ and the corresponding null transverse vector field to $\mathscr{S}(\zeta)$ is as follows:

$$N = \zeta - \frac{1}{2}\overline{g}(\zeta, \zeta)\xi. \tag{2.5}$$

A null hypersurface M equipped with a rigging ζ is said to be normalized and is denoted (M, ζ) (the latter is called a normalization of the null hypersurface). A normalization (M, ζ) is said to be closed (resp. conformal) if the rigging ζ is closed, i.e., the 1-form α is closed (resp. ζ is a conformal vector field, i.e., there exists a function ρ on the domain of ζ, such that $L_\zeta \overline{g} = 2\rho\overline{g}$). We say that ζ is a *null rigging* for M if the restriction of ζ to the null hypersurface M is a null vector at each point in M.

Let ζ be a rigging for a null hypersurface in a Lorentzian manifold $(\overline{M}, \overline{g})$. The screen distribution $\mathscr{S}(\zeta) = \ker \omega$ is integrable whenever ω is closed, in particular if the rigging is closed. On a normalized null hypersurface (M, ζ), the Gauss and Weingarten formulas are given by the following:

$$\overline{\nabla}_X Y = \nabla_X Y + B(X, Y)N, \tag{2.6}$$
$$\overline{\nabla}_X N = -A_N X + \tau(X)N, \tag{2.7}$$
$$\nabla_X PY = \overset{\star}{\nabla}_X PY + C(X, PY)\xi, \tag{2.8}$$
$$\nabla_X \xi = -\overset{\star}{A}_\xi X - \tau(X)\xi, \tag{2.9}$$

for any $X, Y \in \Gamma(TM)$, where $\overline{\nabla}$ denotes the Levi-Civita connection on $(\overline{M}, \overline{g})$, ∇ denotes the connection on M induced from $\overline{\nabla}$ through the projection along the null transverse vector field N, and $\overset{\star}{\nabla}$ denotes the connection on the screen distribution $\mathscr{S}(\zeta)$ induced from ∇ through the projection morphism P of $\Gamma(TM)$ onto $\Gamma(\mathscr{S}(\zeta))$ with respect to the decomposition (2.1). The $(0, 2)$ tensor B is the null second fundamental form on TM, $\overset{\star}{A}_\xi$ is the shape operator on TM with respect to the rigged vector field ξ, and τ is a 1-form on TM defined by the following:

$$\tau(X) = \overline{g}(\overline{\nabla}_X N, \xi).$$

The null second fundamental form B is symmetric, whereas the tensor C is not in general. The following holds:

$$B(X, Y) = g(\overset{\star}{A}_\xi X, Y), \quad C(X, PY) = g(A_N X, Y) \ \ \forall X, Y \in \Gamma(TM), \tag{2.10}$$

and

$$B(X, \xi) = 0, \quad \overset{\star}{A}_\xi \xi = 0. \tag{2.11}$$

It follows from (2.11) that the integral curves of ξ are pregeodesics in both $\overline{M}$ and (M, ∇), as $\overline{\nabla}_\xi \xi = \nabla_\xi \xi = -\tau(\xi)\xi$.

A null hypersurface M is said to be *totally umbilic* (resp. *totally geodesic*) if there exists a smooth function ρ on M, such that at each $p \in M$ and for all $u, v \in T_pM$, $B(p)(u,v) = \rho(p)\overline{g}(u,v)$ (resp. B vanishes identically on M). These are intrinsic notions on any null hypersurface in the sense that they are independent of the normalization. Note that M is *totally umbilic* (resp. *totally geodesic*) if and only if $\overset{\star}{A}_\xi = \rho P$ (resp. $\overset{\star}{A}_\xi = 0$). It is noteworthy to mention that the shape operators $\overset{\star}{A}_\xi$ and A_N are $\mathscr{S}(\zeta)$-valued.

The induced connection ∇ is torsion-free, but it does not preserve $\overline{g}$ except M is totally geodesic. In fact, we have for all tangent vector fields X, Y, and Z in TM:

$$(\nabla_X \overline{g})(Y,Z) = B(X,Y)\omega(Z) + B(X,Z)\omega(Y). \tag{2.12}$$

The trace of $\overset{\star}{A}_\xi$ is the null mean curvature of M, explicitly given by

$$H_p = \sum_{i=2}^{n+1} \overline{g}(\overset{\star}{A}_\xi(e_i), e_i) = \sum_{i=2}^{n+1} B(e_i, e_i),$$

being $(e_2, \ldots, e_{n+1})$ an orthonormal basis of $\mathscr{S}(\zeta)$ at p.

The (shape) operator $\overset{\star}{A}_\xi$ is self-adjoint as the null second fundamental form B is symmetric. However, this is not the case for the operator A_N as it is shown in the following lemma.

Lemma 2.2 [5]. *For all $X, Y \in \Gamma(TM)$:*

$$\overline{g}(A_N X, Y) - \overline{g}(A_N Y, X) = \tau(X)\alpha(Y) - \tau(Y)\alpha(X) - \mathrm{d}\alpha(X,Y). \tag{2.13}$$

In case the normalization is closed, the 1-form τ is related to the shape operator of M as follows.

Lemma 2.3. *Let (M, ζ) be a closed normalization of a null hypersurface M in a Lorentzian manifold. Then*

$$\tau = -\overline{g}(A_N \xi, \ \cdot\ ) + \tau(\xi)\alpha. \tag{2.14}$$

In particular if $\tau(\xi) = 0$, then $A_N \xi = -\tau^{\sharp_{\overline{g}}}$.

Proof. By the closedness of α, the condition

$$X \cdot \alpha(Y) - Y \cdot \alpha(X) - \alpha([X,Y]) = 0$$

is equivalent to

$$\overline{g}(\overline{\nabla}_X N, Y) = \overline{g}(\overline{\nabla}_Y N, X).$$

Then, by the Weingarten formula, we get

$$\overline{g}(-A_N X, Y) + \tau(X)\alpha(Y) = \overline{g}(-A_N Y, X) + \tau(Y)\alpha(X).$$

In this relation, take $Y = \xi$ to get

$$\tau(X) = -\overline{g}(A_N \xi, X) + \tau(\xi)\alpha(X)$$

which gives the desired formula.

Assume now that $\tau(\xi) = 0$. Then

$$\tau(X) = -\overline{g}(A_N\xi, X) = -\widetilde{g}(A_N\xi, X),$$

for all $X \in TM$, as $A_N\xi \in \mathscr{S}(\zeta)$. Hence, $A_N\xi = -\tau^{\sharp_{\widetilde{g}}}$. □

3. Compact Null Hypersurfaces

The existence of a rigging vector for a null hypersurface is the first step in the rigging technique. It is clear that, in general, it is not possible to choose a rigging vector field, so it is interesting to identify the situations where you cannot choose it. Despite the trivial cases where there is an obstruction due to the existence of a nowhere zero vector, the rigged vector, on a compact null hypersurface which forces it to have zero Euler characteristic, there are more subtle situations that we explore here. Given a compact null hypersurface M in a Lorentzian manifold $(\overline{M}, \overline{g})$, we give some restriction on the geometric properties of the admissible rigging of M due to the topology of M that can prevent the existence of some kind of normalization.

Theorem 3.1. *Let $(\overline{M}, \overline{g})$ be a Lorentzian manifold and M a compact null hypersurface in $\overline{M}$ with trivial first De Rham cohomology group $H^1(M, \mathbb{R})$. Then, M admits no closed normalization.*

Proof. Suppose that M is compact and that $b_1(M) = 0$. If there exists a closed rigging ζ, then the 1-form $\omega = i^\star\alpha$ is closed and there exists a function f on M, such that $df = \omega$; that is, $\widetilde{\nabla} f = \xi$. As a consequence, we have $\widetilde{g}(\widetilde{\nabla} f, \widetilde{\nabla} f) = 1$ which is not possible as f has at least one critical point on the compact manifold M. □

This allows us to prove the following result.

Corollary 3.2. *Let $(\overline{M}, \overline{g})$ be a Lorentzian manifold with a closed timelike vector field. Then, there is no compact simply connected null hypersurface in $\overline{M}$.*

Proof. Let M be a compact null hypersurface in $\overline{M}$. Since $(\overline{M}, \overline{g})$ has a closed timelike vector field say ζ, the later can be used (due to signature considerations) as a rigging for M. If we suppose, in addition, $\pi_1(M) = 0$, which implies that $b_1(M) = 0$, we get a contradiction using above Theorem 3.1. We conclude that if M is compact, then $b_1(M) \geq 1$, and hence, M is not simply connected. □

Remark 3.3. In fact, the above proof shows that if $(\overline{M}, \overline{g})$ has a closed timelike vector field, then there is no compact null hypersurface with trivial first De Rham cohomology group in $\overline{M}$.

Proposition 3.4. *Let $(\overline{M}, \overline{g})$ be a simply connected Lorentzian manifold, then there is no closed normalization for any compact null hypersurface in $\overline{M}$.*

Proof. Suppose that there exists a compact null hypersurface in $\overline{M}$ with a closed normalization ζ, then $\alpha = \overline{g}(\zeta, .)$ is a closed 1-form on $\overline{M}$, and since $\overline{M}$ is simply connected, there exists $f : \overline{M} \longrightarrow \mathbb{R}$, such that $\alpha = df$. This imply that the $\widetilde{g}$-equivalent 1-form ω to the rigged vector field ξ satisfies

$$\omega = i^\star \alpha = i^\star df = d(i^\star f) = d(f \circ i).$$

It follows that $\widetilde{\nabla}(f \circ i) = \xi$ and then $\widetilde{g}(\widetilde{\nabla}(f \circ i), \widetilde{\nabla}(f \circ i)) = 1$ which is a contradiction, because $f \circ i$ has at least one critical point. □

Definition 3.5 [4]. A normalized null hypersurface (M, ζ) of a semi-Riemannian manifold is screen conformal if the shape operators A_N and $\overset{\star}{A}_\xi$ are related by

$$A_N = \varphi \overset{\star}{A}_\xi, \tag{3.1}$$

where φ is a non-vanishing smooth function on M.

As stated in the following theorem, such class of lightlike hypersurfaces has a geometry which is essentially the same as that of their chosen screen distribution.

Theorem 3.6 [4]. *Let $(M, \mathscr{S}(\zeta))$ be a screen conformal lightlike hypersurface of a semi-Riemannian manifold $(\overline{M}, \overline{g})$. Then, the screen distribution is integrable. Moreover, M is totally geodesic (resp. totally umbilical or minimal) in $\overline{M}$ if and only if any leaf M' of $\mathscr{S}(\zeta)$ is totally geodesic (resp. totally umbilical or minimal) in $\overline{M}$ as a codimension 2 non-degenerate submanifold.*

Our aim is to show that, in a four-dimensional Lorentzian manifold, compact null hypersurfaces with finite fundamental group cannot admit such normalizations. We prove first the following:

Proposition 3.7. *Let ζ be a rigging for a compact null hypersurface M in a Lorentzian manifold $(\overline{M}, \overline{g})$ of constant curvature. If the screen $\mathscr{S}(\zeta)$ is conformal and the first De Rham cohomology group $H^1(M, \mathbb{R})$ is trivial, then M is totally geodesic.*

Proof. Since M is screen conformal, there exists a non-vanishing function ρ defined on M, such that $A_N = \rho \overset{\star}{A}_\xi$. From the Gauss–Codazzi equations, see [11, Page 95, Eq. (3.12)]:

$$\begin{aligned} \overline{g}(\overline{R}(X, Y)\xi, N) = {} & C(Y, \overset{\star}{A}_\xi X) - C(X, \overset{\star}{A}_\xi Y) \\ & -2d\tau(X, Y), \quad \forall X, Y \in \Gamma(TM). \end{aligned} \tag{3.2}$$

However, the left-hand side of (3.2) vanishes, since $\overline{M}$ has constant curvature, and ξ is orthogonal to both X and Y. Moreover

$$C(Y, \overset{\star}{A}_\xi X) - C(X, \overset{\star}{A}_\xi Y) = \overline{g}(A_N Y, \overset{\star}{A}_\xi X) - \overline{g}(A_N X, \overset{\star}{A}_\xi Y) = 0,$$

since $A_N = \rho \overset{\star}{A}_\xi$. Using the fact that $H^1(M, \mathbb{R})$ is trivial, there exists a function (say) ϕ defined on M, such that $\tau = d\phi$. Define a new rigging vector field by $\widehat{\zeta} = \exp(-\phi)\zeta$, so $\widehat{N} = \exp(-\phi)N$. Moreover, it follows (from [2, Lemma 2.1]) that $\widehat{\tau} = \tau + d(\ln(\exp(-\phi)) = 0$ as $\tau = d\phi$. Denote, respectively, by $\widehat{\xi}, \widehat{H}, \overset{\star}{A}_{\widehat{\xi}}$ the rigged vector field, the mean curvature function, and the screen shape operator form of $\widehat{\zeta}$, we have [5, Remark 3]

$$\overline{Ric}(\widehat{\xi}) = \widehat{\xi}.(\widehat{H}) + \widehat{\tau}(\widehat{\xi})\widehat{H} - |\overset{\star}{A}_{\widehat{\xi}}|^2.$$

However, $\overline{Ric}(\widehat{\xi}) = 0$, since $\overline{M}$ has constant curvature and $\widehat{\tau}(\widehat{\xi}) = 0$, it follows that $\widehat{\xi}.(\widehat{H}) - |\overset{\star}{A}_{\widehat{\xi}}|^2 = 0$. Using the inequality $|\overset{\star}{A}_{\widehat{\xi}}|^2 \geq \frac{1}{n}\widehat{H}^2$, we obtain $\widehat{\xi}.(\widehat{H}) - \frac{1}{n}\widehat{H}^2 \geq 0$, and since $\widehat{\xi}$ is complete (M being compact), we get that $\widehat{H} = 0$. From the relation $\widehat{\xi}.(\widehat{H}) - |\overset{\star}{A}_{\widehat{\xi}}|^2 = 0$, it follows that $|\overset{\star}{A}_{\widehat{\xi}}|^2 = 0$ which leads to $\overset{\star}{A}_{\widehat{\xi}} = 0$. We conclude that M is totally geodesic. □

We can get now the following result.

Proposition 3.8. *Let $(\overline{M}^4, \overline{g})$ be a 4-dimensional Lorentzian manifold of constant curvature and M be a compact null hypersurface. If M has finite fundamental group, then there is no normalization, such that M is screen conformal.*

Proof. Let M be as above. Suppose that there is a normalization, such that M is screen conformal. Since M has finite fundamental group, the first De Rham cohomology group $H^1(M, \mathbb{R})$ is trivial. It follows from Proposition 3.7 that M is totally geodesic. Elsewhere, M being screen conformal, $\mathscr{S}(\zeta)$ is integrable and induces a foliation on M. We show that the leaves of the screen distribution $\mathscr{S}(\zeta)$ are totally geodesic in $(M, \widetilde{g})$. For this, recall (from [20], Proposition 3.7) that for X and Y in $\mathscr{S}(\zeta)$:

$$\widetilde{\nabla}_X Y = \overset{\star}{\nabla}_X Y - \widetilde{g}(\widetilde{\nabla}_X \xi, Y)\xi,$$

but we also have

$$\widetilde{g}(\widetilde{\nabla}_X \xi, Y) + \widetilde{g}(\widetilde{\nabla}_Y \xi, X) = L_\xi \widetilde{g}(X, Y) = -2B(X, Y).$$

Now, since the screen structure $\mathscr{S}(\zeta)$ is integrable, we have $\widetilde{g}(\widetilde{\nabla}_X \xi, Y) = \widetilde{g}(\widetilde{\nabla}_Y \xi, X)$. It follows that $\widetilde{g}(\widetilde{\nabla}_X \xi, Y) = -B(X, Y)$, which implies that

$$\widetilde{\nabla}_X Y = \overset{\star}{\nabla}_X Y + B(X, Y)\xi.$$

In other words, the second fundamental form of each leaf of $\mathscr{S}(\zeta)$ in $(M, \widetilde{g})$ is B, and then, each of them is totally geodesic in $(M, \widetilde{g})$ as M is totally geodesic in $(\overline{M}^4, \overline{g})$. It follows that there exits a totally geodesic codimension one foliation on the compact 3-manifold M, and hence, M must have infinite fundamental group (see [7]), which is a contradiction. □

4. Completeness of $(M, \widetilde{g})$

On a normalized null hypersurface in a Lorentzian manifold, there is a bridge between the Riemannian geometry of the couple $(M, \widetilde{g})$ and the null geometry of M. The key is to use Riemannian techniques, so it is worth to investigate on its completeness. We consider first this problem in some particular

Lorentzian manifold (Robertson–Walker spaces and generalized Robertson–Walker spaces) and finish with the case of arbitrary Lorentzian manifold.

It is known that a totally umbilic null hypersurface with a closed normalization splits locally as a twisted product; the decomposition being global if M is simply connected and the rigged vector field complete [20, Theorem 5.3]. We show here that if, moreover, it admits a closed conformal rigging in an ambient space form, the local twisted product structure of the rigged metric is, in fact, a warped product. Elsewhere, we show that, in a Robertson–Walker space case, using a specific rigging, we also get warped decomposition of totally umbilic null hypersurfaces. This allows us to state some sufficient conditions for $(M, \widetilde{g})$ to be complete.

Theorem 4.1. *Let $(\overline{M}^{n+2}, \overline{g})$ be a Lorentzian manifold with constant curvature (with $n \geq 2$) and M be a totally umbilic null hypersurface admitting a closed conformal normalization ζ. Then, given $p \in M$, the Riemannian structure$(M, \widetilde{g})$ is locally isometric to a warped product $(\mathbb{R} \times S, dr^2 + f^2 g_0)$, where S is the leaf of $\mathscr{S}(\zeta)$ through p, and g_0 is a conformal metric to $g_{|S}$. Moreover, if M is simply connected and the rigged vector field ξ complete, the decomposition is global.*

Proof. Using [20, Theorem 5.3], the only point which we are going to show is the warped decomposition of $(M, \widetilde{g})$.

In [20, Theorem 4.8] it is shown that, for $U, V \in TM$, the following holds:

$$R(U,V)\xi - \widetilde{R}(U,V)\xi = \overline{g}(\overline{R}(U,V)\xi, N)\xi - \tau(U)\overset{\star}{A}_\xi(V) + \tau(V)\overset{\star}{A}_\xi(U).$$

We also know that, if ζ is closed and conformal, the 1−form τ vanishes identically. Using the Gauss–Codazzi equation, we have $R(U,V)\xi = \overline{R}(U,V)\xi$. Finally, $\overline{R}(U,V)\xi = 0$, since $(\overline{M}, \overline{g})$ has constant curvature and the above equality becomes $\widetilde{R}(U,V)\xi = 0$ for all tangent vector fields U and V. Then, $\widetilde{Ric}(X, \xi) = 0$ for all $\mathscr{S}(\zeta)$−valued vector field X (in fact, for all tangent vector field X). The result follows from the mixed Ricci flat condition [12, Theorem 1]. □

Remark 4.2. The global decomposition of $(M, \widetilde{g})$ as warped product still holds in Theorem 4.1 if M is not simply connected, but the rigging is a gradient vector field (see [20, Remark 5.4]).

In case $\overline{M} = I \times_f L$ is a Robertson–Walker space, we use the classical rigging $\zeta = f\frac{\partial}{\partial t}$ which is a gradient conformal vector field to get the following.

Proposition 4.3. *Let $\overline{M} = I \times_f L$ be a Robertson–Walker space and M a totally umbilic null hypersurface equipped with the (natural) rigging $\zeta = f\frac{\partial}{\partial t}$.*

1. *Then, $(M, \widetilde{g})$ is locally isometric to a warped product. Moreover, if ξ is complete, the decomposition is global.*
2. *If ξ is complete and the screen distribution $\mathscr{S}(\zeta)$ (which is integrable) has compact leaves, then $(M, \widetilde{g})$ is complete.*

Proof. 1. Since $\overline{M} = I \times_f L$ is a Robertson–Walker space and L being of constant curvature c, we know that $\overline{R}(U,V)W = \frac{(f')^2+c}{f^2}(\overline{g}(V,W)U - \overline{g}(U,W)V)$ and $\overline{R}(U,V)\frac{\partial}{\partial t} = 0$ for all U, V, W tangent to the factor L. Using the classical rigging $\zeta = f\frac{\partial}{\partial t}$ which is closed (in fact a gradient) and conformal, decompose the associated rigged vector field as $\xi = a\frac{\partial}{\partial t} + X_0$ with $X_0 \in TL$; we have $\overline{g}(X_0, X) = 0\ \forall\ X \in \mathscr{S}(\zeta)$. Remark also that $\mathscr{S}(\zeta) \subset TL$. Taking into account the above considerations, we get: $\overline{R}(Y,X)\xi = 0, \forall\ X, Y \in \mathscr{S}(\zeta)$. Following the proof of previous Theorem 4.1, we get that $\widetilde{R}(Y,X)\xi = 0, \forall\ X, Y \in \mathscr{S}(\zeta)$, and then, $\widetilde{Ric}(X,\xi) = 0$ for all $\mathscr{S}(\zeta)-$valued vector field X. The conclusion follows as in the last theorem. Taking into account Remark 4.2, the global decomposition holds if ξ is complete.

2. From point 1, $(M, \widetilde{g})$ is globally isometric to a warped product $(\mathbb{R} \times S, dr^2 + f^2 g_0)$, and being S compact, it is complete. □

We study now the $\widetilde{g}$-completeness of null hypersurfaces in generalized Robertson–Walker spaces. Let $\overline{M} = I \times_f L$ be a generalized Robertson–Walker space and M a null hypersurface of $\overline{M}$. Take h to be any primitive of $-f$. Then, $\overline{\nabla}h = f\frac{\partial}{\partial t}$. Using $\zeta = f\frac{\partial}{\partial t}$ as a rigging of M, we get $\widetilde{\nabla}(h \circ i) = \xi$ and $\widetilde{g}(\widetilde{\nabla}(h \circ i), \widetilde{\nabla}(h \circ i)) = 1$, where i is the canonical inclusion of M in $\overline{M}$. Recall, from [14,15], the following important fact: A Riemannian manifold (M, g) is complete if and only if it supports a proper C^3 function say f, such that $g(\nabla f, \nabla f)$ is bounded. Hence, if h is proper on M, then $(M, \widetilde{g})$ is complete. We have shown the following:

Proposition 4.4. *Let $\overline{M} = I \times_f L$ be a generalized Robertson–Walker space and M be a null hypersurface equipped with the rigging $\zeta = f\frac{\partial}{\partial t}$, such that $h \circ i$ is a proper function on M (where h is any primitive of $-f$). Then, its rigged Riemannian structure $(M, \widetilde{g})$ is complete.*

Remark 4.5. In case L is compact, $h : \overline{M} \longrightarrow \mathbb{R}$ is proper if and only if $\overline{M}$ is null complete. Recall also that if $h : \overline{M} \longrightarrow \mathbb{R}$ is proper and M is a closed subset, then $h \circ i : M \longrightarrow \mathbb{R}$ is also proper.

Proposition 4.6. *Let $\overline{M} = I \times_f L$ be a generalized Robertson–Walker space with compact Riemannian factor L. If $\overline{M}$ is null complete, any topologically closed null hypersurface M in $\overline{M}$ is $\widetilde{g}$-complete for the usual rigging $\zeta = f\frac{\partial}{\partial t}$.*

Example 4.7. Consider $\overline{M} = \mathbb{R} \times_{t^2+1} L$ with L compact. It is null complete, and then, any (topologically) closed null hypersurface M in $\overline{M}$ is $\widetilde{g}$-complete for the usual rigging $\zeta = f\frac{\partial}{\partial t}$.

For GRW spaces with complete Riemannian factors, we show the following.

Theorem 4.8. *Let $\overline{M} = \mathbb{R} \times_f L$ be a generalized Robertson–Walker space with complete Riemannian factor (L, g_0) and M be a topologically closed null hypersurface of $\overline{M}$. Then, the Riemannian structure $(M, \widetilde{g})$ induced by the rigging $\zeta = \sqrt{2}\frac{\partial}{\partial t}$ is complete.*

Proof. The Lorentzian metric on $\overline{M}$ is given by $\overline{g} = -\mathrm{d}t^2 + f^2 g_0$. Then, using the rigging $\zeta = \sqrt{2}\dfrac{\partial}{\partial t}$ and the first equality in (2.4), we get

$$\breve{g} = \mathrm{d}t^2 + f^2 g_0,$$

which shows that $(\overline{M}, \breve{g})$ is a complete Riemannian manifold as (L, g_0) is complete. Then, since M is topologically closed, using the second equality in (2.4), we see that $(M, \widetilde{g})$ is a complete Riemannian manifold. □

The following theorem gives some sufficient conditions to get a complete induced Riemannian structure on a given null hypersurface in any Lorentzian manifold. It is an improvement of Proposition 4.3 point 2.

Theorem 4.9. *Let $(\overline{M}^{n+2}, \overline{g})$ be a Lorentzian manifold and (M, ζ) be a closed normalization of a connected non-compact null hypersurface. If ξ is complete and $\mathscr{S}(\zeta)$ has compact leaves, then $(M, \widetilde{g})$ is complete.*

Proof. Let Φ be the flow of ξ. Since ξ is complete, closed with compact orthogonal leaves:

$$\begin{aligned} \Phi : \mathbb{R} \times L &\longrightarrow M \\ (t, p) &\longmapsto \Phi_t(p) \end{aligned}$$

is a diffeomorphism, where L is a leaf of $\mathscr{S}(\zeta)$ [18, Proof of Lemma 3.1], [19, Theorem 4.1]. Suppose the inverse of Φ decomposes as

$$\begin{aligned} \Phi^{-1} : M &\longrightarrow \mathbb{R} \times L \\ x &\longmapsto (f(x), \psi(x)), \end{aligned}$$

then we have $\widetilde{\nabla} f = \xi$, and since ξ is nowhere-vanishing, f is a submersion. Since $\left|\widetilde{\nabla} f\right| = 1$ and ξ is complete, there exists a diffeomorphism $F : \mathbb{R} \times f^{-1}(0) \to M$, [13, Theorem 6.2]. Moreover, $pr_1 : \mathbb{R} \times f^{-1}(0) \to \mathbb{R}$ being the projection on the first factor, we have $f = pr_1 \circ F^{-1}$. By hypothesis, M is connected and F is a diffeomorphism, so $f^{-1}(0)$ is connected too. It follows that $f^{-1}(0)$ is a leaf of $\mathscr{S}(\zeta)$ which is compact, so pr_1 is a proper map. Thus, f is a proper map. Since its gradient is bounded, we conclude that $(M, \widetilde{g})$ is complete. □

Theorem 4.10. *Let $(\overline{M}, \overline{g})$ be a Lorentzian manifold furnished with a proper function f whose gradient is timelike everywhere. Then, for any topologically closed null hypersurface in $\overline{M}$, the rigging $\zeta = \overline{\nabla} f$ makes $(M, \widetilde{g})$ complete.*

Proof. Let us denote by h the restriction of f on M. Since M is closed in $\overline{M}$, h is also a proper function on M. Considering the rigging $\zeta = \overline{\nabla} f$, a straightforward argument shows that $\widetilde{\nabla} h = \xi$, so $\widetilde{g}(\widetilde{\nabla} h, \widetilde{\nabla} h) = 1$. It follows that h is a proper function on M whose gradient is bounded, then $(M, \widetilde{g})$ is complete. □

5. Applications

Let (M, ζ) be a normalized null hypersurface of a Lorentzian manifold $\overline{M}$, we show several results under the hypothesis that $(M, \widetilde{g})$ is a complete Riemannian manifold. In the first part, we show that the non-normalized null mean curvature of M is strongly controlled by the Ricci curvature of $\overline{M}$ evaluated on the associated rigged vector field ξ. We also investigate about mean curvature of null hypersurfaces all of whose screen principal curvatures are constant. The second part deals with null hypersurface with semi-definite shape. Non-existence of closed geodesic in $(M, \widetilde{g})$ is proved for some special cases which allows us to give a classification theorem, e.g., Theorem 5.13 and Corollary 5.14. Finally, we investigate about the existence of topologically closed totally geodesic null hypersurfaces in Robertson–Walker spaces.

5.1. Ricci Estimates and Mean Curvature Boundedness

Given a normalized null hypersurface (M, ζ) of a Lorentzian manifold $\overline{M}$, we prove some results about M under hypothesis on the Ricci curvature of $\overline{M}$ evaluated on the associated rigged vector field ξ.

Theorem 5.1. *Let $(\overline{M}, \overline{g})$ be a Lorentzian manifold and (M, ζ) be a closed normalization of a null hypersurface, such that $\tau(\xi) = 0$. Assume M to be $\widetilde{g}-$complete and there exists a non-negative constant k, such that $\overline{Ric}(\xi) \geq -k$. Then, we have $|H| \leq k$ where H stands for the (non-normalized) mean curvature of M.*

The proof uses the following.

Theorem 5.2 [24]. *Let (M, g) be a complete connected Riemannian manifold, such that there exists $f : M \longrightarrow \mathbb{R}$ satisfying $|\nabla f| = 1$ and $Ric(\nabla f, \nabla f) \geq -k$ (k a non-negative constant), then $|\Delta f| \leq k$.*

We give now the proof of Theorem 5.1.

Proof. Let $\overline{\pi} : (\overline{M}', \overline{g}') \to (\overline{M}, \overline{g})$ be the semi-Riemannian universal covering of $\overline{M}$. Define $M' = \overline{\pi}^{-1}(M)$ which is a null hypersurface, because $\overline{\pi}$ is a local isometry and call $i' : M' \to \overline{M}'$ the canonical inclusion. The closed rigging ζ can be lifted to a closed rigging ζ' on M'. Call $\alpha' = \overline{\pi}^* \alpha$ its equivalent 1-form being α the equivalent 1-form to ζ. The rigged metric on M' is $\widetilde{g}' = i'^*(\overline{g}' + \alpha' \otimes \alpha')$. Using the following commutative diagram:

$$\begin{array}{ccc} \overline{M}' & \xrightarrow{\overline{\pi}} & \overline{M} \\ \uparrow{\scriptstyle i'} & & \uparrow{\scriptstyle i} \\ M' & \xrightarrow{\pi} & M \end{array}$$

where π is the canonical projection from M' onto M, it is clear that $\omega' = \pi^* \omega$, where ω is the equivalent 1-form to the rigged field ξ on M, and $\widetilde{g}' = \pi^* \widetilde{g}$. If we call ξ' the rigged vector field on M' induced by ζ', we have $\pi_* \xi' = \xi \circ \pi$.

Using that π is a local isometry, we have $\tau'(\xi') = 0$; $\widetilde{g}'$ is complete and $\overline{Ric}'(\xi') \geq -k$.

Since $\overline{M}'$ is simply connected and ζ' is closed, we know that there exists $f: M' \longrightarrow \mathbb{R}$, such that $\widetilde{\nabla}' f = \xi'$, and thus, $\widetilde{g}'(\widetilde{\nabla}' f, \widetilde{\nabla}' f) = 1$. Moreover, for a closed normalization, we have (see [20])

$$\overline{Ric}'(\xi') = \widetilde{Ric}'(\xi') + \tau'(\xi')H'$$

being H' the null mean curvature of M'. Since $\tau'(\xi') = 0$, we get $\overline{Ric}'(\xi') = \widetilde{Ric}'(\xi')$. From this, we get $\widetilde{Ric}'(\widetilde{\nabla}' f) = \widetilde{Ric}'(\xi') \geq -k$. Finally, using the $\widetilde{g}'$-completeness and Theorem 5.2, we have $|\widetilde{\Delta}' f| \leq k$ on each connected component of M', so on M' itself. However, $H' = -\widetilde{\operatorname{div}}'(\widetilde{\nabla}' f) = \varepsilon \widetilde{\Delta}' f$ ($\varepsilon = \pm 1$ according to the sign convention of the Laplacian), then $|H'|_{\widetilde{g}'} \leq k$. Using again that π is a local isometry and $H' = \pi^* H$, we get $|H| \leq k$. □

Corollary 5.3. *Let $(\overline{M}^{n+2}, \overline{g})$ be a Lorentzian manifold and (M, ζ) be a closed normalization of a totally umbilic null hypersurface with umbilicity factor ρ (i.e., $B = \rho g$), such that $\tau(\xi) = 0$. If there exists a non-negative constant k, such that $\overline{Ric}(\xi) \geq -k$ and M is $\widetilde{g}$-complete, then it holds $|\rho| \leq \frac{k}{n}$.*

Proof. The proof is straightforward using $H = n\rho$. □

Since any Riemannian metric on a compact manifold is complete, the following also holds:

Corollary 5.4. *Let $(\overline{M}, \overline{g})$ be a Lorentzian manifold and (M, ζ) be a closed normalization of a compact null hypersurface, such that $\tau(\xi) = 0$. Assume there exists a non-negative constant k, such that $\overline{Ric}(\xi) \geq -k$. Then, we have $|H| \leq k$, where H stands for the (non-normalized) mean curvature of M.*

Theorem 5.5. *Let $(\overline{M}^{n+2}, \overline{g})$ be a simply connected Lorentzian manifold and (M, ζ) be a closed normalization of a non-compact null hypersurface, such that $\tau(\xi) = 0$. Suppose that there exists a positive constant k, such that $\overline{Ric}(\xi) \geq -nk^2$. If M is $\widetilde{g}-$complete and $|H| = nk$, then the hypersurface M endowed with the Riemannian structure $\widetilde{g}$ is isometric to the warped product $\mathbb{R} \times_{e^{\pm kt}} Z$, where Z inherits a Riemannian structure from $\overline{M}$. In particular, M is totally umbilic.*

The proof makes use of the following.

Theorem 5.6 [23, Theorem $1 \cdot 1$]. *Let (M, g) be a complete connected Riemannian manifold, such that there exists $f : M \longrightarrow \mathbb{R}$ satisfying $|\nabla f| = 1$. Suppose that $Ric(\nabla f, \nabla f) \geq -n\frac{\phi''(f(x))}{\phi(f(x))}$ (resp $Ric(\nabla f, \nabla f) \geq -n\frac{(\phi^\star)''(f(x))}{\phi^\star(f(x))}$). If $\Delta f = -n\frac{\phi'(f(x))}{\phi(f(x))}$ (resp $\Delta f = n\frac{\phi'(-f(x))}{\phi(-f(x))}$), then*

$$\begin{aligned} \Phi : \mathbb{R} \times_\phi Z &\longrightarrow M \\ (s, p) &\longmapsto \psi_s(p) \end{aligned}$$

is an isometry (resp

$$\begin{aligned} \Phi : \mathbb{R} \times_{\phi^\star} Z &\longrightarrow M \\ (s, p) &\longmapsto \psi_s(p) \end{aligned}$$

is an isometry), where ψ_s is the flow of ∇f, $\phi : \mathbb{R} \longrightarrow \mathbb{R}^+$ a smooth positive function, $\phi^\star(t) = \phi(-t)$, and $Z = f^{-1}(0)$.

Proof of Theorem 5.5. We know that there exists $f : \overline{M} \longrightarrow \mathbb{R}$, such that $\widetilde{\nabla} f = \xi$, and then, $\widetilde{g}(\widetilde{\nabla} f, \widetilde{\nabla} f) = 1$. Let $\phi(t) = e^{kt}$ $(t \in \mathbb{R})$. It follows that $\frac{\phi''(f(x))}{\phi(f(x))} = k^2$ and $\frac{\phi'(f(x))}{\phi(f(x))} = k$. We deduce using the assumption on the Ricci curvature that $\overline{Ric}(\xi) = \widetilde{Ric}(\xi) \geq -n\frac{\phi''(f(x))}{\phi(f(x))}$, that is, $\widetilde{Ric}(\widetilde{\nabla} f, \widetilde{\nabla} f) \geq -n\frac{\phi''(f(x))}{\phi(f(x))}$. Moreover, if $H = -nk$, then $H = \widetilde{\Delta} f = -n\frac{\phi'(f(x))}{\phi(f(x))}$. Using Theorem 5.6, we conclude that $(M, \widetilde{g})$ is isometric to the warped product $\mathbb{R} \times_{e^{kt}} Z$, where $Z = f^{-1}(0)$ is endowed with the induced Riemannian metric. If $H = nk$, $(M, \widetilde{g})$ is isometric to the warped product $\mathbb{R} \times_{e^{-kt}} Z$. Finally, since each Z with the Riemannian structure induced from the warped structure is totally umbilic in $(M, \widetilde{g})$, the null hypersurface M is totally umbilic [20, Corollary 3.14]. □

Theorem 5.7. *Let $(\overline{M}, \overline{g})$ be a simply connected Lorentzian manifold and (M, ζ) be a $\widetilde{g}$-complete closed normalization of a null hypersurface M, all of whose screen principal curvatures are constant. Then, it holds $|H| \leq |\overset{*}{A}_\xi|^2$.*

Proof. We know that $\overline{Ric}(\xi) = \xi(H) + \tau(\xi)H - |\overset{*}{A}_\xi|^2$ [5]. Observe that, since the screen principal curvatures are constant, H and $|\overset{*}{A}_\xi|$ are constant quantities, and that for closed normalizations, $\widetilde{Ric}(\xi) = \overline{Ric}(\xi) - \tau(\xi)H$. Then, $\widetilde{Ric}(\xi) = -|\overset{*}{A}_\xi|^2 = constant$. Using the fact that $\overline{M}$ is simply connected and ζ is closed, we know there exists $f : M \longrightarrow \mathbb{R}$, such that $\widetilde{\nabla} f = \xi$, and thus, $\widetilde{g}(\widetilde{\nabla} f, \widetilde{\nabla} f) = 1$. By Theorem 5.2, it follows that $|\widetilde{\Delta} f| = |H| \leq |\overset{*}{A}_\xi|^2$. □

Corollary 5.8. *Let $(\overline{M}^{n+2}, \overline{g})$ be a simply connected Lorentzian manifold and (M, ζ) be a $\widetilde{g}$-complete closed normalization of a non-totally geodesic null hypersurface M all of whose screen principal curvatures are non-negative constants. Then, at least one of them is greater or equal to 1.*

Proof. Since all the eigenvalues of $\overset{\star}{A}_\xi$ are non-negative, the inequality in Theorem 5.7 becomes $H \leq |\overset{*}{A}_\xi|^2$. Let denote the eigenvalues by λ_i. If we suppose that all of them are less than 1, then we have $\lambda_i \geq \lambda_i^2$. However, as M is non-totally geodesic, there exist i_0, such that λ_{i_0} is positive, and then, $\lambda_{i_0} > \lambda_{i_0}^2$. It follows that $H > |\overset{*}{A}_\xi|^2$, which is in contradiction with $H \leq |\overset{*}{A}_\xi|^2$. □

Corollary 5.9. *Let $(\overline{M}^{n+2}, \overline{g})$ be a simply connected Lorentzian manifold and (M, ζ) a $\widetilde{g}$-complete closed normalization of a proper totally umbilic null hypersurface M with constant umbilicity factor ρ. Then, $|\rho| \geq 1$.*

Proof. Because ρ is constant, all the eigenvalues of $\overset{*}{A}_\xi$ are constant. Using Theorem (5.7), we have $|H| \leq |\overset{*}{A}_\xi|^2$. However, since M is proper totally umbilic, $H = n\rho \neq 0$ and $|\overset{*}{A}_\xi|^2 = n\rho^2$, so the inequality becomes $n|\rho| \leq n\rho^2$, and we get $|\rho| \geq 1$. □

5.2. Null Hypersurfaces with Semi-definite Shape B

Positive definiteness of the second fundamental form of hypersurfaces have many consequences in Riemannian geometry. A well-known theorem due to Hadamard [17] (see also [16, Theorem 2.4]) states that if the second fundamental form of a compact immersed hypersurface M of a Euclidean space is positive definite, then M is embedded as the boundary of a convex body. This also implies an equivalence between definiteness of the second fundamental form and the fact that M is orientable and its (spherical) Gauss map is a diffeomorphism. Equivalently, the Gaussian curvature of M is nowhere-vanishing. In the present section, we present some facts about definiteness of the second fundamental form B and geodesics relative to the rigged Riemannian structure. Recall that if a complete Riemannian manifold supports a convex function, the latter is constant along any closed geodesic [6, Proposition 2.1].

Proposition 5.10. *Let (M,ζ) be a closed normalization of a null hypersurface M in a simply connected Lorentzian manifold $(\overline{M},\overline{g})$. Assume that M is $\widetilde{g}$-complete and B restricts to a definite form on $\mathscr{S}(\zeta)$. Then, $(M,\widetilde{g})$ contains no closed geodesics.*

Proof. By a change of rigging $\zeta \longleftarrow -\zeta$ if necessary, we can suppose without loss of generality that M is connected and that the restriction of B to $\mathscr{S}(\zeta)$ is negative definite which implies that B is negative semi-definite on$M$. In addition, by the simply connectedness of $\overline{M}$ and the closedness of the normalization, we know that there exists $f : M \longrightarrow \mathbb{R}$, such that $\widetilde{\nabla} f = \xi$. Let us remark that fibers of f are leaves of $\mathscr{S}(\zeta)$ (see proof of Theorem 4.9). Using [20, Proposition 3.15.], we have $\widetilde{Hess} f(U,V) = -B(U,V)$ for all $U,V \in TM$. B being negative semi-definite and f is a convex function on M. Suppose that there exists a closed geodesic γ in M. Then, $f \circ \gamma$ is a constant say c (as stated above [6]) and γ is contained in the leaf $f^{-1}(c)$ of $\mathscr{S}(\zeta)$, and hence, $\gamma\prime \in \mathscr{S}(\zeta)$, and then, $\widetilde{Hess} f((\gamma\prime, \gamma\prime) > 0$, which gives the contradiction as $\widetilde{Hess} f(\gamma\prime, \gamma\prime) = (f \circ \gamma)\prime\prime = 0$; $f \circ \gamma$ being constant. We conclude that $(M,\widetilde{g})$ contains no closed geodesics. □

Remark 5.11. The proposition remain true if $\overline{M}$ is not simply connected, but the first De Rham cohomology group $H^1(M,\mathbb{R})$ is trivial or the one form ω is exact, so that the rigged vector field ξ is a gradient vector field.

Since, for proper totally umbilic null hypersurfaces, the restriction of B to the screen structure $\mathscr{S}(\zeta)$ is always definite form, we easily deduce the following.

Corollary 5.12. *Let (M,ζ) be a closed normalization of a proper totally umbilic null hypersurface M in a simply connected Lorentzian manifold $(\overline{M},\overline{g})$, such that M is $\widetilde{g}$-complete. Then, $(M,\widetilde{g})$ contains no closed geodesics.*

The next result gives a restriction on the topology of proper totally umbilic null surface (in 3-dimensional Lorentzian manifold) which can admit a $\widetilde{g}$-complete Riemannian metric for a given rigging.

Theorem 5.13. *Let (M, ζ) be a closed normalization of a null surface M non-totally geodesic at any point in a simply connected 3-dimensional Lorentzian manifold $(\overline{M}^3, \overline{g})$, such that M is $\widetilde{g}$-complete; then, M is homeomorphic to the plane or the cylinder.*

Proof. From Corollary 5.12, the null hypersurface $(M, \widetilde{g})$ contains no closed geodesics. It follows from the classification of complete surfaces without closed geodesic (see [21, Theorem 3.2]) that M is homeomorphic to the plane or the cylinder. □

Corollary 5.14. *Let $\overline{M} = \mathbb{R} \times_f L$ be a 3-dimensional generalized Robertson–Walker (GRW) space with complete Riemannian factor (L, g_0). Any topologically closed null surface and non-totally geodesic at any point is homeomorphic to the plane or the cylinder.*

Proof. Let M be a topologically closed proper totally umbilic null surface. Consider the normalizing rigging $\zeta = \sqrt{2}\frac{\partial}{\partial t}$ for M. Then, from Theorem 4.8, $(M, \widetilde{g})$ is complete. Moreover, the $1-$form ω is exact. From Remark 5.11 and Theorem 5.13, M is homeomorphic to the plane or the cylinder. □

5.3. Totally Geodesic Null Hypersurfaces in Robertson–Walker Spaces

Totally geodesic null hypersurfaces are intensively used in general relativity as they represent horizons of various sorts (Non-expanding horizon, isolated horizon, Killing horizon, etc.). We investigate here the existence of totally geodesic null hypersurfaces in Robertson–Walker spaces. For this, we use the Hilbert theorem which we recall here.

Theorem 5.15 [9,10]. *Let Σ be a complete surface with negative constant Gauss curvature K. Then, there exists no isometric immersion $f : \Sigma \rightarrow \mathbb{M}^3(c)$ (with $K < -1$ for $c = -1$), where $\mathbb{M}^3(c)$ stands for the simply connected complete Riemannian 3-space with constant sectional curvature $c = -1, 0, 1$.*

We prove the following.

Theorem 5.16. *Let $\overline{M} = \mathbb{R} \times_f \mathbb{M}^3(c)$ be a Robertson–Walker space. Then, the followings hold:*

1. *If $c = 0$ or $c = -1$ and f is strictly monotone, then there is no topologically closed totally geodesic null hypersurface in $\overline{M}$.*
2. *If $c = 1$ and $f'(t) > 1 \,\forall\, t$, then there is no topologically closed totally geodesic null hypersurface in $\overline{M}$.*

Proof. Let $\overline{M} = \mathbb{R} \times_f \mathbb{M}^3(c)$ be a Robertson–Walker space. Suppose that there exists a topologically closed totally geodesic null hypersurface M in $\overline{M}$. Consider the normalizing rigging $\zeta = \sqrt{2}\frac{\partial}{\partial t}$ for M. Let $\Pi : \mathbb{R} \times \mathbb{M}^3(c) \rightarrow \mathbb{R}$ be the projection on the first factor. Then, a leaf of $\mathscr{S}(\zeta)$ is the intersection of M with a fiber of Π, and hence, a leaf of $\mathscr{S}(\zeta)$ is a closed subset contained in some slice $\{t_0\} \times \mathbb{M}^3(c)$. Let us call g the Riemannian metric on $\mathbb{M}^3(c)$. Since the Lorentzian metric on $\overline{M}$ is given by $\overline{g} = -\mathrm{d}t^2 + f^2 g$, using the equalities in (2.4), we get that $\widetilde{g}_{|_S} = f^2(t_0)g$, so that $(S, \widehat{g} = \frac{1}{f^2(t_0)}\widetilde{g}_{|_S})$ is a complete surface isometrically immersed in $\mathbb{M}^3(c)$. Elsewhere, the following relation [20] holds:

$$\overline{K}(X,Y) = \widetilde{K}^S(X,Y) - C(X,X)B(Y,Y) - B(X,X)C(Y,Y) + 2C(X,Y)B(X,Y)$$

$\forall\ X, Y \in \mathscr{S}(\zeta)$, where $\overline{K}(X,Y)$ is the sectional curvature in $(\overline{M}, \overline{g})$ and $\widetilde{K}^S(X,Y)$ is the induced sectional curvature from $(M, \widetilde{g})$; that is, $\widetilde{K}^S(X,Y)$ is the Gauss curvature of $(S, \widetilde{g}_{|_S})$. Since the null hypersurface M is totally geodesic, $B = 0$. Hence, $\widetilde{K}^S(X,Y) = \overline{K}(X,Y) = \frac{c-(f')^2}{f^2}$. Note that S being contained in some slice $\{t_0\} \times \mathbb{M}^3(c)$, $\widetilde{K}^S(X,Y) = \frac{c-(f')^2(t_0)}{f^2(t_0)}$ and is constant. Finally, since $\widehat{g} = \frac{1}{f^2(t_0)}\widetilde{g}_{|_S}$, the Gauss curvature of the surface $(S, \widehat{g})$ is $\widehat{K} = c-(f')^2(t_0)$ If $c = 0$ or $c = -1$ and f is strictly monotone or $c = 1$ and $f'(t) > 1\,\forall\ t$, then $(S, \widehat{g})$ has negative constant Gauss curvature (with $\widehat{K} < -1$, in case $c = -1$). The contradiction follows from the Hilbert Theorem. $\square$

In [8, TheoremIV.1], Galloway shows that if a Lorentzian manifold is null complete and satisfies the null convergence condition, then any null line is contained in a smooth (topologically) closed achronal totally geodesic null hypersurface. Therefore, under null completeness and null convergence condition hypothesis, the absence of topologically closed totally geodesic null hypersurface implies the absence of null line. The following holds:

Corollary 5.17. *Let $\overline{M} = \mathbb{R} \times_f \mathbb{M}^3(c)$ be a null complete Robertson–Walker space satisfying the null convergence condition. Then, we have:*

1. *If $c = 0$ or $c = -1$ and f is strictly monotone, $\overline{M}$ contains no lightlike line.*
2. *If $c = 1$ and $f'(t) > 1\,\forall\ t$, $\overline{M}$ contains no lightlike line.*

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