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On the Existence of Ground State Solutions for Fractional Schrödinger–Poisson Systems with General Potentials and Super-quadratic Nonlinearity

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Abstract. In this article, we are concerned with the following fractional Schrödinger–Poisson system:

$$\begin{cases} (-\Delta)^s u + V(x)u + \phi u = f(u) & \text{in } \mathbb{R}^3, \\ (-\Delta)^t \phi = u^2 & \text{in } \mathbb{R}^3, \end{cases}$$

where $0 < s \leq t < 1$, $2s + 2t > 3$, and $f \in C(\mathbb{R}, \mathbb{R})$. Under more relaxed assumptions on potential $V(x)$ and $f(x)$, we obtain the existence of ground state solutions for the above problem by adopting some new tricks. Our results here extend the existing study.

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1. Introduction

In the present paper, we deal with the existence of ground state solutions for the following fractional Schrödinger–Poisson problem:

$$\begin{cases} (-\Delta)^s u + V(x)u + \phi u = f(u) & \text{in } \mathbb{R}^3, \\ (-\Delta)^t \phi = u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $0 < s \leq t < 1$, $2s + 2t > 3$, and $(-\Delta)^s$ is the fractional Laplacian of order s. Here, the fractional Laplacian $(-\Delta)^s$ is defined, up to normalization factors, by the following singular integral:

$$(-\Delta)^s u(x) = C_s \text{P.V.} \int_{\mathbb{R}^3} \frac{u(x) - u(y)}{|x - y|^{3+2s}} \mathrm{d}y,$$

where P.V. is a commonly used abbreviation for "in the principle value sense" and C_s is a dimensional constant that depends on s. Via the Fourier transform

$\mathcal{F}$, $(-\Delta)^s$ can also be computed by the following:

$$(-\Delta)^\alpha u = \mathcal{F}^{-1}(|\xi|^{2\alpha}(\mathcal{F}u)), \quad \forall\, \xi \in \mathbb{R}^3,$$

(see [13] and the references therein for further details on fractional Laplacian). V and f satisfy

(V_1) $V \in L^\infty(\mathbb{R}^3)$ and $\inf_{x\in\mathbb{R}^3} V(x) > 0$;

(F_1) $f \in C(\mathbb{R}, \mathbb{R})$, and there exists constants $C_0 > 0$ and $p \in (2, 2_s^*)$, where $2_s^* = \frac{6}{3-2s}$ is the fractional critical Sobolev exponent, such that

$$|f(u)| \le C_0(1 + |u|^{p-1}), \quad \forall\, u \in \mathbb{R};$$

(F_2) $f(u) = o(u)$ as $u \to 0$.

Recently, fractional Laplacian equations have concrete applications in many fields, such as thin obstacle problem, optimization, finance, phase transitions, anomalous diffusion, and so on. For previous related results, see [1,4,5,8,9,12,14,18,21–23,27,31,32] and the references therein.

System (1.1) is called a fractional Schrödinger–Poisson system, which is also called fractional Schrödinger–Maxwell system, because it consists of a fractional Schrödinger equation coupled with a Poisson term. It is well known that a great attention has been devoted to the fractional and non-local integro-differential operators like (1.1), for the thought-provoking theoretical structure and their impressive applications in many fields. In fact, the fractional Laplacian $(-\Delta)^s$ is a non-local operator in the fractional Schrödinger equation, which is obvious a difficulty. And then, Caffarelli and Silvestrein made greatest achievement in overcoming this difficulty by the extension theorem in [7]. The authors used some extension to transform the non-local problem into a local problem, and established some existence and nonexistence of Dirichlet problem involving the fractional Laplacian on bounded domain. Furthermore, a great deal of progress has been made to the fractional Laplacian equations after the work [7].

If $s = t = 1$, $x \in \mathbb{R}^3$, System (1.1) reduces to the classical Schrödinger–Poisson system:

$$\begin{cases} -\Delta u + V(x)u + \phi u = f(u) & \text{in } \mathbb{R}^3, \\ -\Delta\phi = u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.2}$$

which was first introduced by Benci and Fortunato in [3] to describe the interaction of a charge particle with an electromagnetic field. The existence and multiplicity of solutions of System (1.2) had been investigated extensively by many authors in the past several years; we refer the interested readers to see [2,5,16,19,24,28,30,33] and the references therein. The literature mainly focuses on the study of System (1.2) with $V(x) \equiv 1$ or $V(x) = \overline{V}(|x|)$, and f satisfies the following assumptions of Ambrosetti–Rabinowitz type and 4-superlinear as follows:

(AR) $f(u)u \ge 4F(u) \ge 0$, $\forall u \in \mathbb{R}$, where $F(u) = \int_0^u f(s)\mathrm{d}s$;

(SF) $\lim_{|u|\to\infty} \frac{F(u)}{u^4} = \infty$.

In fact, for (AR) and (SF), it is easy to verify the Mountain Pass geometry and the boundedness of (PS) or $(\mathrm{C})_\mathrm{c}$ sequences.

When $V(x) = 1$ as follows:

$$\begin{cases} -\Delta u + u + \phi u = f(u) & \text{in } \mathbb{R}^3, \\ -\Delta \phi = u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.3}$$

more specially, the case that $f(u) = |u|^{p-2}u$ associated with (1.3) has been paid much attention by various authors. In detail, for $p \in (4, 6)$, in [6,10], a radial positive solution of (1.3) was obtained and the corresponding energy functional was proved to attain a local minimum at zero by Mountain Pass Theorem. On the other hand, for the aim of obtaining the nonexistence of nontrivial solutions of (1.3) for $p \leq 2$ or $p \geq 6$, a related Pohozaev equality was provided in [11]. Later, Ruiz [19] proved the existence of a positive radial solution for $3 < p \leq 4$, and the nonexistence of any nontrivial solution for $2 < p \leq 3$. Obviously, the result fills the gap $p \in (2, 4]$ left in the previous study. Ruiz's approach in [19] is to get the minimizer on the Nehari–Pohozaev manifold $\mathcal{M}$, which is defined as a linear combination of the Nehari manifold and the Pohozaev manifold. However, we should also notice that Ruiz's method cannot be applied for general nonlinearity f. Then, Sun and Ma [20] proved that (1.3) admits a least energy solution if f satisfies (F_1), (F_2) and the following assumption of Ambrosetti–Rabinowitz type (AR$'$) there exists $\mu > 3$, such that $f(u)u \geq \mu F(u) > 0$ for $u \in \mathbb{R}\backslash\{0\}$.

Actually, Sun and Ma [20] employed Jeanjean's monotonicity trick [17] to get a bounded (PS) sequence, then adopted Pohozaev identity and global compactness lemma to obtain a series of nontrivial critical points, which were used to construct a special (PS) sequence, and then proved the boundedness of the special (PS) sequence, and hence, got a nontrivial critical point of the initial problem. More recently, by exploiting some new tricks with mild conditions on potential V and f, Tang and Chen [25] made a substantial improvement to the main results in [20].

When $f(u) = \mu|u|^{q-2}u + |u|^{2_s^*-2}u$, $\mu \in \mathbb{R}^+$ is a parameter, $q \in (2, 2_s^*)$, $s, t \in (0, 1)$ and $2s + 2t > 3$, taking advantage of Pohozaev–Nehari manifold, the arguments of Brezis–Nirenberg, the monotonic tricks and global compactness lemma, Teng [26] investigated the existence of a nontrivial ground state solution for System (1.1). Moreover, in the situation, where the nonlinearity $f(u) = |u|^{p-2}u$ has subcritical growth, $p \in (3, 2_s^*)$, $t = s \in (\frac{3}{4}, 1)$, V satisfies (V_1) and the following assumptions:

- (V_2) $V(\infty) := \liminf_{|y|\to\infty} V(y) \geq (\not\equiv) V(x)$.
- (V_3) $V(x)$ is weakly differentiable, and satisfies $(\nabla V(x), x) \in L^\infty(\mathbb{R}^3) \cup L^{\frac{2_s^*}{2_s^*-2}}(\mathbb{R}^3)$:

$$2sV(x) + (\nabla V(x), x) \geq 0 \quad \text{a.e. } x \in \mathbb{R}^3.$$

With the similar spirit of [30], Teng [29] studied the existence of ground state solutions, which is a minimizer of the reduced functional restricted on the manifold introduced in [19]. It is worth mentioning that the approach is invalid for generally nonlinear case.

It is natural to ask whether or not the existence results got in those classical contexts can be extended to non-local fractional systems. Motivated by the results mentioned above, especially [15, 25], our main goal in the paper is to prove the existence of a ground solution for System (1.1), which makes a substantial improvement to the main results in [29].

Before stating our main results, we shall introduce the following assumptions on the potential V and the nonlinearity f as follows:

(V_4) $V(x)$ is weakly differentiable, and satisfies $(\nabla V(x), x) \in L^{\infty}(\mathbb{R}^3)$, and for some $\varrho_0 > 0$

$$2sV(x) + (\nabla V(x), x) \geq \varrho_0 \text{ a.e. } x \in \mathbb{R}^3,$$

(F_3) $\lim_{|u|\to\infty} \frac{F(u)}{|u|^3} = \infty$.

(F_4) $[(s+t)f(u)u - 3F(u)]/|u|^{(4s+2t)/(s+t)}$ is a nondecreasing function of u on $\mathbb{R}\backslash\{0\}$.

(F_5) There exist $\kappa > \frac{3}{2s}$ and $C_1 > 0$, such that

$$\frac{f(u)}{u} > \frac{\gamma_0}{2} \Rightarrow \left|\frac{f(u)}{u}\right|^{\kappa} \leq C_1\Big[(s+t)f(u)u - (4s+2t)F(u)\Big],$$

where γ_0 is Sobolev imbedding constant, such that $\gamma_0\|u\|_2^2 \leq \|u\|^2$ for $u \in H^s(\mathbb{R}^3)$.

Remark 1.1. There are indeed functions which satisfy (V_1)−(V_4). An example is given by $V(x) = V_1 - \frac{1}{|x|+1}$, where $V_1 > 1$ is a positive constant.

We are now in a position to state the main results of this paper.

Theorem 1.2. *Suppose that* (V_1)–(V_3) *and* (F_1)–(F_5) *hold. Then, System (1.1) has a ground state solution* $u_0 \in H^s(\mathbb{R}^3)\backslash\{0\}$.

Theorem 1.3. *Suppose that* (V_1), (V_2), (V_4) *and* (F_1)−(F_4) *hold. Then, System (1.1) has a ground state solution* $u_0 \in H^s(\mathbb{R}^3)\backslash\{0\}$.

The plan of this paper is as follows. In Sect. 2, we present some notations and preliminaries. In Sect. 3, we prove Theorems 1.2 and 1.3, respectively.

2. Preliminary Results

Throughout the paper, we denote by $\|\cdot\|_p$ the usual norm of the space $L^p(\mathbb{R}^3)$ and by $\widehat{u}$ the usual Fourier transform of u, the letters c_i, C, C_i stand for different positive constants. Moreover, we set $u_\tau = \tau^{s+t}u(\tau x)$. Next, we establish the variational setting of System (1.1) in fractional Sobolev spaces.

A complete introduction to fractional Sobolev spaces can be found in [13]. For fixed $\alpha \in (0,1)$, we define the homogeneous fractional Sobolev space $\mathcal{D}^{\alpha,2}(\mathbb{R}^3)$ as follows:

$$\mathcal{D}^{\alpha,2}(\mathbb{R}^3) = \left\{u \in L^{2^*_\alpha}(\mathbb{R}^3)\Big| |\xi|^{\alpha}\widehat{u}(\xi) \in L^2(\mathbb{R}^3)\right\},$$

which is the completion of $C_0^{\infty}(\mathbb{R}^3)$ with the norm:

$$\|u\|_{\mathcal{D}^{\alpha,2}} = \int_{\mathbb{R}^3} |\xi|^{2\alpha}|\widehat{u}(\xi)|^2 \mathrm{d}\xi.$$

From Plancherel's theorem we have $\|u\|_2 = \|\widehat{u}\|_2$, and then $\||\xi|^\alpha\widehat{u}\|_2 = \|(-\Delta)^{\frac{\alpha}{2}}u\|_2$. The fractional Sobolev space $H^\alpha(\mathbb{R}^3)$ can be described through the Fourier transform, that is

$$H^\alpha(\mathbb{R}^3) = \left\{u \in L^2(\mathbb{R}^3)\Big| \int_{\mathbb{R}^3}\Big(|\xi|^{2\alpha}|\widehat{u}(\xi)|^2 + |\widehat{u}(\xi)|^2\Big)\mathrm{d}\xi < +\infty\right\}.$$

In this case, the inner product and the norm are defined, respectively, as

$$(u, v) = \int_{\mathbb{R}^3}\Big(|\xi|^{2\alpha}\widehat{u}(\xi)\widehat{v}(\xi) + \widehat{u}(\xi)\widehat{v}(\xi)\Big)\mathrm{d}\xi,$$

and

$$\|u\|_{H^\alpha} = \left(\int_{\mathbb{R}^3}\Big(|\xi|^{2\alpha}|\widehat{u}(\xi)|^2 + |\widehat{u}(\xi)|^2\Big)\mathrm{d}\xi\right)^{\frac{1}{2}}.$$

Hence

$$\|u\|_{H^\alpha} = \left(\int_{\mathbb{R}^3}(|(-\Delta)^{\frac{\alpha}{2}}u(x)|^2 + |u(x)|^2)\mathrm{d}x\right)^{\frac{1}{2}}, \quad \forall\, u \in H^\alpha(\mathbb{R}^3).$$

For simplicity, we denote $\|\cdot\|$ by $\|\cdot\|_{H^\alpha}$ in the sequel.

In terms of finite differences, the fractional Sobolev space $H^\alpha(\mathbb{R}^3)$ can also be defined as follows:

$$H^\alpha(\mathbb{R}^3) = \left\{u \in L^2(\mathbb{R}^3)\Big|\frac{|u(x) - u(y)|}{|x - y|^{\alpha+\frac{3}{2}}} \in L^2(\mathbb{R}^3 \times \mathbb{R}^3)\right\}$$

endowed with the natural norm:

$$\|u\|_{H^\alpha} = \left(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u(x) - u(y)|^2}{|x - y|^{2\alpha+3}}\mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^3}|u|^2\mathrm{d}x\right)^{\frac{1}{2}}.$$

In addition, in light of Propositions 3.4 and 3.6 in [13], we have

$$\|(-\Delta)^{\frac{\alpha}{2}}u\|_2^2 = \int_{\mathbb{R}^3}|\xi|^{2\alpha}|\widehat{u}(\xi)|^2\mathrm{d}\xi = \frac{1}{C(\alpha)}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u(x) - u(y)|^2}{|x - y|^{2\alpha+3}}\mathrm{d}x\mathrm{d}y.$$

By [13], $H^\alpha(\mathbb{R}^3) \hookrightarrow L^q(\mathbb{R}^3)$ is continuous for $q \in [2, 2^*_\alpha]$ and $H^\alpha(\mathbb{R}^3) \hookrightarrow L^q_{\mathrm{loc}}(\mathbb{R}^3)$ is compact for $q \in [2, 2^*_\alpha)$, and for any $\alpha \in (0, 1)$, there exists a best constant $\mathcal{S}_\alpha > 0$, such that

$$\mathcal{S}_\alpha = \inf_{u\in\mathcal{D}^{\alpha,2}}\frac{\int_{\mathbb{R}^3}|(-\Delta)^{\frac{\alpha}{2}}u|^2\mathrm{d}x}{\left(\int_{\mathbb{R}^3}|u(x)|^{2^*_\alpha}\mathrm{d}x\right)^{\frac{2}{2^*_\alpha}}}. \tag{2.1}$$

Next, we assume that $s, t \in (0, 1)$. Observe that if $4s + 2t \geq 3$, then it follows that $2 \leq \frac{12}{3+2t} \leq \frac{6}{3-2s}$ and thus $H^s(\mathbb{R}^3) \hookrightarrow L^{\frac{12}{3+2t}}(\mathbb{R}^3)$. For $u \in H^s(\mathbb{R}^3)$, the linear functional $\mathcal{L}_u : \mathcal{D}^{t,2}(\mathbb{R}^3) \to \mathbb{R}$ is defined by

$$\mathcal{L}_u(v) = \int_{\mathbb{R}^3}u^2v\mathrm{d}x. \tag{2.2}$$

The Hölder inequality and (2.2) imply that

$$|\mathcal{L}_u(v)| \leq \left(\int_{\mathbb{R}^3}|u(x)|^{\frac{12}{3+2t}}\mathrm{d}x\right)^{\frac{3+2t}{6}}\left(\int_{\mathbb{R}^3}|v(x)|^{2^*_t}\mathrm{d}x\right)^{\frac{1}{2^*_t}} \leq C\|u\|^2\|v\|_{\mathcal{D}^{t,2}}. \tag{2.3}$$

Then, by the Lax–Milgram theorem, there exists a unique $\phi_u^t \in \mathcal{D}^{t,2}(\mathbb{R}^3)$, such that

$$\int_{\mathbb{R}^3} (-\Delta)^{\frac{t}{2}} \phi_u^t (-\Delta)^{\frac{t}{2}} v \mathrm{d}x = \int_{\mathbb{R}^3} u^2 v \mathrm{d}x, \quad \forall\, v \in \mathcal{D}^{t,2}(\mathbb{R}^3), \tag{2.4}$$

that is, ϕ_u^t is a weak solution of

$$(-\Delta)^t \phi_u^t = u^2, \quad x \in \mathbb{R}^3,$$

and the representation formula holds:

$$\phi_u^t(x) = c_t \int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|^{3-2t}} \mathrm{d}y, \quad x \in \mathbb{R}^3, \tag{2.5}$$

which is called t-Riesz potential, where

$$c_t = \pi^{-\frac{3}{2}} 2^{-2t} \frac{\Gamma(3-2t)}{\Gamma(t)}.$$

Throughout the sequel, we often omit the constant c_t in (2.5) for convenience. Substituting ϕ_u^t in (1.1), it leads to the following fractional Schrödinger equation, when $V(x) = 1$:

$$(-\Delta)^s u + u + \phi_u^t u = f(u), \quad x \in \mathbb{R}^3, \tag{2.6}$$

whose solutions can be obtained by seeking critical points of the functional $\varphi : H^s(\mathbb{R}^3) \to \mathbb{R}$ defined by

$$\varphi(u) = \frac{1}{2} \int_{\mathbb{R}^3} \left(|(-\Delta)^{\frac{s}{2}} u|^2 + u^2\right) \mathrm{d}x + \frac{1}{4} \int_{\mathbb{R}^3} \phi_u^t u^2 \mathrm{d}x - \int_{\mathbb{R}^3} F(u) \mathrm{d}x.$$

From (2.1)–(2.3), we can deduce that

$$\begin{aligned} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{t}{2}} \phi_u^t|^2 \mathrm{d}x &= \int_{\mathbb{R}^3} \phi_u^t u^2 \mathrm{d}x \le \left(\int_{\mathbb{R}^3} |u(x)|^{\frac{12}{3+2t}} \mathrm{d}x\right)^{\frac{3+2t}{6}} \left(\int_{\mathbb{R}^3} |\phi_u^t|^{2_t^*} \mathrm{d}x\right)^{\frac{1}{2_t^*}} \\ &\le \frac{1}{\sqrt{\mathcal{S}_t}} \left(\int_{\mathbb{R}^3} |u(x)|^{\frac{12}{3+2t}} \mathrm{d}x\right)^{\frac{3+2t}{6}} \|\phi_u^t\|_{\mathcal{D}^{t,2}} \le C\|u\|^2 \|\phi_u^t\|_{\mathcal{D}^{t,2}}. \end{aligned} \tag{2.7}$$

Therefore, (F_1) and (F_2) imply that φ is well defined in $H^s(\mathbb{R}^3)$ and $\varphi \in C^1(H^s(\mathbb{R}^3), \mathbb{R})$. For (V_1), we define the functional in $H^s(\mathbb{R}^3)$ as follows:

$$\Phi(u) = \frac{1}{2} \int_{\mathbb{R}^3} \left(|(-\Delta)^{\frac{s}{2}} u|^2 + V(x) u^2\right) \mathrm{d}x + \frac{1}{4} \int_{\mathbb{R}^3} \phi_u^t u^2 \mathrm{d}x - \int_{\mathbb{R}^3} F(u) \mathrm{d}x,$$

which is also well defined in $H^s(\mathbb{R}^3)$ and $\Phi \in C^1(H^s(\mathbb{R}^3), \mathbb{R})$ with derivative given by

$$\langle \Phi'(u), v\rangle = \int_{\mathbb{R}^3} \left((-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} v + V(x) uv + \phi_u^t uv - f(u)v\right) \mathrm{d}x, \quad \forall\, v \in H^s(\mathbb{R}^3).$$

Evidently, the critical points of Φ are weak solutions of System (1.1).

Lemma 2.1. (see [23, Lemma 2.4]) *Assume that* $\{u_n\}$ *is bounded in* $H^\alpha(\mathbb{R}^N)$ *and*

$$\lim_{n\to\infty} \sup_{y \in \mathbb{R}^N} \int_{B_R(y)} |u_n|^2 \mathrm{d}x = 0,$$

where $R > 0$. *Then,* $u_n \to 0$ *in* $L^q(\mathbb{R}^N)$ *for* $2 < q < 2_\alpha^*$.

Proposition 2.2. [17] *Let X be a Banach space and $\Lambda \subset \mathbb{R}^+$ be an interval. Consider a family I_λ of C^1 functionals on X with the form:*

$$I_\lambda(u) = A(u) - \lambda B(u), \quad \forall\ \lambda \in \Lambda,$$

where $B(u) \geq 0$, $\forall u \in X$, and such that either $A(u) \to +\infty$ or $B(u) \to +\infty$ as $\|u\| \to \infty$. If there exist v_1, $v_2 \in X$, such that

$$c_\lambda := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I_\lambda(\gamma(t)) > \max\Big\{I_\lambda(v_1), I_\lambda(v_2)\Big\}, \quad \forall \lambda \in \Lambda,$$

where $\Gamma = \Big\{\gamma \in C([0,1], X) : \gamma(0) = v_1, \gamma(1) = v_2\Big\}$.

Then, for almost every $\lambda \in \Lambda$, there exists a sequence $\{v_n\} \subset X$, such that

(i) $\{v_n\}$ *is bounded in* X.
(ii) $I_\lambda(v_n) \to c_\lambda$.
(iii) $I'_\lambda(v_n) \to 0$ *in the dual* X^* *of* X.

Moreover, the map $\lambda \to c_\lambda$ is non-increasing and left continuous.

Next, we introduce two families of functional defined by

$$\Phi_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^3} |(-\Delta)^{s/2}u|^2 \mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^3} V(x)u^2 \mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u(x)u^2 \mathrm{d}x - \lambda \int_{\mathbb{R}^3} F(u)\mathrm{d}x \tag{2.8}$$

and

$$\Phi_\lambda^\infty(u) = \frac{1}{2}\int_{\mathbb{R}^3} |(-\Delta)^{s/2}u|^2 \mathrm{d}x + \frac{V(\infty)}{2}\int_{\mathbb{R}^3} u^2 \mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u(x)u^2 \mathrm{d}x - \lambda \int_{\mathbb{R}^3} F(u)\mathrm{d}x, \tag{2.9}$$

for $\lambda \in [1/2, 1]$.

Lemma 2.3. *Assume that* (V_1)–(V_3), (F_1) *and* (F_2) *hold. Let u be a critical point of Φ_λ in $H^s(\mathbb{R}^3)$, then we have the following Pohozaev-type identity:*

$$\begin{aligned}\mathcal{P}_\lambda(u) :&= \frac{3-2s}{2}\int_{\mathbb{R}^3} |(-\Delta)^{s/2}u|^2 \mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^3} \big[3V(x) + (\nabla V(x), x)\big]u^2 \mathrm{d}x \\ &\quad + \frac{3+2t}{4}\int_{\mathbb{R}^3} \phi_u(x)u^2 \mathrm{d}x - 3\lambda \int_{\mathbb{R}^3} F(u)\mathrm{d}x = 0.\end{aligned} \tag{2.10}$$

With the virtue of Pohozaev-type identity, we set $J_\lambda(u) := (s+t)\langle \Phi'_\lambda(u), u\rangle - \mathcal{P}_\lambda(u)$, then

$$\begin{aligned}J_\lambda(u) &= \frac{4s+2t-3}{2}\int_{\mathbb{R}^3} |(-\Delta)^{s/2}u|^2 \mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^3} \big[(2s+2t-3)V(x) - (\nabla V(x), x)\big]u^2 \mathrm{d}x \\ &\quad + \frac{4s+2t-3}{4}\int_{\mathbb{R}^3} \phi_u(x)u^2 \mathrm{d}x - \lambda \int_{\mathbb{R}^3} \big[(s+t)f(u)u - 3F(u)\big]\mathrm{d}x,\end{aligned} \tag{2.11}$$

for $\lambda \in [1/2, 1]$. Moreover, we also let

$$J_\lambda^\infty(u) = \frac{4s+2t-3}{2}\int_{\mathbb{R}^3} |(-\Delta)^{s/2}u|^2 \mathrm{d}x + \frac{(2s+2t-3)V(\infty)}{2}\|u\|_2^2 + \frac{4s+2t-3}{4}\int_{\mathbb{R}^3} \phi_u(x)u^2 \mathrm{d}x - \lambda \int_{\mathbb{R}^3} \big[(s+t)f(u)u - 3F(u)\big]\mathrm{d}x, \tag{2.12}$$

for $\lambda \in [1/2, 1]$.

To state our results, we define

$$\mathcal{M} := \Big\{u \in H^s(\mathbb{R}^3)\backslash\{0\} : J_\lambda(u) = 0\Big\},$$
$$\mathcal{M}_\lambda^\infty := \Big\{u \in H^s(\mathbb{R}^3)\backslash\{0\} : J_\lambda^\infty(u) = 0\Big\},$$

and

$$m_\lambda^\infty := \inf_{u\in\mathcal{M}_\lambda^\infty} \Phi_\lambda^\infty(u).$$

Similar to Lemma 3.2 in [15], we have the following lemma.

Lemma 2.4. *Assume that* (F_1), (F_2) *and* (F_4) *hold. Then*

$$\Phi_\lambda^\infty(u) \geq \Phi_\lambda^\infty(u_\tau) + \frac{1-\tau^{4s+2t-3}}{4s+2t-3}J_\lambda^\infty(u) + \lambda h(\tau)\|u\|_2^2,\ \forall\ u \in H^s(\mathbb{R}^3),\quad \tau \geq 0,\ 0 \leq \lambda \leq 1, \tag{2.13}$$

where

$$h(\tau) := \frac{s}{4s+2t-3} - \frac{\tau^{2s+2t-3}}{2}\Big(1 - \frac{2s+2t-3}{4s+2t-3}\tau^{2s}\Big).$$

In view of Theorem 1.1 and Remark 3.11 in [15], Φ_1^∞ has a minimizer u_1^∞ on $\mathcal{M}_1^\infty$, that is to say:

$$u_1^\infty \in \mathcal{M}_1^\infty, \quad (\Phi_1^\infty)'(u_1^\infty) = 0 \text{ and } \quad m_1^\infty = \Phi_1^\infty(u_1^\infty). \tag{2.14}$$

Lemma 2.5. *Suppose that* (V_1)–(V_3) *and* (F_1)–(F_3) *hold. Then*

(i) *There exists* $\widetilde{\tau} > 0$ *independent of* λ*, such that* $\Phi_\lambda((u_1^\infty)_{\widetilde{\tau}}) < 0$ *for all* $\lambda \in [1/2, 1]$.

(ii) *There exists a positive constant* κ_0 *independent of* λ*, such that for all* $\lambda \in [1/2, 1]$*:*

$$c_\lambda := \inf_{\gamma\in\Gamma}\max_{\theta\in[0,1]} \Phi_\lambda(\gamma(\theta)) \geq \kappa_0 > \max\{\Phi_\lambda(0), \Phi_\lambda((u_1^\infty)_{\widetilde{\tau}})\},$$

where

$$\Gamma = \{\gamma \in C([0,1], H^s(\mathbb{R}^3)) : \gamma(0) = 0, \gamma(1) = (u_1^\infty)_{\widetilde{\tau}}\}.$$

(iii) *if* $(s+t)f(u)u \geq 3F(u) \geq 0$ *for* $u \geq 0$*, then* c_λ *and* m_λ^∞ *are non-increasing on* $\lambda \in [1/2, 1]$.

The proof of Lemma 2.5 is standard, so we omit it.

Lemma 2.6. *Suppose that* (V_1)–(V_3) *and* (F_1)–(F_4) *hold. Then there exists a* $\bar{\lambda} \in [1/2, 1)$*, such that* $c_\lambda < m_\lambda^\infty$ *for* $\lambda \in [\bar{\lambda}, 1]$.

Proof. We can easily check that $\Phi_\lambda((u_1^\infty)_\tau)$ is continuous with respect to $\tau \in [0,\infty)$. Then, for any $\lambda \in [1/2,1)$, we can choose $\tau_\lambda \in (0,\widetilde{\tau})$, such that

$$\Phi_\lambda((u_1^\infty)_{\tau_\lambda}) = \max_{\tau\in[0,\widetilde{\tau}]} \Phi_\lambda((u_1^\infty)_\tau).$$

Since $\Phi_{1/2}((u_1^\infty)_\tau) \to -\infty$ as $\tau \to \infty$, then there exists $\overline{\tau}$, such that

$$\Phi_{1/2}((u_1^\infty)_\tau) \le \Phi_1(u_1^\infty) - 1, \quad \forall\, \tau \ge \overline{\tau}. \tag{2.15}$$

By (2.8) and the definition of τ_λ, we obtain

$$\Phi_1(u_1^\infty) \le \Phi_\lambda(u_1^\infty) \le \Phi_\lambda((u_1^\infty)_{\tau_\lambda}) \le \Phi_{1/2}((u_1^\infty)_{\tau_\lambda}), \quad \forall\, \lambda \in [1/2,1],$$

which, together with (2.15), implies $\tau_\lambda < \overline{\tau}$ for $\lambda \in [1/2,1]$. Let $\beta_0 = \inf_{\lambda\in[1/2,1]} \tau_\lambda$. If $\beta_0 = 0$, then there exists a sequence $\{\lambda_n\} \subset [1/2,1]$, such that

$$\lambda_n \to \lambda_0 \in [1/2,1] \quad \text{and} \quad \tau_{\lambda_n} \to 0.$$

Then, we get

$$0 < c_1 \le c_{\lambda_n} \le \Phi_{\lambda_n}((u_1^\infty)_{\tau_{\lambda_n}}) = o(1),$$

which implies $\beta_0 > 0$. Therefore

$$0 < \beta_0 \le \tau_\lambda < \overline{\tau}, \quad \forall\, \lambda \in [1/2,1]. \tag{2.16}$$

Set

$$\overline{\lambda} := \max\left\{\frac{1}{2}, 1 - \frac{\beta_0^{2s+2t} \min_{\beta_0\le\vartheta\le\overline{\tau}} \int_{\mathbb{R}^3} \left[V(\infty) - V(\vartheta^{-1}x)\right]|u_1^\infty|^2 \mathrm{d}x}{2\int_{\mathbb{R}^3} F(\overline{\tau}^{s+t} u_1^\infty)\mathrm{d}x}\right\}, \tag{2.17}$$

then we have $1/2 \le \overline{\lambda} < 1$. By (2.8), (2.9), (2.13), (2.16), (2.17), and Lemma 2.5 (iii), we have

$$\begin{aligned}
m_\lambda^\infty &\ge m_1^\infty \\
&= \Phi_1^\infty(u_1^\infty) \ge \Phi_1^\infty((u_1^\infty)_{\tau_\lambda}) \\
&= \Phi_\lambda((u_1^\infty)_{\tau_\lambda}) - \frac{1-\lambda}{\tau_\lambda^3}\int_{\mathbb{R}^3} F(\tau_\lambda^{s+t} u_1^\infty)\mathrm{d}x \\
&\quad + \frac{\tau_\lambda^{2s+2t-3}}{2}\int_{\mathbb{R}^3} \left[V(\infty) - V(\tau_\lambda^{-1}x)\right]|u_1^\infty|^2\mathrm{d}x \\
&> c_\lambda - \frac{1-\lambda}{\beta_0^3}\int_{\mathbb{R}^3} F(\overline{\tau}^{s+t} u_1^\infty)\mathrm{d}x \\
&\quad + \frac{\beta_0^{2s+2t-3}}{2}\min_{\beta_0\le\vartheta\le\overline{\tau}}\int_{\mathbb{R}^3} \left[V(\infty) - V(\vartheta^{-1}x)\right]|u_1^\infty|^2\mathrm{d}x \\
&\ge c_\lambda, \quad \forall\, \lambda \in [\overline{\lambda},1].
\end{aligned}$$

□

Lemma 2.7. *Suppose that* (V_1)–(V_3) *and* (F_1)–(F_3) *hold. Let* $\{u_n\}$ *be a bounded* (PS) *sequence of* Φ_λ, *for* $\lambda \in [1/2,1]$. *Then, there exists a subsequence of* $\{u_n\}$, *still denoted by* $\{u_n\}$ *for convenience, an integer* $l \in \mathbb{N}\cup\{0\}$, $w^k \in H^s(\mathbb{R}^3)$ *for* $1 \le k \le l$, *such that*

(i) $u_n \rightharpoonup u_0$ *with* $\Phi'_\lambda(u_0) = 0$.
(ii) $w^k \ne 0$ *and* $\langle(\Phi_\lambda^\infty)' w^k, w^k\rangle = 0$ *for* $1 \le k \le l$.

(iii) $\Phi_\lambda(u_n) \to \Phi_\lambda(u_0) + \Sigma_{i=1}^{l} \Phi_\lambda^\infty(w^i)$,

where we agree that in the case $l = 0$ the above holds without w^k.

It is clear that (3.13) and (3.14) of Lemma 3.8 in [15] hold. So we can prove Lemma 2.7 in a standard way, and we omit it here.

Lemma 2.8. *Suppose that* $(V_1)-(V_3)$ *and* $(F_1)-(F_4)$ *hold. Then, for almost every* $\lambda \in [\bar{\lambda}, 1]$*, there exists* $u_\lambda \in H^s(\mathbb{R}^3)\backslash\{0\}$*, such that*

$$\Phi'_\lambda(u_\lambda) = 0, \quad \Phi_\lambda(u_\lambda) = c_\lambda. \tag{2.18}$$

Proof. From (F_1), (F_3) and Lemma 2.5, we get that $\Phi_\lambda(u)$ satisfies the assumptions of Proposition 2.2 with $X = H^s(\mathbb{R}^3)$ and $I_\lambda = \Phi_\lambda$. So for almost every $\lambda \in [1/2, 1]$, there exists a bounded sequence $\{u_n(\lambda)\} \subset H^s(\mathbb{R}^3)$, denoted by $\{u_n\}$ for simplicity, such that

$$\Phi_\lambda(u_n) \to c_\lambda > 0, \quad \|\Phi'_\lambda(u_n)\| \to 0.$$

From Lemma 2.7, there exist $l \in \mathbb{N} \cup \{0\}$ and $u_\lambda \in H^s(\mathbb{R}^3)$, such that $\Phi'_\lambda(u_\lambda) = 0$ and

$$u_n \rightharpoonup u_\lambda \text{ in } H^s(\mathbb{R}^3), \quad \Phi_\lambda(u_n) \to \Phi_\lambda(u_\lambda) + \Sigma_{i=1}^{l} \Phi_\lambda^\infty(w^i),$$

where $\{w^i\}_{i=1}^{l}$ are critical points of Φ_λ^∞. Since $\Phi'_\lambda(u_\lambda) = 0$, we get $J_\lambda(u_\lambda) = 0$. Combining (2.8) and (2.11), one has

$$\begin{aligned}\Phi_\lambda(u_\lambda) &= \Phi_\lambda(u_\lambda) - \frac{1}{4s+2t-3} J_\lambda(u_\lambda) \\ &= \frac{1}{2(4s+2t-3)} \int_{\mathbb{R}^3} \big[2sV(x) + (\nabla V(x), x)\big] u_\lambda^2 \mathrm{d}x \\ &\quad + \frac{\lambda}{4s+2t-3} \int_{\mathbb{R}^3} \big[(s+t) f(u_\lambda) u_\lambda - (4s+2t) F(u_\lambda)\big] \mathrm{d}x \\ &\geq 0.\end{aligned}$$

If $l \neq 0$, then we have

$$c_\lambda = \lim_{n\to\infty} \Phi_\lambda(u_n) = \Phi_\lambda(u_\lambda) + \Sigma_{i=1}^{l} \Phi_\lambda^\infty(w^i) \geq m_\lambda^\infty, \quad \forall\, \lambda \in [\bar{\lambda}, 1],$$

which contradicts with Lemma 2.6. Thus, $l = 0$, and then from Lemma 2.7, we get that $u_n \to u_\lambda$ in $H^s(\mathbb{R}^3)$ and $\Phi_\lambda(u_\lambda) = c_\lambda$. □

3. Existence of Ground State Solutions

In this section, we are going to show that System (1.1) possesses ground state solutions.

Proof of Theorem 1.2. From Lemma 2.8, we know that for a.e. $\lambda \in [1/2, 1]$, there has a nontrivial critical point $u_\lambda \in H^s(\mathbb{R}^3)$ of Φ_λ, with $\Phi'_\lambda(u_\lambda) = 0$ and $\Phi_\lambda(u_\lambda) = c_\lambda$. We can choose a sequence $\lambda_n \in [1/2, 1]$ satisfying $\lambda_n \to 1$, then there exists a sequence of nontrivial critical points $\{u_{\lambda_n}\}$ for Φ_{λ_n}, denoted by $\{u_n\}$ for convenience, such that $\Phi'_{\lambda_n}(u_n) = 0$ and $\Phi_{\lambda_n}(u_n) = c_{\lambda_n}$.

By (V_3), (2.8) and (2.11), we obtain

$$\begin{aligned} c_{\lambda_n} &= \Phi_{\lambda_n}(u_n) - \frac{1}{4s+2t-3} J_{\lambda_n}(u_n) \\ &= \frac{1}{2(4s+2t-3)} \int_{\mathbb{R}^3} \left[2sV(x) + (\nabla V(x), x)\right] u_n^2 \mathrm{d}x \\ &\quad + \frac{\lambda_n}{4s+2t-3} \int_{\mathbb{R}^3} \left[(s+t) f(u_n) u_n - (4s+2t) F(u_n)\right] \mathrm{d}x \\ &\geq \frac{\lambda_n}{4s+2t-3} \int_{\mathbb{R}^3} \left[(s+t) f(u_n) u_n - (4s+2t) F(u_n)\right] \mathrm{d}x. \end{aligned} \tag{3.1}$$

Then, we need to prove the boundness of $\{u_n\}$ in $H^s(\mathbb{R}^3)$. Arguing indirectly, assume that $\|u_n\| \to \infty$. Let $v_n = u_n/\|u_n\|$, then $\|v_n\| = 1$. If

$$\delta := \limsup_{n\to\infty} \sup_{y\in\mathbb{R}^3} \int_{B_2(y)} |v_n|^2 \mathrm{d}x = 0,$$

By the virtue of Lemma 2.1, we have $u_n \to 0$ in $L^q(\mathbb{R}^3)$ for all $q \in (2, 2_s^*)$. Set $\kappa' = \kappa/(\kappa-1)$ and

$$\Omega_n := \left\{x \in \mathbb{R}^3 : \frac{f(u_n)}{u_n} \leq \frac{\gamma_0}{2}\right\}.$$

Then we get

$$\int_{\Omega_n} \frac{f(u_n)}{u_n} v_n^2 \mathrm{d}x \leq \frac{\gamma_0}{2} \|v_n\|_2^2 \leq \frac{1}{2}. \tag{3.2}$$

On the other hand, from (F_5), (3.1) and the Hölder inequality, one has

$$\begin{aligned} \int_{\mathbb{R}^3\setminus\Omega_n} \frac{f(u_n)}{u_n} v_n^2 \mathrm{d}x &\leq \left[\int_{\mathbb{R}^3\setminus\Omega_n} \left|\frac{f(u_n)}{u_n}\right|^\kappa \mathrm{d}x\right]^{1/\kappa} \|v_n\|_{2\kappa'}^2 \\ &\leq C_1 \left(\int_{\mathbb{R}^3\setminus\Omega_n} \left[(s+t) f(u_n) u_n - (4s+2t) F(u_n)\right] \mathrm{d}x\right)^{1/\kappa} \|v_n\|_{2\kappa'}^2 \\ &\leq C_2 \|v_n\|_{2\kappa'}^2 = o(1). \end{aligned} \tag{3.3}$$

For (2.16), (2.17) and $\Phi'_{\lambda_n}(u_n) = 0$, we obtain

$$\begin{aligned} 1 &\leq \frac{1}{\|u_n\|^2} \left[\int_{\mathbb{R}^3} \left(|(-\Delta)^{s/2} u_n|^2 + V(x) u_n^2\right) \mathrm{d}x + \int_{\mathbb{R}^3} \phi_{u_n}^t u_n^2 \mathrm{d}x\right] \\ &= \lambda_n \int_{\mathbb{R}^3} \frac{f(u_n)}{u_n} v_n^2 \mathrm{d}x \\ &= \lambda_n \int_{\Omega_n} \frac{f(u_n)}{u_n} v_n^2 \mathrm{d}x + \lambda_n \int_{\mathbb{R}^3\setminus\Omega_n} \frac{f(u_n)}{u_n} v_n^2 \mathrm{d}x \\ &\leq \frac{1}{2} + o(1). \end{aligned}$$

The contradiction implies that $\delta := \limsup_{n\to\infty} \sup_{y\in\mathbb{R}^3} \int_{B_2(y)} |v_n|^2 \mathrm{d}x > 0$.

Passing to a subsequence, we may assume the existence of $y_n \in \mathbb{R}^3$, such that $\int_{B_2(y_n)} |v_n|^2 \mathrm{d}x > \frac{\delta}{2}$. Set $w_n(x) = v_n(x + y_n)$, then $\|w_n\| = \|v_n\| = 1$, and

$$\int_{B_1(0)} |w_n|^2 \mathrm{d}x > \frac{\delta}{2}. \tag{3.4}$$

Going if necessary to a subsequence, we obtain $w_n \rightharpoonup w$ in $H^s(\mathbb{R}^3)$, $w_n \to w$ in $L^q_{\mathrm{loc}}(\mathbb{R}^3)$, $2 \le q < 2_s^*$, $w_n \to w$ a.e. on $\mathbb{R}^3$. Clearly, (3.4) shows that $w \ne 0$.

Next we set $\tilde{u}_n(x) = u_n(x + y_n)$, then $\tilde{u}_n/\|u_n\| = w_n \to w$ a.e. on $\mathbb{R}^3$, $w \ne 0$. For $x \in \{y \in \mathbb{R}^3 : w(y) \ne 0\}$, we have $\lim_{n\to\infty} |\tilde{u}_n(x)| = \infty$. It follows from (2.8), (2.10) and $\langle \Phi'_{\lambda_n}(u_n), u_n\rangle = 0$ that

$$\begin{aligned}&\frac{s}{3-2t}\int_{\mathbb{R}^3} |(-\Delta)^{s/2}u_n|^2 \mathrm{d}x - \frac{1}{2(3-2t)}\int_{\mathbb{R}^3} (\nabla V(x), x)u_n^2 dx - \Phi_{\lambda_n}(u_n)\\ &= \frac{\lambda_n t}{3-2t}\int_{\mathbb{R}^3} \Big[f(u_n)u_n - 2F(u_n)\Big]\mathrm{d}x.\end{aligned} \tag{3.5}$$

For (4.21) in [15], together with (F_3) and (3.5), we have that

$$\begin{aligned} o(1) &\ge \frac{1}{\|u_n\|^3}\left[\frac{s}{3-2t}\int_{\mathbb{R}^3} |(-\Delta)^{s/2}u_n|^2 \mathrm{d}x - \frac{1}{2(3-2t)}\int_{\mathbb{R}^3} (\nabla V(x), x)u_n^2 \mathrm{d}x - \Phi_{\lambda_n}(u_n)\right]\\ &= \frac{t\lambda_n}{(3-2t)\|u_n\|^3}\int_{\mathbb{R}^3}\Big[f(u_n)u_n - 2F(u_n)\Big]\mathrm{d}x\\ &= \frac{t\lambda_n}{(3-2t)\|\tilde{u}_n\|^3}\int_{\mathbb{R}^3}\Big[f(\tilde{u}_n)\tilde{u}_n - 2F(\tilde{u}_n)\Big]\mathrm{d}x\\ &\ge \frac{2st\lambda_n}{(s+t)(3-2t)}\int_{\mathbb{R}^3}\frac{F(\tilde{u}_n)}{|\tilde{u}_n|^3}w_n^3 \mathrm{d}x \to \infty. \end{aligned} \tag{3.6}$$

This implies that $\{u_n\}$ is bounded in $H^s(\mathbb{R}^3)$. The rest proof is standard, and we omit it. □

Proof of Theorem 1.3. Owing to Lemma 2.8, there exist two sequences of $\{\lambda_n\} \subset [\bar{\lambda}, 1]$ and $\{u_{\lambda_n}\} \subset H^s(\mathbb{R}^3)$, denoted by $\{u_n\}$ for convenience, such that

$$\lambda_n \to 1, \quad \Phi'_{\lambda_n}(u_n) = 0, \quad \Phi_{\lambda_n}(u_n) = c_{\lambda_n}. \tag{3.7}$$

By (V_4), (2.8), (2.11) and (3.7), we get

$$\begin{aligned} c_{1/2} \ge c_{\lambda_n} &= \Phi_{\lambda_n}(u_n) - \frac{1}{4s+2t-3}J_{\lambda_n}(u_n)\\ &= \frac{1}{2(4s+2t-3)}\int_{\mathbb{R}^3}\big[2sV(x) + (\nabla V(x), x)\big]u_n^2 \mathrm{d}x\\ &\quad + \frac{\lambda_n}{4s+2t-3}\int_{\mathbb{R}^3}\big[(s+t)f(u_n)u_n - (4s+2t)F(u_n)\big]\mathrm{d}x\\ &\ge \frac{\varrho_0}{2(4s+2t-3)}\|u_n\|_2^2 + \frac{\lambda_n}{4s+2t-3}\int_{\mathbb{R}^3}\big[(s+t)f(u_n)u_n - (4s+2t)F(u_n)\big]\mathrm{d}x. \end{aligned}$$

Which, together with (4.21) in [15], implies the boundedness of $\{\|u_n\|_2\}$. Next, we need to show $\left\{\int_{\mathbb{R}^3} |(-\Delta)^{s/2}u_n|^2 \mathrm{d}x\right\}$ is also bounded. Arguing indirectly, assume that $\int_{\mathbb{R}^3} |(-\Delta)^{s/2}u_n|^2 \mathrm{d}x \to \infty$. Choose $M_0 > 1$, such that

$$c_{\lambda_n} + \int_{\mathbb{R}^3}\Big[(2s+2t-3)(V(\infty) - V(x)) + |(\nabla V(x), x)|\Big]u_n^2 \mathrm{d}x \le M_0. \tag{3.8}$$

We set

$$\tau_n = \min\left\{1, \left(\frac{6M_0}{\int_{\mathbb{R}^3} |(-\Delta)^{s/2}u_n|^2 \mathrm{d}x}\right)^{\frac{1}{4s+2t-3}}\right\}.$$

Then, from (2.8), (2.9), (2.11), (2.12) and (3.8), we have

$$\begin{aligned}
\Phi^{\infty}_{\lambda_n}((u_n)_{\tau_n}) &\le \Phi^{\infty}_{\lambda_n}(u_n) - \frac{1-\tau_n^{4s+2t-3}}{4s+2t-3} J^{\infty}_{\lambda_n}(u_n) \\
&= \Phi_{\lambda_n}(u_n) + \frac{1}{2}\int_{\mathbb{R}^3} \left[V(\infty) - V(x)\right] u_n^2 \mathrm{d}x \\
&\quad - \frac{1-\tau_n^{4s+2t-3}}{4s+2t-3}\Bigg[J_{\lambda_n}(u_n) + \frac{1}{2}\int_{\mathbb{R}^3} \big[(2s+2t-3)(V(\infty) - V(x)) \\
&\quad + (\nabla V(x), x)\big] u_n^2 \mathrm{d}x\Bigg] \\
&\le c_{\lambda_n} + \int_{\mathbb{R}^3} \Big[(2s+2t-3)(V(\infty)-V(x)) + \big|(\nabla V(x), x)\big|\Big] u_n^2 \mathrm{d}x \\
&\le M_0.
\end{aligned} \tag{3.9}$$

It is similar to the proof of (3.26) in [15], we can provide a contradiction by (3.9). Hence $\{\int_{\mathbb{R}^3} |(-\Delta)^{s/2}u_n|^2 \mathrm{d}x\}$ is also bounded, so $\{u_n\}$ is also bounded in $H^s(\mathbb{R}^3)$. The rest proof is standard, and we omit it. □

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