Mediterr. J. Math. (2018) 15:155
https://doi.org/10.1007/s00009-018-1208-7
1660-5446/18/040001-22
published online June 16, 2018

Mediterranean Journal of Mathematics

Asymptotically Almost Periodicity for a Class of Weyl–Liouville fractional Evolution Equations

Junfei Cao, Amar Debbouche and Yong Zhou

Abstract. This paper is devoted to study a class of abstract fractional evolution equation in a Banach space X:

$$\mathrm{D}_+^\alpha x(t) + Ax(t) = F(t, x(t)), \quad t \in \mathbb{R}, \tag{1}$$

where $0 < \alpha < 1$, $-A$ is the infinitesimal generator of a C_0-semigroup on X, and $F(t, x)$ is an appropriate function defined on phase space; the fractional derivative is understood in the Weyl–Liouville sense. Combining the fixed point theorem due to Krasnoselskii and a decomposition technique, we obtain some new sufficient conditions to ensure the existence of asymptotically almost periodic mild solutions for (1). Our result generalizes and improves some previous results, since the Lipschitz continuity on the nonlinearity $F(t, x)$ with respect to x is not required. An example is also presented as an application to illustrate the feasibility of the abstract result.

Mathematics Subject Classification. 34A08, 43A60, 26A33.

Keywords. Fractional evolution equation, asymptotic almost periodicity, Weyl–Liouville fractional derivative.

1. Introduction

The theory of almost periodic functions was introduced in the literature around 1924–1926 with the pioneering works of the Danish mathematician Bohr [1,2]. Loosely speaking, almost periodic functions are those functions which come arbitrarily close to being periodic when one looks over long enough time scales, they play an important role in describing the phenomena that are similar to the periodic oscillations which can be observed frequently in many fields, such as celestial mechanics, nonlinear vibration, electromagnetic theory, plasma physics, engineering, ecosphere, and so on [3–5].

As a natural extension of almost periodicity, the concept of asymptotic almost periodicity, which was the central issue to be discussed in this paper, was introduced in the literature [6,7] by Fréchet in the early 1940s. Since then, the theory of asymptotically almost periodic functions and their various extensions has attracted a great deal of attention of many mathematicians due to both their mathematical interest and significance as well as applications in physics, mathematical biology, control theory, and so forth. In particular, asymptotically almost periodic functions have been utilized to study various ordinary differential equations, partial differential equations, functional differential equations, integro-differential equations, as well as stochastic differential equations (see, for instance, [8–14] and the references therein), and due to their significance and applications in control theory, mathematical biology, physics, etc., the study of asymptotically almost periodic solutions to various differential equations becomes an attractive topic in the qualitative theory of differential equations.

With motivation coming from a wide range of engineering and physical applications, fractional differential equations have recently attracted great attention of mathematicians and scientists. This kind of equation is a generalization of ordinary differential equations to arbitrary noninteger orders. Fractional differential equations find numerous applications in the field of viscoelasticity, feedback amplifiers, electrical circuits, electro analytical chemistry, fractional multipoles, neuron modeling encompassing different branches of physics, chemistry, and biological sciences [15–18]. Many physical processes appear to exhibit fractional order behavior that may vary with time or space. In recent years, there has been a significant development in ordinary and partial differential equations involving fractional derivatives; we only enumerate here the monographs of Kilbas et al. [15], Podlubny [16], Miller [17], Zhou [18], and a series of papers [19–30], and the references therein. The study of almost periodic type solutions to fractional differential equations was initiated by Araya and Lizama [31]. In their work, they investigated the existence and uniqueness of an almost automorphic mild solution of the semilinear fractional differential equation:

$$\mathrm{D}_t^\alpha x(t) = Ax(t) + F(t, x(t)), \quad t \in \mathbb{R}, \quad 1 < \alpha < 2,$$

when A is a generator of an α-resolvent family and D_t^α is the Riemann–Liouville fractional derivative. For more on almost periodic type solutions to fractional differential equations, one can refer to [32–39] and the references therein. Especially, very recently, in [39], Mu, Zhou, and Peng considered the fractional differential equation in an ordered Banach space X:

$$\mathrm{D}_+^\alpha x(t) + Ax(t) = F(t, x(t)), \quad t \in \mathbb{R}, \tag{2}$$

where $0 < \alpha < 1$, $-A$ is the infinitesimal generator of a C_0-semigroup on X, and $F(t, x)$ is an appropriate function defined on phase space; the fractional derivative is understood in the Weyl–Liouville sense. Applying Fourier transform, they first gave reasonable definition of mild solutions of Eq. (2). Then, they established the existence and uniqueness results for the corresponding linear fractional evolution equations, and accurately estimated the spectral radius of resolvent operator. Finally, they established some sufficient conditions for the existence and uniqueness of periodic solutions, asymptotically periodic solutions, asymptotically almost periodic solutions, asymptotically almost automorphic solutions, and other types of bounded solutions when $F(t,x)$ satisfies some ordered or Lipschitz conditions.

To the best of our knowledge, much less is known about the existence of asymptotically almost periodic solutions to Eq. (2) when the nonlinearity $F(t,x)$ as a whole loses the Lipschitz continuity with respect to x. Motivated by the above mentioned works, the purpose of this paper is to establish a new existence result of asymptotically almost periodic mild solutions to the Eq. (2). In our result, the nonlinearity $F(t,x)$ does not have to satisfy a Lipschitz condition with respect to x (see Remark 3.5). As can be seen, the hypotheses in our result are reasonably weak (see Remark 3.7), and our result generalizes those as well as related research and has more broad applications. In particular, as application and to illustrate the feasibility of the abstract result, we will examine some sufficient conditions for the existence of asymptotically almost periodic mild solutions to the fractional partial differential equation given by the following:

$$\partial_t^\alpha u(t,x) = \partial_x^2 u(t,x) + \mu(\sin t + \sin\sqrt{2}t)\sin u(t,x) \\ + \nu e^{-|t|}u(t,x)\sin u^2(t,x), t \in \mathbb{R}, \quad x \in [0,\pi],$$

with Dirichlet boundary conditions $u(t,0) = u(t,\pi) = 0$, $t \in \mathbb{R}$, where μ and ν are constants.

The rest of this paper is organized as follows. In Sect. 2, some concepts, the related notations, and some useful lemmas are introduced. In Sect. 3, we present some criteria ensuring the existence of asymptotically almost periodic mild solutions. An example is given to illustrate the feasibility of the abstract result in Sect. 4.

2. Preliminaries

This section is concerned with some notations, definitions, lemmas, and preliminary facts which are used in what follows.

Throughout this work, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ stand for the set of natural numbers, integral numbers, real numbers, and complex numbers, respectively. Let $(X, \|\cdot\|)$, $(Y, \|\cdot\|_Y)$ be two Banach spaces, $BC(\mathbb{R}, X)$ (resp., $BC(\mathbb{R}\times Y, X)$) is the space of all X-valued bounded continuous functions (resp., jointly bounded continuous functions $F : \mathbb{R}\times Y \to X$). Furthermore, $C_0(\mathbb{R}, X)$ (resp., $C_0(\mathbb{R}\times Y, X)$) is the closed subspace of $BC(\mathbb{R}, X)$ (resp., $BC(\mathbb{R}\times Y, X)$) consisting of functions vanishing at infinity (vanishing at infinity uniformly in any compact subset of Y, and in other words:

$$\lim_{|t|\to+\infty} \|g(t,x)\| = 0 \text{ uniformly for } x \in \mathbb{K},$$

where $\mathbb{K}$ is an any compact subset of Y). Let also $\mathbb{L}(X)$ be the Banach space of all bounded linear operators from X into itself endowed with the norm:

$$\|T\|_{\mathbb{L}(X)} = \sup\{\|Tx\| : x \in X,\ \|x\| = 1\}.$$

First, we recall some basic definitions and results on almost periodic and asymptotically almost periodic functions.

Definition 2.1. [1,2, Bohr] A continuous function $F:\mathbb{R}\to X$ is said to be (Bohr) almost periodic in $t\in\mathbb{R}$ if, for every $\varepsilon>0$, there exists $l(\varepsilon)>0$, such that every interval of length $l(\varepsilon)$ contains a number τ with the property that:

$$\|F(t+\tau)-F(t)\|<\varepsilon \quad \text{for every} \quad t\in\mathbb{R}.$$

The number τ is called an ε-translation number of $F(t)$ and the collection of those functions is denoted by $AP(\mathbb{R},X)$.

Lemma 2.2. [12] *$AP(\mathbb{R},X)$ is a Banach space with the norm $\|F\|_\infty=\sup_{t\in\mathbb{R}}\|F(t)\|$.*

Definition 2.3. [12] A function $F:\mathbb{R}\times Y\to X$ is said to be almost periodic if $F(t,x)$ is almost periodic in $t\in\mathbb{R}$ uniformly for $x\in\mathbb{K}$, where $\mathbb{K}$ is any compact subset of Y.
The collection of those functions is denoted by $AP(\mathbb{R}\times Y,X)$.

Lemma 2.4. [12] *Let $F(t,x)\in AP(\mathbb{R}\times X,X)$ and $\varphi(t)\in AP(\mathbb{R},X)$, then $\Phi(t)=F(t,\varphi(t))$ belongs to $AP(\mathbb{R},X)$.*

Definition 2.5. [6,7, Fréchet] A continuous function $F:\mathbb{R}\to X$ is said to be asymptotically almost periodic if it can be decomposed as $F(t)=G(t)+\Phi(t)$, where

$$G(t)\in AP(\mathbb{R},X),\ \ \Phi(t)\in C_0(\mathbb{R},X).$$

Denote by $AAP(\mathbb{R},X)$ the set of all such functions.

Lemma 2.6. *[12] $AAP(\mathbb{R},X)$ is also a Banach space with the norm $\|\cdot\|_\infty$.*

Definition 2.7. [12] A function $F:\mathbb{R}\times Y\to X$ is said to be asymptotically almost periodic if it can be decomposed as $F(t,x)=G(t,x)+\Phi(t,x)$, where

$$G(t,x)\in AP(\mathbb{R}\times Y,X),\quad \Phi(t,x)\in C_0(\mathbb{R}\times Y,X).$$

Denote by $AAP(\mathbb{R}\times Y,X)$ the set of all such functions.

Next, we recall the definitions of some fractional derivatives and integrals which are used in this paper (see [40]).

Definition 2.8. [40] Let $f\in L^p(\mathbb{R}/2\pi\mathbb{Z})$ $(1\le p<+\infty)$ be a periodic function with period 2π and the property that its integral over a period vanishes. The Weyl fractional integral of order α is defined as follows:

$$(I^\alpha_\pm f)(t)=\frac{1}{2\pi}\int_0^{2\pi}\Psi^\alpha_\pm(t-s)f(s)\mathrm{d}s,$$

where

$$\Psi^\alpha_\pm(t)=\sum_{k=-\infty,k\neq0}^{\infty}\frac{e^{ikt}}{(\pm ik)^\alpha}\quad \text{for } 0<\alpha<1.$$

The above Weyl definition is accordant with the Riemann–Liouville definition [15]:

$$(I_+^{\alpha} f)(t) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{t} (t-s)^{\alpha-1} f(s)\mathrm{d}s, \ (I_-^{\alpha} f)(t) = \frac{1}{\Gamma(\alpha)} \int_{t}^{\infty} (t-s)^{\alpha-1} f(s)\mathrm{d}s$$

for 2π periodic functions whose integrals over a period vanish, see [39].

Definition 2.9. [40] The Weyl–Liouville fractional derivative is defined as

$$(D_{\pm}^{\alpha} f)(t) = \pm\frac{\mathrm{d}}{\mathrm{d}t}(I_{\pm}^{1-\alpha} f)(t) \quad \text{for } 0 < \alpha < 1.$$

It is shown that the Weyl–Liouville derivative $(0 < \alpha < 1)$

$$(D_+^{\alpha} f)(t) = \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \int_{-\infty}^{t} (t-s)^{-\alpha} f(s)\mathrm{d}s$$

coincides with the Caputo, Riemann–Liouville, and Grunwald–Letnikov derivative with lower limit $-\infty$ [16]. It is known that $D_{\pm}^{\alpha+\beta} = D_{\pm}^{\alpha}(D_{\pm}^{\beta})$ for any $\alpha, \beta \in \mathbb{R}$, where $D_{\pm}^0 = Id$ denotes the identity operator and $(-1)^n D_-^n = D_+^n = \mathrm{d}^n/\mathrm{d}t^n$ holds with $n \in \mathbb{N}$, see [17].

Let us now recall the definitions and properties of semigroups of linear operators (see [41] for details) and a new operator which was introduced in [39].

Assume that $-A$ is the infinitesimal generator of a C_0-semigroup $\{T(t)\}_{t\geq 0}$. If there are $M \geq 0$ and $\nu \in \mathbb{R}$, such that $\|T(t)\|_{\mathbb{L}(X)} \leq Me^{\nu t}$, then

$$(\lambda I + A)^{-1}x = \int_0^{\infty} e^{-\lambda t} T(t)x\mathrm{d}t, \ \mathrm{Re}\lambda > \nu, \quad x \in X.$$

A C_0-semigroup $\{T(t)\}_{t\geq 0}$ is said to be uniformly exponentially stable if there exist two constants $M, \delta > 0$, such that

$$\|T(t)\| \leq Me^{-\delta t} \quad \text{for all } t \geq 0. \tag{3}$$

In addition, a C_0-semigroup $\{T(t)\}_{t\geq 0}$ is said to be uniformly bounded if there exists a constant $M > 0$, such that

$$\|T(t)\| \leq M \quad \text{for all} \quad t \geq 0. \tag{4}$$

Let

$$V(t) = \alpha \int_0^{\infty} \theta \zeta_{\alpha}(\theta) T(t^{\alpha}\theta)\mathrm{d}\theta, \quad t \geq 0, \tag{5}$$

where $\{T(t)\}_{t\geq 0}$ is a C_0-semigroup and $\zeta_{\alpha}(\theta)$ is a probability density function with

$$\zeta_{\alpha}(\theta) = \frac{1}{\pi\alpha} \sum_{n=0}^{\infty} (-1)^{n-1} \theta^{n-1} \frac{\Gamma(n\alpha+1)}{n!} \sin(n\pi\alpha).$$

One has the following results.

Lemma 2.10. [39]

1. *Assume that* $\{T(t)\}_{t\geq 0}$ *is a uniformly bounded* C_0*-semigroup satisfying* (4)*. Then, for any fixed* $t \geq 0$*,* $V(t)$ *is a linear and bounded operator, that is, for any* $x \in X$*, one has* $\|V(t)x\| \leq M\|x\|/\Gamma(\alpha)$*.*
2. *If* $\{T(t)\}_{t\geq 0}$ *is a* C_0*-semigroup, then* $\{V(t)\}_{t\geq 0}$ *is strongly continuous.*
3. *If* $\{T(t)\}_{t\geq 0}$ *is exponentially stable satisfying (3), then*
$$\|V(t)\|_{\mathbb{L}(X)} \leq ME_{\alpha,\alpha}(-\delta t^{\alpha}) \text{ for all } t \geq 0.$$

In the following, we recall the definitions and properties of Mittag–Lefer functions [15]:

$$E_{\alpha}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k + 1)}, \quad E_{\alpha,\alpha}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha(k+1))}, \quad t \in \mathbb{C}.$$

These functions have the following properties for $\alpha \in (0,1)$ and $t \in \mathbb{R}$:

Lemma 2.11. [42]

1. $E_{\alpha}(t) > 0$, $E_{\alpha,\alpha}(t) > 0$.
2. [43] $(E_{\alpha}(t))' = (1/\alpha)E_{\alpha,\alpha}(t)$.
3. [44,45] $\lim_{t\to-\infty} E_{\alpha}(t) = \lim_{t\to-\infty} E_{\alpha,\alpha}(t) = 0$.

Now, we present the following compactness criterion, which is a special case of the general compactness result of Theorem 2.1 in [37].

Lemma 2.12. [46] *A set* $D \subset C_0(\mathbb{R}, X)$ *is relatively compact if*

1. D *is equicontinuous.*
2. $\lim_{|t|\to+\infty} x(t) = 0$ *uniformly for* $x \in D$*.*
3. *the set* $D(t) := \{x(t) : x \in D\}$ *is relatively compact in* X *for every* $t \in \mathbb{R}$*.*

The following Krasnoselskii's fixed point theorem plays a key role in the proofs of our main results, which can be found in many books.

Lemma 2.13. [47] *Let* B *be a bounded closed and convex subset of* X*, and* J_1, J_2 *be maps of* B *into* X*, such that* $J_1x + J_2y \in B$ *for every pair* $x, y \in B$*. If* J_1 *is a contraction and* J_2 *is completely continuous, then the equation* $J_1x + J_2x = x$ *has a solution on* B*.*

3. Asymptotically Almost Periodic Mild Solutions

In this section, we study the existence of asymptotically almost periodic mild solutions for the fractional evolution equations in a Banach space X of the form:

$$\mathrm{D}_{+}^{\alpha}x(t) + Ax(t) = F(t, x(t)), \ t \in \mathbb{R}, \tag{6}$$

where the fractional derivative is understood in the Weyl–Liouville sense, $0 < \alpha < 1$, $-A$ is the infinitesimal generator of a C_0-semigroup on X, and $F : \mathbb{R} \times X \to X$ is a given function to be specified later.

In [39], applying Fourier transform, Mu, Zhou, and Peng proved the following Lemma.

Lemma 3.1. *Assume that $-A$ generates an exponentially stable C_0-semigroup $\{T(t)\}_{t\geq 0}$. If $x:\mathbb{R}\to X$ is a function satisfying the equation $D_+^\alpha x(t)+Ax(t)=F(t)$ for $t\in\mathbb{R}$, then x satisfies the integral equation:*

$$x(t)=\int_{-\infty}^{t}(t-s)^{\alpha-1}V(t-s)F(s)\mathrm{d}s,\ t\in\mathbb{R},$$

where $V(t)$ is defined by (5).

The above Lemma motivates the following definition of mild solution to Eq. (6), which is a reasonable definition and also given in [39], it is essential for us.

Definition 3.2. A function $x:\mathbb{R}\to X$ is said to be a mild solution to Eq. (6) if

$$x(t)=\int_{-\infty}^{t}(t-s)^{\alpha-1}V(t-s)F(s,x(s))\mathrm{d}s,\quad t\in\mathbb{R},$$

where $V(t)$ is given by (5).

In the proof of our result, we need the following auxiliary results.

Lemma 3.3. *Assume that $\{T(t)\}_{t\geq 0}$ is exponentially stable satisfying (3). Given $Y(t)\in AP(\mathbb{R},X)$. Let*

$$\Phi_1(t):=\int_{-\infty}^{t}(t-s)^{\alpha-1}V(t-s)Y(s)\mathrm{d}s,\quad t\in\mathbb{R}.$$

Then, $\Phi_1(t)\in AP(\mathbb{R},X)$.

Proof. First, from Lemmas 2.10(3), 2.11, and $E_\alpha(0)=1$, it follows that:

$$\begin{aligned}\|\Phi_1(t)\|&=\left\|\int_{-\infty}^{t}(t-s)^{\alpha-1}V(t-s)Y(s)\mathrm{d}s\right\|\\&\leq\|Y\|_\infty\left\|\int_{-\infty}^{t}(t-s)^{\alpha-1}V(t-s)\mathrm{d}s\right\|_{\mathbb{L}(X)}\\&\leq M\|Y\|_\infty\int_{-\infty}^{t}(t-s)^{\alpha-1}E_{\alpha,\alpha}(-\delta(t-s)^\alpha)\mathrm{d}s\\&=M\|Y\|_\infty E_\alpha(-\delta(t-s)^\alpha)\Big|_{-\infty}^{t}\\&=\frac{M\|Y\|_\infty}{\delta},\end{aligned}$$

which implies $\Phi_1(t)$ is well defined and continuous on $\mathbb{R}$. Since $Y(t)\in AP(\mathbb{R},X)$, then for every $\varepsilon>0$, there exists $l(\varepsilon)>0$, such that every interval of length $l(\varepsilon)$ contains a number τ with the property that:

$$\|Y(s+\tau)-Y(s)\|<\varepsilon\text{ for every }s\in\mathbb{R},$$

which together with Lemmas 2.10(3), 2.11, and $E_\alpha(0) = 1$, implies that

$$\begin{aligned}\|\Phi_1(t+\tau) - \Phi_1(t)\| &= \left\| \int_{-\infty}^{t+\tau} (t+\tau-s)^{\alpha-1} V(t+\tau-s) Y(s) \mathrm{d}s \right. \\ &\quad \left. - \int_{-\infty}^{t} (t-s)^{\alpha-1} V(t-s) Y(s) \mathrm{d}s \right\| \\ &= \left\| \int_{-\infty}^{t} (t-s)^{\alpha-1} V(t-s) Y(s+\tau) \mathrm{d}s \right. \\ &\quad \left. - \int_{-\infty}^{t} (t-s)^{\alpha-1} V(t-s) Y(s) \mathrm{d}s \right\| \\ &\leq \int_{-\infty}^{t} \|(t-s)^{\alpha-1} V(t-s) [Y(s+\tau) - Y(s)]\| \mathrm{d}s \\ &\leq \varepsilon \left\| \int_{-\infty}^{t} (t-s)^{\alpha-1} V(t-s) \mathrm{d}s \right\|_{\mathbb{L}(X)} \\ &\leq M\varepsilon \int_{-\infty}^{t} (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\delta(t-s)^\alpha) \mathrm{d}s \\ &= M\varepsilon E_\alpha(-\delta(t-s)^\alpha) \Big|_{-\infty}^{t} = \frac{M\varepsilon}{\delta},\end{aligned}$$

which implies that $\Phi_1(t) \in AP(\mathbb{R}, X)$. □

Lemma 3.4. *Assume that* $\{T(t)\}_{t\geq 0}$ *is exponentially stable satisfying* (3). *Given* $Z(t) \in C_0(\mathbb{R}, X)$. *Let*

$$\Phi_2(t) := \int_{-\infty}^{t} (t-s)^{\alpha-1} V(t-s) Z(s) \mathrm{d}s, \quad t \in \mathbb{R}.$$

Then, $\Phi_2(t) \in C_0(\mathbb{R}, X)$.

Proof. First, similar to the proof of Lemma 3.3, it is easy to see that $\Phi_2(t)$ is well defined and continuous on $\mathbb{R}$. Since $Z(t) \in C_0(\mathbb{R}, X)$, one can choose an $N_1 > 0$, such that $\|Z(t)\| < \varepsilon$ for all $t > N_1$. This together with Lemma 2.10(3), Lemma 2.11, and $E_\alpha(0) = 1$, enables us to conclude that for all $t > N_1$:

$$\begin{aligned}\|\Phi_2(t)\| &\leq \left\| \int_{-\infty}^{N_1} (t-s)^{\alpha-1} V(t-s) Z(s) \mathrm{d}s \right\| \\ &\quad + \left\| \int_{N_1}^{t} (t-s)^{\alpha-1} V(t-s) Z(s) \mathrm{d}s \right\| \\ &\leq \|Z\|_\infty \left\| \int_{-\infty}^{N_1} (t-s)^{\alpha-1} V(t-s) \mathrm{d}s \right\|_{\mathbb{L}(X)} \\ &\quad + \varepsilon \left\| \int_{N_1}^{t} (t-s)^{\alpha-1} V(t-s) \mathrm{d}s \right\|_{\mathbb{L}(X)}\end{aligned}$$

$$\begin{aligned}
&\le M\|Z\|_\infty \int_{-\infty}^{N_1} (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\delta(t-s)^\alpha)\mathrm{d}s\\
&\quad + M\varepsilon \int_{N_1}^{t} (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\delta(t-s)^\alpha)\mathrm{d}s\\
&= M\|Z\|_\infty E_\alpha(-\delta(t-s)^\alpha)\Big|_{-\infty}^{N_1} + M\varepsilon E_\alpha(-\delta(t-s)^\alpha)\Big|_{N_1}^{t}\\
&\le M\|Z\|_\infty E_\alpha(-\delta(t-N_1)^\alpha) + \frac{M\varepsilon}{\delta},
\end{aligned}$$

which together with Lemma 2.11(3), implies that $\lim_{t\to+\infty}\|\Phi_2(t)\| = 0$.

On the other hand, from $Z(t) \in C_0(\mathbb{R}, X)$, it follows that there exists an $N_2 > 0$, such that $\|Z(t)\| < \varepsilon$ for all $t < -N_2$. This together with Lemma 2.10(3), Lemma 2.11 and $E_\alpha(0) = 1$, enables us to conclude that for all $t < -N_2$:

$$\begin{aligned}
\|\Phi_2(t)\| &= \left\| \int_{-\infty}^{t} (t-s)^{\alpha-1} V(t-s) Z(s)\mathrm{d}s \right\|\\
&\le \varepsilon \left\| \int_{-\infty}^{t} (t-s)^{\alpha-1} V(t-s)\mathrm{d}s \right\|_{\mathbb{L}(X)}\\
&\le M\varepsilon \int_{-\infty}^{t} (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\delta(t-s)^\alpha)\mathrm{d}s\\
&= M\varepsilon E_\alpha(-\delta(t-s)^\alpha)\Big|_{-\infty}^{t} = \frac{M\varepsilon}{\delta},
\end{aligned}$$

which implies that $\lim_{t\to-\infty}\|\Phi_2(t)\| = 0$. □

Now, we are in position to state and prove our main result. For that, let us introduce the following assumptions:

(H_1) $F(t,x) = F_1(t,x) + F_2(t,x) \in AAP(\mathbb{R}\times X, X)$ with

$$F_1(t,x) \in AP(\mathbb{R}\times X, X),\ F_2(t,x) \in C_0(\mathbb{R}\times X, X),$$

and there exists a constant $L > 0$, such that

$$\|F_1(t,x) - F_1(t,y)\| \le L\|x-y\| \quad \text{for all } t\in\mathbb{R},\ x,y\in X. \tag{7}$$

(H_2) There exist a function $\beta(t) \in C_0(\mathbb{R}, \mathbb{R}^+)$ and a nondecreasing function $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$, such that for all $t\in\mathbb{R}$ and $x \in X$ with $\|x\| \le r$:

$$\|F_2(t,x)\| \le \beta(t)\Phi(r) \ \text{ and } \ \liminf_{r\to+\infty} \frac{\Phi(r)}{r} = \rho_1. \tag{8}$$

Remark 3.5. Assuming that $F(t,x)$ satisfies the assumption (H_1), it is noted that $F(t,x)$ does not have to meet the Lipschitz continuity with respect to x. Such class of asymptotically almost periodic functions $F(t,x)$ is more complicated than those with Lipschitz continuity and little is known about them.

Lemma 3.6. *Given* $F(t,x) = F_1(t,x) + F_2(t,x) \in AAP(\mathbb{R}\times X, X)$ *with*

$$F_1(t,x) \in AP(\mathbb{R}\times X, X), \quad F_2(t,x) \in C_0(\mathbb{R}\times X, X).$$

Then, it yields that

$$\sup_{t\in\mathbb{R}} \|F_1(t,x) - F_1(t,y)\| \le \sup_{t\in\mathbb{R}} \|F(t,x) - F(t,y)\|, \quad x, y \in X. \tag{9}$$

Proof. To show this result, it suffices to verify that

$$\{F_1(t,x) - F_1(t,y) : t \in \mathbb{R}\} \subset \overline{\{F(t,x) - F(t,y) : t \in \mathbb{R}\}}, \quad x, y \in X.$$

If this is not the case, then, for fixed $x, y \in X$, there exist some $t_0 \in \mathbb{R}$ and $\varepsilon > 0$, such that

$$\|(F_1(t_0,x) - F_1(t_0,y)) - (F(t,x) - F(t,y))\| \ge 3\varepsilon \quad \text{for all } t \in \mathbb{R}.$$

It is clear that $\lim_{t\to+\infty} \|F_2(t,x) - F_2(t,y)\| = 0$, which implies that there exists a positive number T, such that for all $t \ge T$:

$$\|F_2(t,x) - F_2(t,y)\| < \varepsilon. \tag{10}$$

Since $F_1(t,x) \in AP(\mathbb{R}\times X, X)$, one can take $l = l(\varepsilon) > 0$, such that $[T, T+l]$ of length l contains at least a τ with the properties:

$$\|F_1(t_0+\tau,x) - F_1(t_0,x)\| < \varepsilon, \quad \|F_1(t_0+\tau,y) - F_1(t_0,y)\| < \varepsilon,$$

which enable us to find that

$$\begin{aligned}\|F_2(t_0+\tau,x) - F_2(t_0+\tau,y)\| \ge\ & \|F(t_0+\tau,x) - F(t_0+\tau,y) - F_1(t_0,x) + F_1(t_0,y)\| \\ & -\|F_1(t_0+\tau,x) - F_1(t_0,x)\| \\ & -\|F_1(t_0+\tau,y) - F_1(t_0,y)\| \ge \varepsilon,\end{aligned}$$

which contradicts (10), completing the proof. □

Remark 3.7. In Lemma 3.6, (9) implies that when $F(t,x)$ meets the Lipschitz continuity with respect to x with Lipschitz constant L, then $F_1(t,x)$ satisfies (7). Note that in [39], to be able to apply the well-known Banach contraction principle, a Lipschitz condition for the nonlinearity $F(t,x)$ of Eq. (6) is needed. Thus, our condition in the assumption (H_1) is weaker than those of [39].

Let $\beta(t)$ be the function involved in assumption (H_2). Define

$$\sigma(t) := \int_{-\infty}^{t} \beta(s)(t-s)^{\alpha-1} E_{\alpha,\alpha}(-\delta(t-s)^{\alpha})\mathrm{d}s, \quad t \in \mathbb{R}.$$

Lemma 3.8. $\sigma(t) \in C_0(\mathbb{R}, \mathbb{R}^+)$.

Proof. Since $\beta(t) \in C_0(\mathbb{R}, \mathbb{R}^+)$, one can choose a $T_1 > 0$, such that $\|\beta(t)\| < \varepsilon$ for all $t > T_1$. This together with Lemmas 2.10(3), 2.11, and $E_\alpha(0) = 1$ enables us to conclude that for all $t > T_1$:

$$\begin{aligned}\sigma(t) &\le \int_{-\infty}^{T_1} \beta(s)(t-s)^{\alpha-1}E_{\alpha,\alpha}(-\delta(t-s)^{\alpha})\mathrm{d}s \\ &\quad + \int_{T_1}^{t} \beta(s)(t-s)^{\alpha-1}E_{\alpha,\alpha}(-\delta(t-s)^{\alpha})\mathrm{d}s \\ &\le \|\beta\|_\infty \int_{-\infty}^{T_1} (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\delta(t-s)^{\alpha})\mathrm{d}s \\ &\quad + M\varepsilon \int_{T_1}^{t} (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\delta(t-s)^{\alpha})\mathrm{d}s \\ &= M\|\beta\|_\infty E_\alpha(-\delta(t-s)^\alpha)\Big|_{-\infty}^{T_1} + M\varepsilon E_\alpha(-\delta(t-s)^\alpha)\Big|_{T_1}^{t} \\ &\le M\|\beta\|_\infty E_\alpha(-\delta(t-T_1)^\alpha) + \frac{M\varepsilon}{\delta},\end{aligned}$$

which, together with Lemma 2.11(3), implies $\lim_{t\to+\infty}\sigma(t)=0$.

On the other hand, from $\beta(t)\in C_0(\mathbb{R},\mathbb{R}^+)$, it follows that there exists a $T_2>0$, such that $\|\beta(t)\|<\varepsilon$ for all $t<-T_2$. This together with Lemma 2.10(3), Lemma 2.11, and $E_\alpha(0)=1$ enables us to conclude that for all $t<-T_2$,

$$\sigma(t)\le \varepsilon\int_{-\infty}^{t}(t-s)^{\alpha-1}E_{\alpha,\alpha}(-\delta(t-s)^\alpha)\mathrm{d}s = \varepsilon E_\alpha(-\delta(t-s)^\alpha)\Big|_{-\infty}^{t} = \frac{M\varepsilon}{\delta},$$

which implies $\lim_{t\to-\infty}\sigma(t)=0$. □

Theorem 3.9. *Assume that $-A$ generates an exponentially stable C_0-semigroup $\{T(t)\}_{t\ge 0}$ satisfying (3). Let $F:\mathbb{R}\times X\to X$ satisfy the hypotheses (H_1) and (H_2). Put $\rho_2 := \sup_{t\in\mathbb{R}}\sigma(t)$. Then, Eq. (6) has at least one asymptotically almost periodic mild solution whenever*

$$ML\delta^{-1} + M\rho_1\rho_2 < 1. \tag{11}$$

Proof. Consider the coupled system of integral equations:

$$\begin{cases} v(t) = \int_{-\infty}^{t}(t-s)^{\alpha-1}V(t-s)F_1(s,v(s))\mathrm{d}s, \quad t\in\mathbb{R}, \\ \omega(t) = \int_{-\infty}^{t}(t-s)^{\alpha-1}V(t-s)[F_1(s,v(s)+\omega(s))-F_1(s,v(s))]\mathrm{d}s \\ \qquad + \int_{-\infty}^{t}(t-s)^{\alpha-1}V(t-s)F_2(s,v(s)+\omega(s))\mathrm{d}s, \quad t\in\mathbb{R}. \end{cases} \tag{12}$$

If $(v(t),\omega(t))\in AP(\mathbb{R},X)\times C_0(\mathbb{R},X)$ is a solution to system (12), then $x(t):=v(t)+\omega(t)\in AAP(\mathbb{R},X)$ and it is a solution to the integral equation

$$x(t) = \int_{-\infty}^{t}(t-s)^{\alpha-1}V(t-s)F(s,x(s))\mathrm{d}s, \quad t\in\mathbb{R},$$

that is, $x(t)$ is an asymptotically almost periodic mild solution to Eq. (6). Hence, the problem has shifted to show that system (12) has at least a solution in $AP(\mathbb{R}, X) \times C_0(\mathbb{R}, X)$.

Define a mapping Λ on $AP(\mathbb{R}, X)$ by

$$(\Lambda v)(t) = \int_{-\infty}^{t} (t-s)^{\alpha-1} V(t-s) F_1(s, v(s)) \mathrm{d}s, \quad t \in \mathbb{R}.$$

First, since the function $s \to F_1(s, v(s))$ is bounded on $\mathbb{R}$ and

$$\begin{aligned} \|[\Lambda v](t)\| &= \left\| \int_{-\infty}^{t} (t-s)^{\alpha-1} V(t-s) F_1(s, v(s)) \mathrm{d}s \right\| \\ &\leq \|F_1\|_\infty \left\| \int_{-\infty}^{t} (t-s)^{\alpha-1} V(t-s) \mathrm{d}s \right\|_{\mathbb{L}(X)} \\ &\leq M \|F_1\|_\infty \int_{-\infty}^{t} (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\delta(t-s)^\alpha) \mathrm{d}s \\ &= M \|F_1\|_\infty E_\alpha(-\delta(t-s)^\alpha) \Big|_{-\infty}^{t} \\ &= \frac{M \|F_1\|_\infty}{\delta}, \end{aligned}$$

which implies that $(\Lambda v)(t)$ exists. Moreover, from $F_1(t, x) \in AP(\mathbb{R} \times X, X)$ satisfying (7), together with Lemma 2.4, it follows that

$$F_1(\cdot, v(\cdot)) \in AP(\mathbb{R}, X) \text{ for every } v(\cdot) \in AP(\mathbb{R}, X).$$

This, together with Lemma 3.3, implies that Λ is well defined and maps $AP(\mathbb{R}, X)$ into itself.

In the sequel, we verify Λ is continuous.

Let $v_n(t), v(t)$ be in $AP(\mathbb{R}, X)$ with $v_n(t) \to v(t)$ as $n \to \infty$, then one has

$$\begin{aligned} \|[\Lambda v_n](t) - [\Lambda v](t)\| &= \left\| \int_{-\infty}^{t} (t-s)^{\alpha-1} V(t-s) \Big[F_1(s, v_n(s)) - F_1(s, v(s)) \Big] \mathrm{d}s \right\| \\ &\leq L \int_{-\infty}^{t} \|(t-s)^{\alpha-1} V(t-s)[v_n(s) - v(s)]\| \mathrm{d}s \\ &\leq ML \|v_n - v\|_\infty \int_{-\infty}^{t} (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\delta(t-s)^\alpha) \mathrm{d}s \\ &= ML \|v_n - v\|_\infty E_\alpha(-\delta(t-s)^\alpha) \Big|_{-\infty}^{t} = \frac{ML \|v_n - v\|_\infty}{\delta}. \end{aligned}$$

Therefore, as $n \to \infty$, $\Lambda v_n \to \Lambda v$, and hence, Λ is continuous.

Next, we prove that Λ is a contraction on $AP(\mathbb{R}, X)$ and has a unique fixed point $v(t) \in AP(\mathbb{R}, X)$.

Let $v_1(t), v_2(t)$ be in $AP(\mathbb{R}, X)$, similar to the above proof of the continuity of Λ, one has

$$\|[\Lambda v_1](t) - [\Lambda v_2](t)\| \leq \frac{ML}{\delta} \|v_1 - v_2\|_\infty,$$

which implies

$$\|[\Lambda v_1](t) - [\Lambda v_2](t)\|_\infty \le \frac{ML}{\delta}\|v_1 - v_2\|_\infty.$$

This together with (11) proves that Λ is a contraction on $AP(\mathbb{R}, X)$. Thus, the Banach's fixed point theorem implies that Λ has a unique fixed point $v(t) \in AP(\mathbb{R}, X)$.

For the above $v(t)$, define $\Gamma := \Gamma^1 + \Gamma^2$ on $C_0(\mathbb{R}, X)$ as follows:

$$(\Gamma^1\omega)(t) = \int_{-\infty}^{t} (t-s)^{\alpha-1} V(t-s)[F_1(s, v(s) + \omega(s)) - F_1(s, v(s))]\mathrm{d}s, \quad t \in \mathbb{R},$$
$$(\Gamma^2\omega)(t) = \int_{-\infty}^{t} (t-s)^{\alpha-1} V(t-s)F_2(s, v(s) + \omega(s))\mathrm{d}s, \quad t \in \mathbb{R}.$$

First, from (7), it follows that

$$\|F_1(s, v(s) + \omega(s)) - F_1(s, v(s))\| \le L\|\omega(s)\| \quad \text{for all } s \in \mathbb{R},\ \omega(s) \in,$$

which implies that

$$F_1(\cdot, v(\cdot) + \omega(\cdot)) - F_1(\cdot, v(\cdot)) \in C_0(\mathbb{R}, X) \quad \text{for every } \omega(\cdot) \in C_0(\mathbb{R}, X).$$

According to (8), one has

$$\|F_2(s, v(s) + \omega(s))\| \le \beta(s)\Phi\Big(r + \sup_{s\in\mathbb{R}} \|v(s)\|\Big)$$

for all $s \in \mathbb{R}$ and $\omega(s) \in X$ with $\|\omega(s)\| \le r$, then

$$F_2(\cdot, v(\cdot) + \omega(\cdot)) \in C_0(\mathbb{R}, X) \quad \text{as } \beta(\cdot) \in C_0(\mathbb{R}, \mathbb{R}^+).$$

Those, together with Lemma 3.4, yield that Γ is well defined and maps $C_0(\mathbb{R}, X)$ into itself.

To complete the proof, it suffices to prove that Γ has at least one fixed point in $C_0(\mathbb{R}, X)$.

Set $\Omega_r := \{\omega(t) \in C_0(\mathbb{R}, X) : \|\omega\|_\infty \le r\}$. In view of (8) and (11), it is not difficult to see that there exists a constant $k_0 > 0$, such that

$$\frac{ML}{\delta}k_0 + M\rho_2\Phi\left(k_0 + \sup_{s\in\mathbb{R}} \|v(s)\|\right) \le k_0.$$

This enables us to conclude that for any $t \in \mathbb{R}$ and $\omega_1(t), \omega_2(t) \in \Omega_{k_0}$:

$$\begin{aligned}
&\|(\Gamma^1\omega_1)(t) + (\Gamma^2\omega_2)(t)\| \\
&\quad \leq \left\| \int_{-\infty}^{t} (t-s)^{\alpha-1} V(t-s)[F_1(s, v(s)+\omega_1(s)) - F_1(s, v(s))]\mathrm{d}s \right\| \\
&\qquad + \left\| \int_{-\infty}^{t} (t-s)^{\alpha-1} V(t-s) F_2(s, v(s)+\omega_2(s))\mathrm{d}s \right\| \\
&\quad \leq \int_{-\infty}^{t} \|(t-s)^{\alpha-1} V(t-s)[F_1(s, v(s)+\omega_1(s)) - F_1(s, v(s))]\|\mathrm{d}s \\
&\qquad + \int_{-\infty}^{t} \|(t-s)^{\alpha-1} V(t-s) F_2(s, v(s)+\omega_2(s))\|\mathrm{d}s \\
&\quad \leq L \int_{-\infty}^{t} \|(t-s)^{\alpha-1} V(t-s)\omega_1(s)\|\mathrm{d}s \\
&\qquad + \Phi\left(\|\omega_2\|_\infty + \sup_{s\in\mathbb{R}} \|v(s)\| \right) \int_{-\infty}^{t} \|\beta(s)(t-s)^{\alpha-1} V(t-s)\|\mathrm{d}s \\
&\quad \leq L\|\omega_1\|_\infty \left\| \int_{-\infty}^{t} (t-s)^{\alpha-1} V(t-s)\mathrm{d}s \right\|_{\mathbb{L}(X)} \\
&\qquad + M\Phi\left(\|\omega_2\|_\infty + \sup_{s\in\mathbb{R}} \|v(s)\| \right) \int_{-\infty}^{t} \beta(s)(t-s)^{\alpha-1} E_{\alpha,\alpha}(-\delta(t-s)^\alpha)\mathrm{d}s \\
&\quad \leq ML\|\omega_1\|_\infty \int_{-\infty}^{t} (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\delta(t-s)^\alpha)\mathrm{d}s \\
&\qquad + M\sigma(t)\Phi\left(\|\omega_2\|_\infty + \sup_{s\in\mathbb{R}} \|v(s)\| \right) \\
&\quad \leq ML\|\omega_1\|_\infty E_\alpha(-\delta(t-s)^\alpha)\Big|_{-\infty}^{t} + M\rho_2\Phi\left(\|\omega_2\|_\infty + \sup_{s\in\mathbb{R}} \|v(s)\| \right) \\
&\quad = \frac{ML\|\omega_1\|_\infty}{\delta} + M\rho_2\Phi\left(\|\omega_2\|_\infty + \sup_{s\in\mathbb{R}} \|v(s)\| \right) \leq \frac{ML}{\delta} k_0 \\
&\qquad + M\rho_2\Phi\left(k_0 + \sup_{s\in\mathbb{R}} \|v(s)\| \right) \\
&\quad \leq k_0,
\end{aligned}$$

which implies that $(\Gamma^1\omega_1)(t) + (\Gamma^2\omega_2)(t) \in \Omega_{k_0}$. Thus, Γ maps Ω_{k_0} into itself.

In the following, we show that Γ^1 is a contraction on Ω_{k_0}. For any $\omega_1(t), \omega_2(t) \in \Omega_{k_0}$, from (7), it follows that

$$\begin{aligned}
&\|[F_1(s, v(s)+\omega_1(s)) - F_1(s, v(s))] - [F_1(s, v(s)+\omega_2(s)) \\
&\quad - F_1(s, v(s))]\| \leq L\|\omega_1(s) - \omega_2(s)\|.
\end{aligned}$$

Thus

$$\begin{aligned}
&\|(\Gamma^1\omega_1)(t)-(\Gamma^1\omega_2)(t)\|\\
&\quad=\bigg\|\int_{-\infty}^{t}(t-s)^{\alpha-1}V(t-s)\Big[\Big(F_1(s,v(s)+\omega_1(s))-F_1(s,v(s))\Big)\\
&\qquad-\Big(F_1(s,v(s)+\omega_2(s))-F_1(s,v(s))\Big)\Big]\mathrm{d}s\bigg\|\\
&\quad\le L\int_{-\infty}^{t}\|(t-s)^{\alpha-1}V(t-s)[\omega_1(s)-\omega_2(s)]\|\mathrm{d}s\\
&\quad\le ML\|\omega_1-\omega_2\|_\infty\int_{-\infty}^{t}(t-s)^{\alpha-1}E_{\alpha,\alpha}(-\delta(t-s)^\alpha)\mathrm{d}s\\
&\quad=ML\|\omega_1-\omega_2\|_\infty E_\alpha(-\delta(t-s)^\alpha)\Big|_{-\infty}^{t}=\frac{ML\|\omega_1-\omega_2\|_\infty}{\delta},
\end{aligned}$$

which implies that

$$\|(\Gamma^1\omega_1)(t)-(\Gamma^1\omega_2)(t)\|_\infty\le\frac{ML}{\delta}\|\omega_1-\omega_2\|_\infty.$$

Thus, in view of (11), one obtains the conclusion.

From our assumption, it is clear that Γ^2 is a continuous mapping from Ω_{k_0} to Ω_{k_0}. Thus, to apply the well-known Krasnoselskii's fixed point theorem (see Lemma 2.13) to obtain a fixed point of Γ, one needs to verify that Γ^2 is completely continuous on Ω_{k_0}.

Given $\varepsilon>0$. Let $\{\omega_k\}_{k=1}^{+\infty}\subset\Omega_{k_0}$ with $\omega_k\to\omega_0$ in $C_0(\mathbb{R},X)$ as $k\to+\infty$. Since $\sigma(t)\in C_0(\mathbb{R},\mathbb{R}^+)$ which follows from Lemma 3.8, one may choose a $t_1>0$ big enough, such that for all $t\ge t_1$:

$$\Phi\Big(k_0+\|v\|_\infty\Big)\sigma(t)<\frac{\varepsilon}{3M}.$$

In addition, in view of (H_1), we have

$$F_2(s,v(s)+\omega_k(s))\to F_2(s,v(s)+\omega_0(s))\text{ for all }s\in(-\infty,t_1]\text{ as }k\to+\infty,$$

and

$$\|F_2(\cdot,v(\cdot)+\omega_k(\cdot))-F_2(\cdot,v(\cdot)+\omega_0(\cdot))\|\le2\Phi\Big(k_0+\|v\|_\infty\Big)\beta(\cdot)\in L^1(-\infty,t_1].$$

Hence, by the Lebesgue dominated convergence theorem, we deduce that there exists an $N>0$, such that for any $k\ge N$:

$$\begin{aligned}
&M\int_{-\infty}^{t_1}\|(t-s)^{\alpha-1}E_{\alpha,\alpha}(-\delta(t-s)^\alpha)[F_2(s,v(s)+\omega_k(s))\\
&\quad-F_2(s,v(s)+\omega_0(s))]\|\mathrm{d}s\le\frac{\varepsilon}{3}.
\end{aligned}$$

Thus, when $k \geq N$:

$$\begin{aligned}
&\|(\Gamma^2\omega_k)(t) - (\Gamma^2\omega_0)(t)\| \\
&\quad = \left\| \int_{-\infty}^{t} (t-s)^{\alpha-1} V(t-s) F_2(s, v(s) + \omega_k(s)) \mathrm{d}s \right. \\
&\qquad \left. - \int_{-\infty}^{t} (t-s)^{\alpha-1} V(t-s) F_2(s, v(s) + \omega_0(s)) \mathrm{d}s \right\| \\
&\quad \leq \int_{-\infty}^{t_1} \|(t-s)^{\alpha-1} V(t-s)[F_2(s, v(s)+\omega_k(s)) - F_2(s, v(s)+\omega_0(s))]\| \mathrm{d}s \\
&\qquad + \int_{t_1}^{\max\{t,t_1\}} \|(t-s)^{\alpha-1} V(t-s)[F_2(s, v(s)+\omega_k(s)) \\
&\qquad - F_2(s, v(s)+\omega_0(s))]\| \mathrm{d}s \\
&\quad \leq M \int_{-\infty}^{t_1} \|(t-s)^{\alpha-1} E_{\alpha,\alpha}(-\delta(t-s)^{\alpha})[F_2(s, v(s)+\omega_k(s)) \\
&\qquad - F_2(s, v(s)+\omega_0(s))]\| \mathrm{d}s \\
&\qquad + 2M\Phi\Big(k_0 + \|v\|_\infty\Big) \int_{t_1}^{\max\{t,t_1\}} \beta(s)(t-s)^{\alpha-1} E_{\alpha,\alpha}(-\delta(t-s)^{\alpha}) \mathrm{d}s \\
&\quad \leq M \int_{-\infty}^{t_1} \|(t-s)^{\alpha-1} E_{\alpha,\alpha}(-\delta(t-s)^{\alpha})[F_2(s, v(s)+\omega_k(s)) \\
&\qquad - F_2(s, v(s)+\omega_0(s))]\| \mathrm{d}s + 2M\Phi\Big(k_0 + \|v\|_\infty\Big)\sigma(t) \\
&\quad \leq \frac{\varepsilon}{3} + \frac{2\varepsilon}{3} = \varepsilon.
\end{aligned}$$

Accordingly, Γ^2 is continuous on Ω_{k_0}.

In the sequel, we consider the compactness of Γ^2.

Set $B_r(X)$ for the closed ball with center at 0 and radius r in X, $\Delta = \Gamma^2(\Omega_{k_0})$ and $z(t) = \Gamma^2(u(t))$ for $u(t) \in \Omega_{k_0}$. First, for all $\omega(t) \in \Omega_{k_0}$ and $t \in \mathbb{R}$:

$$\begin{aligned}
\|(\Gamma^2\omega)(t)\| &= \left\| \int_{-\infty}^{t} (t-s)^{\alpha-1} V(t-s) F_2(s, v(s) + \omega(s)) \mathrm{d}s \right\| \\
&\leq M\Phi\left(k_0 + \sup_{s\in\mathbb{R}} \|v(s)\|\right) \int_{-\infty}^{t} \beta(s)(t-s)^{\alpha-1} E_{\alpha,\alpha}(-\delta(t-s)^{\alpha}) \mathrm{d}s \\
&= M\sigma(t)\Phi\left(k_0 + \sup_{s\in\mathbb{R}} \|v(s)\|\right),
\end{aligned}$$

in view of $\sigma(t) \in C_0(\mathbb{R}, \mathbb{R}^+)$ which follows from Lemma 3.8, one concludes that

$$\lim_{|t|\to+\infty} (\Gamma^2\omega)(t) = 0 \quad \text{uniformly for } \omega(t) \in \Omega_{k_0}.$$

As

$$\begin{aligned}(\Gamma^2\omega)(t) &= \int_{-\infty}^{t}(t-s)^{\alpha-1}V(t-s)F_2(s,v(s)+\omega(s))\mathrm{d}s\\ &= \int_{0}^{+\infty}\tau^{\alpha-1}V(\tau)F_2(t-\tau,v(t-\tau)+\omega(t-\tau))\mathrm{d}\tau.\end{aligned}$$

Hence, for given $\varepsilon_0>0$, one can choose a $\xi>0$, such that

$$\left\|\int_{\xi}^{+\infty}\tau^{\alpha-1}V(\tau)F_2(t-\tau,v(t-\tau)+\omega(t-\tau))\mathrm{d}\tau\right\|<\varepsilon_0.$$

Thus, we get

$$\begin{aligned}z(t)\in \xi\overline{c(\{\tau^{\alpha-1}V(\tau)F_2(\lambda,v(\lambda)+\omega(\lambda)):0\le\tau\le\xi,t-\xi\le\lambda\le\xi,\|\omega\|_\infty\le r\})}\\ +B_{\varepsilon_0}(X),\end{aligned}$$

where $c(K)$ denotes the convex hull of K. Using that $V(\cdot)$ is strongly continuous which follows from Lemma 2.10(2), we infer that

$$K=\{\tau^{\alpha-1}V(\tau)F_2(\lambda,v(\lambda)+\omega(\lambda)):0\le\tau\le\xi,t-\xi\le\lambda\le\xi,\|\omega\|_\infty\le r\}$$

is a relatively compact set, and $\Delta\subset\xi\overline{c(K)}+B_{\varepsilon_0}(X)$, which implies that Δ is a relatively compact subset of X.

Next, we verify the equicontinuity of the set $\{(\Gamma^2\omega)(t):\omega(t)\in\Omega_{k_0}\}$.

Let $k>0$ be small enough and $t_1,t_2\in\mathbb{R}$, $\omega(t)\in\Omega_{k_0}$. Then, by (8), we have

$$\begin{aligned}&\|(\Gamma^2\omega)(t_2)-(\Gamma^2\omega)(t_1)\|\\ &\quad=\left\|\int_{-\infty}^{t_2}(t_2-s)^{\alpha-1}V(t_2-s)F_2(s,v(s)+\omega(s))\mathrm{d}s\right.\\ &\qquad\left.-\int_{-\infty}^{t_1}(t_1-s)^{\alpha-1}V(t_1-s)F_2(s,v(s)+\omega(s))\mathrm{d}s\right\|\\ &\quad\le\int_{t_1}^{t_2}\|(t_2-s)^{\alpha-1}V(t_2-s)F_2(s,v(s)+\omega(s))\|\mathrm{d}s\\ &\qquad+\int_{-\infty}^{t_1-k}\|[(t_2-s)^{\alpha-1}V(t_2-s)-(t_1-s)^{\alpha-1}V(t_1-s)]F_2(s,v(s)\\ &\qquad+\omega(s))\|\mathrm{d}s\\ &\qquad+\int_{t_1-k}^{t_1}\|[(t_2-s)^{\alpha-1}V(t_2-s)-(t_2-s)^{\alpha-1}V(t_1-s)]F_2(s,v(s)\\ &\qquad+\omega(s))\|\mathrm{d}s\\ &\quad\le M\Phi\Big(k_0+\|v\|_\infty\Big)\int_{t_1}^{t_2}\beta(s)(t_2-s)^{\alpha-1}E_{\alpha,\alpha}(-\delta(t_2-s)^\alpha)\mathrm{d}s\\ &\qquad+\Phi\Big(k_0+\|v\|_\infty\Big)\\ &\qquad\times\sup_{s\in[-\infty,t_1-k]}\|[(t_2-s)^{\alpha-1}V(t_2-s)-(t_2-s)^{\alpha-1}V(t_1-s)]\|\end{aligned}$$

$$
\begin{aligned}
&\times \int_{-\infty}^{t_1-k} \beta(s)\mathrm{d}s + M\Phi\Big(k_0 + \|v\|_\infty\Big) \\
&\times \int_{t_1-k}^{t_1} \big(\beta(s)(t_2-s)^{\alpha-1}E_{\alpha,\alpha}(-\delta(t_2-s)^\alpha) \\
&+ \beta(s)(t_1-s)^{\alpha-1}E_{\alpha,\alpha}(-\delta(t_1-s)^\alpha)\big)\,\mathrm{d}s \\
\to 0 \quad & \text{as } t_2 - t_1 \to 0,\ k \to 0,
\end{aligned}
$$

which implies the equicontinuity of the set $\{(\Gamma^2\omega)(t) : \omega(t) \in \Omega_{k_0}\}$.

Now, an application of Lemma 2.12 justifies the compactness of Γ^2.

Finally, from the Krasnoselskii's fixed point theorem (see Lemma 2.13), it follows that Γ has at least one fixed point in Ω_{k_0}. This proves that system (12) has at least one solution in $AP(\mathbb{R}, X) \times C_0(\mathbb{R}, X)$. □

4. Applications

In this section, we give an example to illustrate the feasibility of the above abstract result.

Consider the following fractional partial differential equation with Dirichlet boundary conditions of the form:

$$
\begin{cases}
\partial_t^\alpha u(t,x) = \partial_x^2 u(t,x) + \mu(\sin t + \sin\sqrt{2}t)\sin u(t,x) \\
\qquad\qquad + \nu e^{-|t|}u(t,x)\sin u^2(t,x), \quad t \in \mathbb{R}, \quad x \in [0,\pi], \\
u(t,0) = u(t,\pi) = 0, \quad t \in \mathbb{R},
\end{cases} \tag{13}
$$

where μ and ν are positive constants.

Take $X = L^2[0,\pi]$ with norm $\|\cdot\|$ and define $A : D(A) \subset X \to X$ given by $Ax = \frac{\partial^2 x(\xi)}{\partial \xi^2}$ with the domain:

$$
\begin{aligned}
D(A) = \Big\{x(\cdot) \in X : x'' \in X, x' \in X \ \text{ is absolutely} \\
\text{continuous on } [0,\pi], x(0) = x(\pi) = 0\Big\}.
\end{aligned}
$$

It is well known that A is self-adjoint, with compact resolvent and is the infinitesimal generator of an analytic semigroup $\{T(t)\}_{t\geq 0}$ satisfying $\|T(t)\| \leq e^{-t}$ for $t > 0$. Let

$$
F_1(t, x(\xi)) := \mu(\sin t + \sin\sqrt{2}t)\sin x(\xi),\ F_2(t, x(\xi)) := \nu e^{-|t|}x(\xi)\sin x^2(\xi).
$$

Then, it is easy to verify that $F_1, F_2 : \mathbb{R} \times X \to X$ are continuous, $F_1(t,x) \in AP(\mathbb{R} \times X, X)$ satisfying

$$
\|F_1(t,x) - F_1(t,y)\| \leq 2\mu\|x - y\| \text{ for all } t \in \mathbb{R},\ x, y \in X,
$$

and

$$
\|F_2(t,x)\| \leq \nu e^{-|t|}\|x\| \quad \text{for all } t \in \mathbb{R},\ x \in X,
$$

which implies that $F_2(t,x) \in C_0(\mathbb{R} \times X, X)$. Furthermore

$$
F(t,x) = F_1(t,x) + F_2(t,x) \in AAP(\mathbb{R} \times X, X).
$$

Thus, (13) can be reformulated as the abstract problem (6) and the assumptions (H_1) and (H_2) hold with

$$L = 2\mu,\ \Phi(r) = r,\ \beta(t) = \nu e^{-|t|},\ \rho_1 = 1,\ \rho_2 \le \nu.$$

Then, from Theorem 3.9, it follows that Eq. (13) has at least one asymptotically almost periodic mild solution whenever $2\mu + \nu < 1$.

Acknowledgements

This research was supported by the Guangdong Province Natural Science Foundation (No.2015A030313896), the Characteristic Innovation Project (Natural Science) of Guangdong Province (No.2016KTSCX094), the Science and Technology Program Project of Guangzhou (No.201707010230).

References

[1] Bohr, H.: Zur Theorie der fastperiodischen Funktionen, I. Acta Math. **45**, 29–127 (1925)

[2] Bohr, H.: Almost Periodic Functions. Chelsea Publishing Company, New York (1947)

[3] Bochner, S.: Beiträge zur Theorie der fastperiodischen Funktionen, I. Math. Ann. **96**, 119–147 (1927)

[4] von Neumann, J.: Almost periodic functions in a group, I. Trans. Am. Math. Soc. **36**, 445–492 (1934)

[5] van Kampen, E.: Almost periodic functions and compact groups. Ann. Math. **37**, 78–91 (1936)

[6] Fréchet, M.: Les fonctions asymptotiquement presque-périodiques continues (in French). C. R. Acad. Sci. Paris **213**, 520–522 (1941)

[7] Fréchet, M.: Les fonctions asymptotiquement presque-périodiques (in French). Revue Sci. (Rev. Rose. Illus.) **79**, 341–354 (1941)

[8] de Andrade, B., Lizama, C.: Existence of asymptotically almost periodic solutions for damped wave equations. J. Math. Anal. Appl. **382**, 761–771 (2011)

[9] Arendt, W., Batty, C.: Asymptotically almost periodic solutions of inhomogeneous Cauchy problems on the half-line. Bull. Lond. Math. Soc. **31**, 291–304 (1999)

[10] Cushing, J.: Forced asymptotically periodic solutions of predator-prey systems with or without hereditary effects. SIAM J. Appl. Math. **30**, 665–674 (1976)

[11] Ruess, W., Phong, V.: Asymptotically almost periodic solutions of evolution equations in Banach spaces. J. Differ. Equations **122**, 282–301 (1995)

[12] Zhang, C.: Almost Periodic Type Functions and Ergodicity. Science Press, Beijing (2003)

[13] Zhao, Z., Chang, Y., Li, W.: Asymptotically almost periodic, almost periodic and pseudo-almost periodic mild solutions for neutral differential equations. Nonlinear Anal. Real World Appl. **11**, 3037–3044 (2010)

[14] Cao, J., Yang, Q., Huang, Z., Liu, Q.: Asymptotically almost periodic solutions of stochastic functional differential equations. Appl. Math. Comput. **218**, 1499–1511 (2011)

[15] Kilbas, A., Srivastava, H., Trujillo, J.: Theory and Applications of Fractional Differential Equations. Elsevier Science B.V, Amsterdam (2006)

[16] Podlubny, I.: Fractional Differential Equations. Academic Press, San Diego (1999)

[17] Miller, K., Ross, B.: An Introduction to the Fractional Calculus and Fractional Diferential Equations, A Wiley Interscience Publication. Wiley, New York (1993)

[18] Zhou, Y.: Basic Theory of Fractional Diferential Equations. World Scientiic, Singapore (2014)

[19] Agarwal, R., Lakshmikantham, V., Nieto, J.: On the concept of solution for fractional differential equations with uncertainty. Nonlinear Anal. Theory Methods Appl. **72**, 2859–2862 (2010)

[20] Benchohra, M., Henderson, J., Ntouyas, S., Ouahab, A.: Existence results for fractional order functional differential equations with infinite delay. J. Math. Anal. Appl. **338**, 1340–1350 (2008)

[21] Debbouche, A., El-Borai, M.M.: Weak almost periodic and optimal mild solutions of fractional evolution equations. Electron. J. Differ. Equations **2009**(46), 1–8 (2009)

[22] El-Borai, M.: Some probability densities and fundamental solutions of fractional evolution equations. Chaos Solitons Fract. **14**, 433–440 (2002)

[23] Li, Y.N., Sun, H.R., Feng, Z.: Fractional abstract Cauchy problem with order $\alpha \in (1, 2)$. Dyn. PDE **13**(2), 155–177 (2016)

[24] Shen, Y., Chen, W.: Laplace transform method for the ulam stability of linear fractional differential equations with constant coefficients. Mediterr. J. Math. **14**(1), 1–17 (2017)

[25] Wang, J.R., Feckan, M., Zhou, Y.: A survey on impulsive fractional differential equations. Fract. Calc. Appl. Anal. **19**, 806–831 (2016)

[26] Zhou, Y., Peng, L.: On the time-fractional Navier–Stokes equations. Comput. Math. Appl. **73**(6), 874–891 (2017)

[27] Zhou, Y., Peng, L.: Weak solution of the time-fractional Navier–Stokes equations and optimal control. Comput. Math. Appl. **73**(6), 1016–1027 (2017)

[28] Zhou, Y., Zhang, L.: Existence and multiplicity results of homoclinic solutions for fractional Hamiltonian systems. Comput. Math. Appl. **73**(6), 1325–1345 (2017)

[29] Zhou, Y., Vijayakumar, V., Murugesu, R.: Controllability for fractional evolution inclusions without compactness. Evol. Equations Control Theory **4**(1), 507–524 (2017)

[30] Zhou, Y., Ahmad, B., Alsaedi, A.: Existence of nonoscillatory solutions for fractional neutral differential equations. Appl. Math. Lett. **72**, 70–74 (2017)

[31] Araya, D., Lizama, C.: Almost automorphic mild solutions to fractional differential equations. Nonlinear Anal. Theory Methods Appl. **69**, 3692–3705 (2008)

[32] Cuevas, C., Lizama, C.: Almost automorphic solutions to a class of semilinear fractional differential equations. Appl. Math. Lett. **21**, 1315–1319 (2008)

[33] Agarwal, R., de Andrade, B., Cuevas, C.: Weighted pseudo-almost periodic solutions of a class of semilinear fractional differential equations. Nonlinear Anal. Real World Appl. **11**, 3532–3554 (2010)

[34] Chang, Y., Zhang, R., N'Guérékata, G.: Weighted pseudo almost automorphic mild solutions to semilinear fractional differential equations. Comput. Math. Appl. **64**, 3160–3170 (2012)

[35] Lizama, C., Poblete, F.: Regularity of mild solutions for a class of fractional order differential equations. Appl. Math. Comput. **224**, 803–816 (2013)

[36] Xia, Z., Fan, M., Agarwal, R.: Pseudo almost automorphy of semilinear fractional differential equations in Banach spaces. Fract. Calc. Appl. Anal. **19**, 741–764 (2016)

[37] Mophou, G.: Weighted pseudo almost automorphic mild solutions to semilinear fractional differential equations. Appl. Math. Comput. **217**, 7579–7587 (2011)

[38] Chang, Y., Luo, X.: Pseudo almost automorphic behavior of solutions to a semi-linear fractional differential equation. Math. Commun. **20**, 53–68 (2015)

[39] Mu, J., Zhou, Y., Peng, L.: Periodic solutions and S-asymptotically periodic solutions to fractional evolution equations. Discrete Dyn. Nat. Soc. **2017**, 12 (2017) (**Article ID 1364532**)

[40] Hilfer, R.: Treefold introduction to fractional derivatives, in Anomalous Transport: Foundations and Applications. Wiley, New York (2008)

[41] Pazy, A.: Semigroups of Linear Operators and Applications to Partial Differential Equations. Springer, New York (1983)

[42] Wei, Z., Li, Q., Che, J.: Initial value problems for fractional differential equations involving Riemann-Liouville sequential fractional derivative. J. Math. Anal. Appl. **367**, 260–272 (2010)

[43] Shu, X., Lai, Y., Chen, Y.: The existence of mild solutions for impulsive fractional partial differential equations. Nonlinear Anal. Theory Methods Appl. **74**, 2003–2011 (2011)

[44] Wei, Z., Dong, W., Che, J.: Periodic boundary value problems for fractional differential equations involving a Riemann–Liouville fractional derivative. Nonlinear Anal. Theory Methods Appl. **73**, 3232–3238 (2010)

[45] Krägeloh, A.: Two families of functions related to the fractional powers of generators of strongly continuous contraction semigroups. J. Math. Anal. Appl. **283**, 459–467 (2003)

[46] Ruess, W., Summers, W.: Compactness in spaces of vector valued continuous functions and asymptotic almost periodicity. Math. Nachr. **135**, 7–33 (1988)

[47] Smart, D.: Fixed Point Theorems. Cambridge University Press, Cambridge (1980)

Junfei Cao
Department of Mathematics
Guangdong University of Education
Guangzhou 510303
People's Republic of China
e-mail: jfcaomath@163.com

Amar Debbouche
Department of Mathematics
Guelma University
24000 Guelma
Algeria
e-mail: amar_debbouche@yahoo.fr

Yong Zhou
Faculty of Mathematics and Computational Science
Xiangtan University
Hunan 411105
People's Republic of China
e-mail: yzhou@xtu.edu.cn

and

Nonlinear Analysis and Applied Mathematics (NAAM) Research Group, Faculty of Science
King Abdulaziz University
Jeddah 21589
Saudi Arabia

Received: January 14, 2018.
Accepted: June 11, 2018.