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# Improved Euler–Maruyama Method for Numerical Solution of the Itô Stochastic Differential Systems by Composite Previous-Current-Step Idea

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**Abstract.** In this paper, by composite previous-current-step idea, we propose two numerical schemes for solving the Itô stochastic differential systems. Our approaches, which are based on the Euler–Maruyama method, solve stochastic differential systems with strong sense. The mean-square convergence theory of these methods are analyzed under the Lipschitz and linear growth conditions. The accuracy and efficiency of the proposed numerical methods are examined by linear and nonlinear stochastic differential equations.

**Mathematics Subject Classification.** Primary 60H10; Secondary 35D35, 41A25.

**Keywords.** Stochastic differential systems, composite previous-current-step idea, strong solution, Euler–Maruyama method, mean-square convergence.

## 1. Introduction

In this paper, we consider two numerical methods for strong solution of the Itô stochastic differential system,

$$\begin{cases} \mathrm{d}X(t) = f(t, X(t))\mathrm{d}t + \sum_{j=1}^{m} g_j(t, X(t))\mathrm{d}W^j(t), \quad t \in [t_0, T], \\ X(t_0) = X_0, \end{cases} \tag{1.1}$$

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where $X \in \mathbb{R}^d$, $f : [t_0, T] \times \mathbb{R}^d \to \mathbb{R}^d$, is a drift vector, $g = (g_1, \ldots, g_m) : [t_0, T] \times \mathbb{R}^d \to \mathbb{R}^{d\times m}$ is a diffusion matrix and $W = (W_1, \ldots, W_m)^T$ is an $m$-dimensional Wiener process whose increment $\Delta W^j(t) = W^j(t + \varDelta) - W^j(t)$ is a Gaussian random variable $\mathcal{N}(0, \varDelta)$.

During the several decades, many efficient methods have been developed for solving different types of SDEs with different properties [3,4,8,12,15–17,19]. The simplest of these methods to approximate solutions of the SDEs (1.1) is the so-called Euler–Maruyama (EM) method with strong order 0.5 [15]. To improve the stability properties of numerical methods for solving SDEs (1.1), some attempts have been made to propose modified EM method. For example, Burrage and Tian [2], consider the composite Euler method, which is a combination of the semi-implicit Euler method and the implicit Euler method. Furthermore, they introduced the implicit Euler–Taylor method, based on relationship between the Itô stochastic integrals and backward stochastic integrals [18]. In 2010, Wang and Li [21] presented two fully explicit methods based on EM method, the drifting split-step forward Euler (DRSSE) and the diffused split-step Euler (DISSE) methods. Higham et al. [9] have analyzed the split-step backward Euler (SSBE) method for solving nonlinear autonomous SDEs. Recently, Hutzenthaler et al. [10,11] developed the tamed EM method, for solution of nonlinear SDEs. In addition, another new approach called the truncated EM method [13,14],

$$Y(t_{k+1}) = Y(t_k) + \varDelta f(q(Y)) + g(q(Y))\Delta W_k,$$

has been developed by Mao to approximate SDEs with one-sided Lipschitz drift coefficient and the linear growth diffusion coefficient. In the truncated EM method,

$$q(Y) = \left(|Y| \wedge \mu^{-1}(h(\varDelta))\right) \tfrac{Y}{|Y|}$$

and $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ is a strictly increasing continuous function such that $\mu(r) \to \infty$ as $r \to \infty$ and

$$\sup_{|x|\le r} (|f(x)| \vee |g(x)|) \le \mu(r), \quad \forall r \ge 0.$$

In this paper, we discuss on methods that use the stage values from only one previous step. For deterministic differential equations the so-called two-step Runge–Kutta methods, this method has been widely studied, for example see [1], which includes also an extensive bibliography. In [5,6] the authors extended the idea of deterministic two-step Runge–Kutta methods to the solution of SDEs.

This paper is organized as follows. In Sect. 2, we introduce the composite previous-current-step (CPCS) idea for original EM method and design two EM methods based this idea. The convergence properties of these methods are discussed in Sect. 3. The numerical results of these methods are discussed in Sect. 4. In the last section, conclusions are given.

## 2. Composite Previous-Current-Step Methods

In this section, we introduce composite general previous-current-step idea for the original EM method, namely

$$\Lambda(t) = \Upsilon(t) + \varDelta f(t, \Phi(t)) + \sum_{j=1}^{m} g_j(t, \Psi(t))\Delta W_i^j, \tag{2.1}$$

where $i = 1, 2, \ldots, N$, $t_i = t_0 + i\varDelta$ which stepsize $\varDelta$ is defined as $\varDelta = t_i - t_{i-1}$ and $\Delta W_i^j = W_{t_{i+1}}^j - W_{t_i}^j$, and $\Lambda(t)$, $\Upsilon(t)$, $\Phi(t)$ and $\Psi(t)$ are combination of the

$$Y_0, Y_1^{[0]}, Y_1^{[1]}, \ldots, Y_1^{[k]}, Y_2^{[0]}, Y_2^{[1]}, \ldots, Y_2^{[k]}, \ldots, Y_N^{[0]}, Y_N^{[1]}, \ldots, Y_N^{[k]}.$$

By the CPCS idea for the EM method (2.1), we present fully explicit EM methods, the CPCS Euler–Maruyama (CPCSEM) method,

$$\begin{aligned} Y_{i+1}^{[k]} = Y_i^{[k]} &+ \varDelta \left( (1-\theta) f\left(t_i, Y_i^{[k-1]}\right) + \theta f\left(t_i, Y_i^{[k]}\right) \right) \\ &+ \sum_{j=1}^{m} \Big( (1-\sigma) g_j \left(t_i, Y_i^{[k-1]}\right) + \sigma g_j \left(t_i, Y_i^{[k]}\right) \Big) \Delta W_i^j, \quad \theta, \sigma \in \mathbb{R} \end{aligned} \tag{2.2}$$

and the Maximum CPCS Euler–Maruyama (MCPCSEM) method,

$$\begin{aligned} Y_{i+1}^{[k]} = Y_i^{[k]} &+ \varDelta f\left(t_i, \max\left(Y_i^{[k-1]}, Y_i^{[k]}\right)\right) \\ &+ \sum_{j=1}^{m} g_j \left(t_i, \max\left(Y_i^{[k-1]}, Y_i^{[k]}\right)\right) \Delta W_i^j, \end{aligned} \tag{2.3}$$

where $Y_i^{[-1]} = Y_i^{[0]} = Y_0$.

## 3. Convergence Properties

In this section, we prove the mean-square convergence of the CPCSEM (2.2) and MCPCSEM (2.3) methods by the following assumption and convergence lemma given in [15].

**Assumption 3.1.** The functions $f$ and $g_j$, $j = 1, \ldots, m$ in SDE (1.1) satisfy in the Lipschitz condition

$$|f(t,x) - f(t,y)|^2 \vee |g_j(t,x) - g_j(t,y)|^2 \leq \mathcal{K}_1 |x-y|^2, \tag{3.1}$$

and linear growth bounds

$$|f(t,x)|^2 \vee |g_j(t,x)|^2 \leq \mathcal{K}_2(1 + |x|^2), \tag{3.2}$$

for the all real variables $x, y$. Here $\mathcal{K}_1$, $\mathcal{K}_2$ are positive constants, and $\vee$ is a maximal operator.

**Lemma 3.2.** *Assume for a one-step discrete time approximation $Y$, the local mean error and mean-square error for all $i = 0, 1, \ldots, N-1$ and $r = 0, 1, \ldots, k$, satisfy the estimates*

$$\left|\mathbb{E}\left[Y_{i+1}^{[k]} - Y^{[k]}(t_{i+1})\middle|Y_i^{[k-r]} = Y^{[k]}(t_i)\right]\right| \leq \mathcal{K}\left(1 + \left|Y_i^{[k]}\right|^2\right)^{1/2} h^{p_1}, \quad (3.3)$$

*and*

$$\left|\mathbb{E}\left[\left(Y_{i+1}^{[k]} - Y^{[k]}(t_{i+1})\right)^2 \middle|Y_i^{[k-r]} = Y^{[k]}(t_i)\right]\right|^{1/2} \leq \mathcal{K}\left(1 + \left|Y_i^{[k]}\right|^2\right)^{1/2} h^{p_2}, \quad (3.4)$$

*where $p_2 \geq \frac{1}{2}$ and $p_1 \geq p_2 + \frac{1}{2}$. Then*

$$\left|\mathbb{E}\left[\left(Y_n^{[k]} - Y^{[k]}(t_n)\right)^2 \middle|Y_0 = Y(t_0)\right]\right|^{1/2} \leq \mathcal{K}\left(1 + |Y_0|^2\right)^{1/2} h^{p_2 - 1/2},$$

*holds for each $n = 0, 1, \ldots, N$. Here $\mathcal{K}$ is independent of $\Delta$, but dependent on the length of the time interval $T - t_0$.*

**Theorem 3.3.** *For $l = 0, 1, \ldots, N$, let $Y_l$ be the numerical approximation of $Y(t_l)$ at time $T$ after $l$ steps with stepsize $\Delta = (T - t_0)/N$. Apply one of the CPCSEM (2.2) and MCPCSEM (2.3) methods to the SDE (1.1) under Assumption 3.1, then we have*

$$\left|\mathbb{E}\left[\left(Y_l^{[k]} - Y^{[k]}(t_l)\right)^2 \middle|Y_0 = Y(t_0)\right]\right|^{1/2} = \mathcal{O}(\Delta^{\frac{1}{2}}).$$

*Proof.* At first we deal with the local mean error. For $i = 0, 1, \ldots N-1$, we have the EM approximation step

$$\widehat{Y}_{i+1}^{[k]} = \widehat{Y}_i^{[k]} + \Delta f(t_i, \widehat{Y}_i^{[k]}) + \sum_{j=1}^{m} g_j(t_i, \widehat{Y}_i^{[k]})\Delta W_i^j, \quad (3.5)$$

with the local mean and mean-square errors

$$\left|\mathbb{E}\left[\left(\widehat{Y}_{i+1}^{[k]} - Y^{[k]}(t_{i+1})\right)\middle|\widehat{Y}_i^{[k]} = Y^{[k]}(t_i)\right]\right| = \mathcal{O}(\Delta^2), \quad (3.6a)$$

$$\left|\mathbb{E}\left[\left(\widehat{Y}_{i+1}^{[k]} - Y^{[k]}(t_{i+1})\right)^2\middle|\widehat{Y}_i^{[k]} = Y^{[k]}(t_i)\right]\right| = \mathcal{O}(\Delta^2), \quad (3.6b)$$

respectively. Then from (3.6a), we have

$$\begin{aligned}\mathcal{H}_1 &= \left|\mathbb{E}\left[\left(Y_{i+1}^{[k]} - Y^{[k]}(t_{i+1})\right)\middle|Y_i^{[k-r]} = Y^{[k]}(t_i)\right]\right| \\ &\leq \left|\mathbb{E}\left[\left(\widehat{Y}_{i+1}^{[k]} - Y^{[k]}(t_{i+1})\right)\middle|\widehat{Y}_i^{[k]} = Y^{[k]}(t_i)\right]\right| \\ &\quad + \left|\mathbb{E}\left[\left(Y_{i+1}^{[k]} - \widehat{Y}_{i+1}^{[k]}\right)\middle|Y_i^{[k-r]} = \widehat{Y}_i^{[k]}\right]\right| \\ &\leq \mathcal{O}(\Delta^2) + \mathcal{H}_2,\end{aligned} \quad (3.7)$$

for the CPCSEM (2.2) and MCPCSEM (2.3) methods, it is easy to see that

$$\begin{aligned}\mathcal{H}_2 &= \left|\mathbb{E}\left[\left(Y_{i+1}^{[k]} - \widehat{Y}_{i+1}^{[k]}\right)\middle|Y_i^{[k-r]} = \widehat{Y}_i^{[k]}\right]\right| \\ &= 0.\end{aligned} \quad (3.8)$$

Consequently, the estimates with $p_1 = 2$ in Theorem 3.1 are satisfied for the CPCSEM (2.2) and MCPCSEM (2.3) methods.

Next, we consider local mean-square error for the CPCSEM (2.2) and MCPCSEM (2.3) methods, by the inequality $(a+b)^2 \leq 2(a^2+b^2)$

$$\begin{aligned}\mathcal{H}_3 &= \mathbb{E}\left[\left(Y_{i+1}^{[k]} - Y^{[k]}(t_{i+1})\right)^2 \Big| Y_i^{[k-r]} = Y^{[k]}(t_i)\right]\\ &\leq 2\mathbb{E}\left[(\widehat{Y}_{i+1}^{[k]} - Y^{[k]}(t_{i+1}))^2 \Big| \widehat{Y}_i^{[k]} = Y^{[k]}(t_i)\right]\\ &\quad + 2\mathbb{E}\left[(Y_{i+1}^{[k]} - \widehat{Y}_{i+1}^{[k]})^2 \Big| Y_i^{[k-r]} = \widehat{Y}_i^{[k]}\right]\\ &\leq 2\mathcal{O}(\Delta^2) + 2\mathcal{H}_4,\end{aligned} \tag{3.9}$$

similarly, for the CPCSEM (2.2) and MCPCSEM (2.3) methods, we obtain

$$\begin{aligned}\mathcal{H}_4 &= \left|\mathbb{E}\left[\left(Y_{i+1}^{[k]} - \widehat{Y}_{i+1}^{[k]}\right)^2 \Big| Y_i^{[k-r]} = \widehat{Y}_i^{[k]}\right]\right|\\ &= 0.\end{aligned} \tag{3.10}$$

Thus we can choose in Lemma 3.1 the exponent $p_2 = 1$ together with $p_1 = 2$ and apply it to finally prove the strong order $p = p_2 - \frac{1}{2} = \frac{1}{2}$ of the CPCSEM (2.2) and MCPCSEM (2.3) methods, as was claimed in Theorem 3.3. □

## 4. Results and Discussion

In this section, we give some numerical examples that demonstrate the computational efficiency of the proposed methods in this paper. To compare their accuracy, we present various examples of linear and nonlinear equations. In some of these examples, despite of the absence of the established assumptions of Theorem 3.3, our scheme is efficient; in other words, these examples show that the analytical conditions of Theorem 3.3 are sufficient for mean-square convergence rather than necessary for an appropriate numerical approximation.

In this paper, to estimate the error in the mean-square sense at time $T = N\Delta$ for various step sizes $\Delta$, we define $\varepsilon_{\text{MS}}$ as follows:

$$\varepsilon_{\text{MS}} = \left(\frac{1}{N}\sum_{i=1}^{N}\left|Y_N^{(i)} - X_{t_N}^{(i)}\right|^2\right)^{\frac{1}{2}}. \tag{4.1}$$

Denoting $Y_N^{(i)}$ and $X_{t_N}^{(i)}$ as the numerical solutions and the exact solution at step point $t_N$ in $i$th simulation, respectively. To simulate the $\Delta W_i^j$ with distribution $\mathcal{N}(0,\Delta)$, we have used random numbers generated by `randn` in MATLAB R2012a.

*Example* 1. For linear test system

$$\mathrm{d}X(t) = \lambda X(t)\mathrm{d}t + \sum_{j=1}^{m}\mu_j X(t)\mathrm{d}W^j(t), \quad X_0 = 1, \tag{4.2}$$

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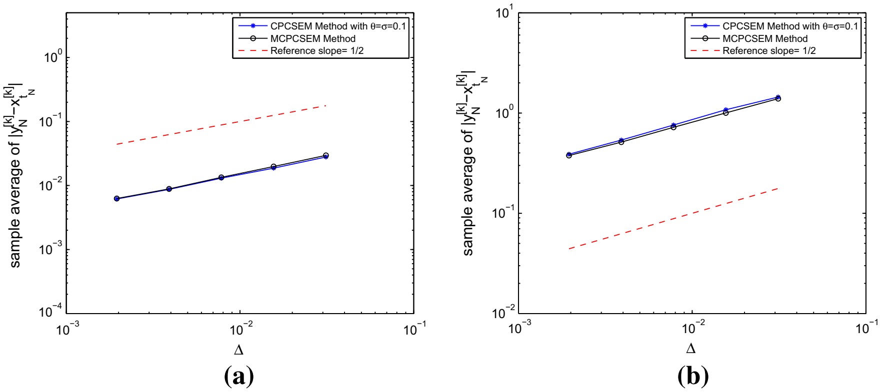


Figure 1. The convergence rate of the CPCSEM with $\theta = \sigma = 0.1$, and MCPCSEM methods for SDE (4.2)

with exact solution

$$X(t) = X_0 \exp\left(\left(\lambda - \tfrac{1}{2}\sum_{j=1}^{m}\mu_j^2\right) t + \sum_{j=1}^{m}\mu_j W_j(t)\right),$$

we show the strong convergence rate of CPCSEM (2.2) and MCPCSEM (2.3) methods at the terminal time $T = N\Delta = 1$, for two groups of parameters

- Case I: $\lambda = -\mu = -\frac{1}{2}$, $N = 1000$ [7],
- Case II: $\lambda = 1.5$, $\mu_1 = -1.0$, $\mu_2 = -\mu_3 = \mu_4 = -\mu_5 = 0.5$ and $N = 5000$.

Figure 1 shows a log–log plot of the sample average approximation $|Y_i^{[k]} - X_{t_i}^{[k]}|$ against $\Delta$, based on the $N$ different discretized Wiener process paths over $[0, 1]$ with the stepsize $\delta t = 2^{-9}$. Denoting $Y_i^{[k]}$ and $X_{t_i}^{[k]}$ as the numerical solutions and the exact solution at step point $t_i$ in $i$th simulation, respectively. For each realization, we have applied the CPCSEM and MCPCSEM methods with five different stepsizes $\Delta = 2^{j-1}\delta t$ for $1 \le j \le 5$. The reference line of slope $\frac{1}{2}$ is added in a dashed line type. This is consistent with the result that the strong error is arbitrarily close to order $\frac{1}{2}$.

*Example* 2. Consider the nonlinear SDE [19],

$$\mathrm{d}X(t) = \left(\tfrac{1}{3}X^{\frac{1}{3}}(t) + 6X^{\frac{2}{3}}(t)\right)\mathrm{d}t + X^{\frac{2}{3}}(t)\mathrm{d}W(t), \tag{4.3}$$

for $t \in [0, 1]$, with the initial value $X_0 = 1$. The exact solution of (4.3) is

$$X(t) = \left(1 + 2t + \frac{W(t)}{3}\right)^3.$$

In Fig. 2, we plot the mean-square errors $\varepsilon_{\mathrm{MS}}$, with the timestep $\Delta = \frac{2^{(1-i)}}{25}$, $i = 1, 2, \ldots, 5$ and $N = 5000$ independent simulations at $t = 1$ for EM,

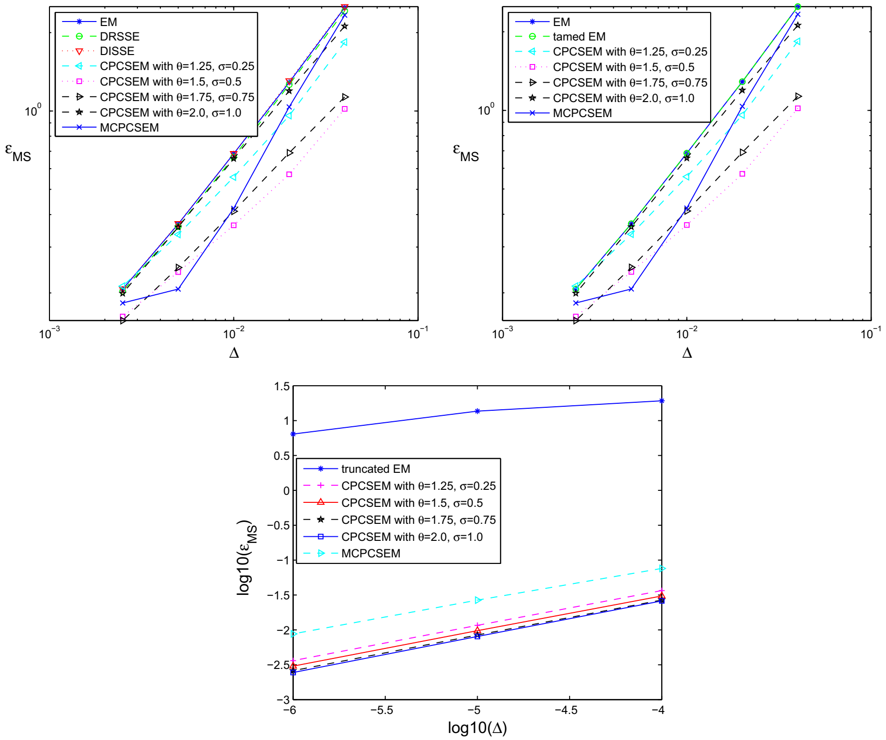


Figure 2. Value of the mean-square errors, $\varepsilon_{\mathrm{MS}}$ (4.1), of the EM, CPCSEM and MCPCSEM, DRSSE [21] and DISSE [21](up-left), tamed EM [10] (up-right) and truncated EM [13,14] (down) methods for nonlinear SDE (4.3).

DRSSE [21] and DISSE [21], tamed EM [10], CPCSEM and MCPCSEM methods. In the second row of Fig. 2, the our methods compared with the truncated EM method [13,14], with $\mu(r) = \frac{1}{3}r^{\frac{1}{3}} + 6r^{\frac{2}{3}}$, $h(\Delta) = \Delta^{-1/4}$, timestep $\Delta = 10^{(-i)}$, $i = 4, 5, 6$ and $N = 1000$. It is obvious that the accuracies of the CPCSEM and MCPCSEM methods are better than of the EM, DRSSE, DISSE, tamed EM and truncated EM methods.

*Example* 3. Third example is a two-dimensional linear SDE system whose Itô form,

$$\mathrm{d}X(t) = UX(t)\mathrm{d}t + VX(t)\mathrm{d}W(t), \quad X_0 = X(t_0), \quad t \in [0,1], \quad X \in \mathbb{R}^2, \tag{4.4}$$

where $U$ and $V$ are the matrices

$$U = \begin{pmatrix} -u & u \\ u & -u \end{pmatrix}, \qquad V = \begin{pmatrix} v & 0 \\ 0 & v \end{pmatrix}. \tag{4.5}$$

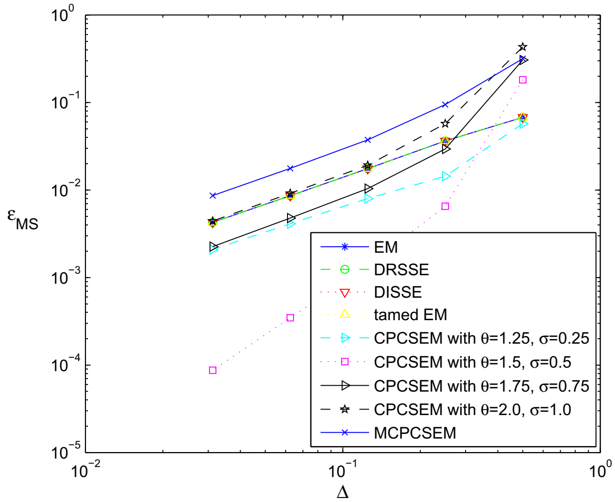


Figure 3. Value of the mean-square errors, $\varepsilon_{\text{MS}}$ (4.1), of the EM, DRSSE [21], DISSE [21], tamed EM [10], CPCSEM and MCPCSEM methods for nonlinear SDE (4.4)–(4.5).

The exact solution of this equation is given by [12],

$$X(t) = P \begin{pmatrix} \exp(\rho^+(t)) & 0 \\ 0 & \exp(\rho^-(t)) \end{pmatrix} P^{-1} X_0, \quad P = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

where $\rho^{\pm}(t) = (u - \frac{1}{2}v^2 \pm u)t + vW(t)$.

Figure 3 gives the mean-square errors $\varepsilon_{\text{MS}}$ of the EM, DRSSE [21], DISSE [21], tamed EM [10], CPCSEM and MCPCSEM methods, for solving (4.4)–(4.5) with $X_0 = [1, 2]^T$, $u = 1$, $v = 0.01$, timestep $\Delta = 2^{(-i)}$, $i = 1, 2, \ldots, 5$ and $N = 5000$ independent simulations. It can be seen that the accuracies of the CPCSEM method with parameters $(\theta, \sigma) = (1.25, 0.25)$, $(1.5, 0.5)$ and $(1.75, 0.75)$ are better than of the EM, DRSSE, DISSE, tamed EM, MCPCSEM methods and the CPCSEM method with parameter $(\theta, \sigma) = (2.0, 1.0)$.

*Example* 4. The last test equation is a stochastic version of the Brusselator system, for modeling unforced periodic oscillations in the certain chemical reactions [20],

$$\begin{aligned} \mathrm{d}X_1(t) = {} & \Big((\alpha - 1)X_1(t) + \alpha X_1^2(t) + (X_1(t) + 1)^2 X_2(t)\Big)\mathrm{d}t \\ & + \gamma X_1(t)(1 + X_1(t))\mathrm{d}W(t), \\ \mathrm{d}X_2(t) = {} & \Big( - \alpha X_1(t) - \alpha X_1^2(t) - (X_1(t) + 1)^2 X_2(t)\Big)\mathrm{d}t \\ & - \gamma X_1(t)(1 + X_1(t))\mathrm{d}W(t). \end{aligned} \tag{4.6}$$

Comparison of numerical simulations of system (4.6) for $0 \le t \le 125$, step-size $\Delta = 0.025$, constant parameters $\alpha = 1.9$, $\gamma = 0.1$, and starting point

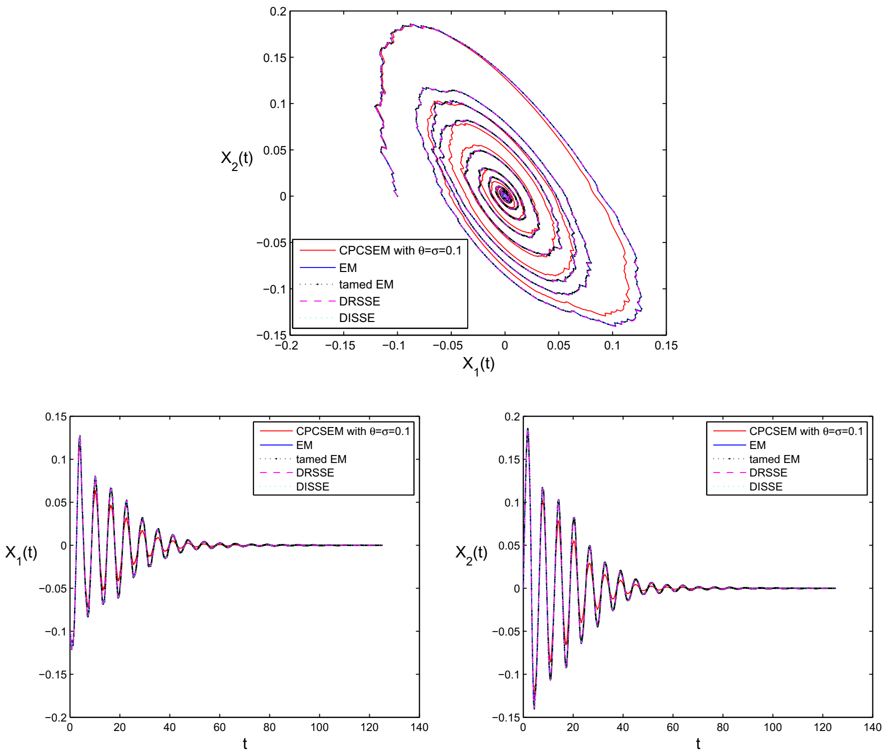


Figure 4. Numerical simulations of the system (4.6) by the EM, DRSSE, DISSE, tamed EM and CPCSEM (with $\theta = \sigma = 0.1$) methods

$(X_1(0), X_2(0)) = (-0.1, 0)$ by the EM, DRSSE, DISSE, tamed EM methods and the CPCSEM method, with $\theta = \sigma = 0.1$, and the MCPCSEM method, are plotted in Figs. 4 and 5, respectively. Observe that the approximate trajectories of the CPCSEM and MCPCSEM methods stay close to the origin, replicating the behavior of the exact solution, and yielding the better approximations than the EM, DRSSE, DISSE and tamed EM methods.

## 5. Conclusions

In this paper, we have constructed the CPCS idea for solving SDEs and derived the CPCSEM and MCPCSEM methods based on the CPCS scheme. Further, the mean-square convergence order of $\frac{1}{2}$ is obtained for the presented methods by fundamental theorem derived in [15]. To highlight the convergence, the numerical results are reported for both one- and two-dimensional SDEs, in linear and nonlinear cases. In addition, we compared our methods against the EM, DRSSE, DISSE, tamed EM and truncated EM methods of

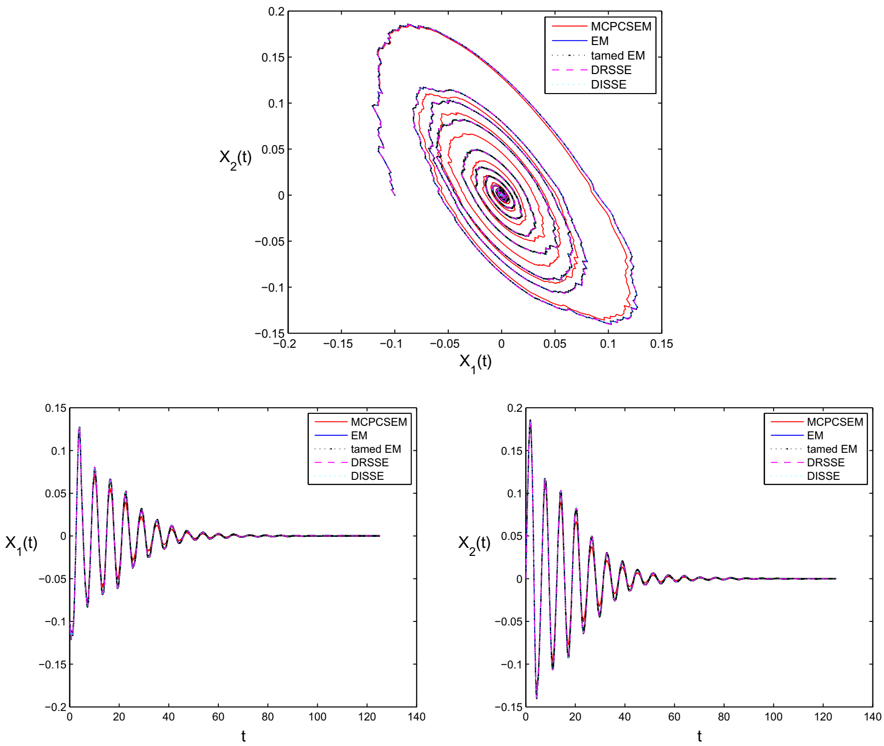


Figure 5. Numerical simulations of the system (4.6) by the EM, DRSSE, DISSE, tamed EM and MCPCSEM methods

the under investigation problems, and numerically showed that the CPCSEM and MCPCSEM methods are computationally more efficient.

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