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Efficient Mittag-Leffler Collocation Method for Solving Linear and Nonlinear Fractional Differential Equations

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Abstract. In this paper, a new approximation method for fractional differential equations based on Mittag-Leffler function is developed. Finite Mittag-Leffler function and its fractional-order derivatives are investigated. An efficient technique for solving linear and nonlinear fractional order differential equations is developed. The proposed method combines Mittag-Leffler collocation method and optimization technique. Error estimation of the approximation is stated and proved. We present numerical results and comparisons of previous treatments to demonstrate the efficiency and applicability of the proposed method. Making use of small number of unknowns, the resulting solution converges to the exact one in the linear case and it has a very small error in the nonlinear case.

Mathematics Subject Classification. 34A08, 33E12, 65N35, 97N20.

Keywords. Fractional differential equations, Mittag-Leffler function, collocation method, error estimation.

1. Introduction

Fractional differential equations arise in different fields of science and engineering (see, for example [1–3]). A deep theory of the existence and uniqueness of solution to linear and nonlinear fractional differential equations has been discussed by many authors (see, for example [4–6]). Numerical treatment of most fractional differential equations becomes in the last two decays wide and flourishing because no exact solution of such problems is available. Pedas and Tamme [7] investigated the numerical solution of fractional differential equations with initial values by piecewise polynomial collocation methods. They studied order of convergence and established super convergence effect for a special choice of collocation points. Yan et al. [8] introduced an accurate numerical technique for solving differential equations of fractional order. They

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introduced two approaches: the first is based on a direct substitution of the fractional differential formula in the given problem. The second method is based on insertion of the fractional differential equation of integral form to obtain a fractional Adams-type method. Mittag-Leffler function becomes a good mathematical formula to represent most observed facts in science and engineering. It arises nowadays in many applications such as fractional relaxation and diffusion problem [9]. Arafa et al. developed an application of the Mittag-Leffler function method for solving linear differential equations with fractional order [10], nonlinear fractional differential equations [11] and Lorenz system, [12]. Yasmin et al. [13] introduced truncated Mittag-Leffler polynomials of exponential-based and discussed their properties. They established the relation between these polynomials and Mittag-Leffler polynomials. Arafa and Rida [14] presented numerical solutions of the fractional order of coupled evolution equations making use of Adomian decomposition method. They obtained approximate and analytic solutions for the problem. They described the fractional derivatives in the Caputo sense and compared given solutions with the traveling wave solutions.

In this article, we develop a new numerical approximation based on Mittag-Leffler function. We state a new suitable formula of finite Mittag-Leffler function and evaluate its derivative of fractional-order. We construct a numerical method depends on Mittag-Leffler collocation approximation and optimization technique. We apply the proposed method for solving linear and nonlinear differential equations of fractional order. We discuss error analysis and derive a formula for the error estimation of the approximation. We apply the proposed method on some different examples to ensure the applicability and efficiency of the proposed method.

This article is organized as follows. In Sect. 2, we introduce some necessary definitions and give some properties of Mittag-Leffler function. In Sect. 3, we define Mittag-Leffler function of integer degree and derive its fractional-order derivatives. In Sect. 4, we develop approaches for handling linear and nonlinear differential equations of fractional order using Mittag-Leffler collocation optimization method (MCOM). In Sect. 5, we estimate the error of the approximation. In Sect. 6, the proposed methods are applied to several examples. Finally, conclusion is drawn in Sect. 7.

2. Basic Concepts

2.1. Some Definitions of Derivative in the Caputo Sense

Definition 2.1. [14] The Caputo fractional derivative D_z^α of order $\alpha > 0$ is given by

$$D_z^\alpha f(z) = \frac{1}{\Gamma(m-\alpha)} \int_0^z (z-t)^{m-\alpha-1} f^{(m-1)} \mathrm{d}t, z > 0, \tag{2.1}$$

where $m - 1 < \alpha \leq m, m \in \mathbb{N}$. □

The Caputo fractional derivative operator satisfies the following properties [15]: For constants $\zeta_k, k = 1, 2, , n$, we have

$$D_z^\alpha \sum_{k=1}^{n} \zeta_k f_k(z) = \sum_{k=1}^{n} \zeta_k D_z^\alpha f_k(z), \tag{2.2}$$

and [14]

$$D_z^\alpha z^n = \frac{\Gamma(n+1)}{\Gamma(n+1-F)} z^{n-\alpha}, n > \alpha - 1. \tag{2.3}$$

In fact If α is an integer, the Caputo differential operator will be identical with the usual differential operator.

2.2. Some Properties and Relations of Mittag-Leffler Function

The Mittag-Leffler function of one-parameter is defined as [16]:

$$E^\xi(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\xi k + 1)}, \xi > 0, z \in \mathbb{R}. \tag{2.4}$$

The Mittag-Leffler function of two-parameter is given by [16]

$$E^{\xi,\eta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\xi k + \eta)}, \xi > 0, \eta > 0, z \in \mathbb{R}. \tag{2.5}$$

As a special cases, we have $E^{\xi,1}(z) = E^\xi(z)$ and $E^{1,1}(z) = E^1(z) = e^z$.

3. Finite Mittag-Leffler Function and its Fractional Derivative

Now, we define two-parameter finite Mittag-Leffler function of any integer n by

$$E_n^{\xi,\eta}(z) = \sum_{k=0}^{n} \frac{z^k}{\Gamma(\xi k + \eta)}, \xi > 0, \eta > 0, z \in \mathbb{R}, \tag{3.1}$$

that is $E^{\xi,\eta}(z) = \frac{z^n}{\Gamma(\xi n+\eta)} + \frac{z^{n-1}}{\Gamma(\xi(n-1)+\eta)} + \cdots + \frac{z}{\Gamma(\xi+\eta)} + \frac{1}{\Gamma(\eta)}$, so, we can write

$$E_n^{\xi,\eta}(z) = \frac{z^n}{\Gamma(\xi n + \eta)} + \text{polynomial of lower degrees}. \tag{3.2}$$

The fractional-order derivative of Mittag-Leffler function (3.1) can be derived making use of (2.3) to be

$$D_z^\alpha E_n^{\xi,\eta}(z) = \sum_{k=0}^{n} \frac{D_z^\alpha z^k}{\Gamma(\xi k + \eta)} = \sum_{k=0}^{n} \frac{\Gamma(k+1)}{\Gamma(k+1-\alpha)} \frac{z^{k-\alpha}}{\Gamma(\xi k + \eta)}, \tag{3.3}$$

$\xi > 0, \eta > 0, z \in \mathbb{R}$.

4. Derivation of the Method

The main aim of this section is to develop numerical treatment for handling linear and nonlinear fractional differential equations. The method, namely Mittag collocation optimization method (MCOM), uses the two- parameters Mittag-Leffler function introduced in Eq. (3.1) as a basis function of approximation.

4.1. Linear Fractional Differential Equations

We consider the linear fractional differential equation with variable coefficients of the form:

$$\sum_{k=1}^{\nu} c_k(z) D_z^{\alpha_k} u(z) = f(z), c_1(z) = 1, \alpha_1 > \alpha_2 > \cdots > \alpha_{\nu-1} > \alpha_\nu = 0, \quad (4.1)$$

and $z \in [0, 1]$. This fractional differential equation is of fractional order α_1. It must be associated with the initial conditions

$$u^{(i)}(0) = \varphi_i, i = 0, 1, \ldots, \lceil \alpha_1 \rceil. \quad (4.2)$$

The ceiling function $\lceil \alpha_1 \rceil$ in this equation denotes the smallest integer that is $\geq \alpha_1$. All coefficients $c_k(z), k = 1, 2, \ldots, \nu$, constants $A = \{\varphi_i\}_{i=0}^{\lceil \alpha_1 \rceil}$ and the functions $f(z)$ are given.

Now, assume that $u(z)$ is approximated as:

$$u_n(z) = \sum_{\ell=0}^{n} a_\ell E_\ell^{\xi,\eta}(z), \quad (4.3)$$

where $A = \{a_\ell\}_{\ell=0}^{n}$ are unknowns and $\{E_\ell^{\xi,\eta}(z)\}_{\ell=0}^{n}$ are defined in equation (3.1).

Now, if we approximate $u(z)$ at selected points $z_j, j = 0, 1, , n$, Eq. (4.1) becomes: $\sum_{k=1}^{\nu} c_k(z) \sum_{\ell=0}^{n} a_\ell D_z^{\alpha_k} E_\ell^{\xi,\eta}(z) = f(z)$,

$$\sum_{\ell=0}^{n} a_\ell \sum_{k=1}^{\nu} c_k(z_j) D_z^{\alpha_k} E_\ell^{\xi,\eta}(z_j) = f(z_j), z_j, j = 0, 1, , n, \quad (4.4)$$

where $D_z^{\alpha} E_\ell^{\xi,\eta}(z)$ is defined in (3.3). The initial conditions (4.2) can be approximated also by

$$\sum_{\ell=0}^{n} a_\ell D_z^{i} E_\ell^{\xi,\eta}(z_0) = \varphi_i, i = 0, 1, \ldots, \lceil \alpha_1 \rceil. \quad (4.5)$$

The unknown values $A = \{a_\ell\}_{\ell=0}^{n}$ can be obtained by solving (4.4) together with (4.5). Since the initial condition (4.2) corresponds to (4.5) is multiple, expressing (4.4)–(4.5) as a square linear system of equation seems to be not available. So, we shall express it as an unconstrained optimization problem with a least squares cost function,

$$\bar{R} = \sum_{j=1}^{n} \left[\sum_{k=1}^{\nu} c_k(z_j) D_z^{\alpha_k} E_\ell^{\xi,\eta}(z_j) - f(z_j) \right]^2 + \sum_{i=0}^{\lceil \alpha_1 \rceil} \left[\sum_{\ell=0}^{n} a_\ell D_z^{i} E_\ell^{\xi,\eta}(z_0) - \varphi_i \right]^2. \quad (4.6)$$

We shall use Leap frog optimization procedure [15] for solving (4.6).

4.2. Nonlinear Fractional Differential Equations

We consider the nonlinear fractional-order differential equation:

$$D_z^{\alpha_1}u(z) = f\big(D_z^{\alpha_2}u(z), D_z^{\alpha_3}u(z), \ldots, D_z^{\alpha_\nu}u(z), u(z), z\big), \alpha_1 > \alpha_2 > \cdots > \alpha_{\nu-1} > \alpha_\nu, z \in [0,1], \tag{4.7}$$

subject to the initial conditions

$$u^{(i)}(0) = \varphi_i, i = 0, 1, \ldots, \lceil\alpha_1\rceil, \tag{4.8}$$

the right hand side function f, is nonlinear in general and constants $\{\varphi_i\}_{i=0}^{\lceil\alpha_1\rceil}$ are given.

Now, if we use Eq. (4.3) to approximate $u(z)$ at selected points $z_j, j = 0, 1, , n$, Eq. (4.8) becomes:

$$\sum_{\ell=0}^{n} a_\ell D_z^{\alpha_1} E_\ell^{\xi,\eta}(z) = f\Bigg(\sum_{\ell=0}^{n} a_\ell D_z^{\alpha_2} E_\ell^{\xi,\eta}(z), \sum_{\ell=0}^{n} a_\ell D_z^{\alpha_3} E_\ell^{\xi,\eta}(z), \ldots, \sum_{\ell=0}^{n} a_\ell D_z^{\alpha_\nu} E_\ell^{\xi,\eta}(z), \sum_{\ell=0}^{n} a_\ell E_\ell^{\xi,\eta}(z), z\Bigg), \tag{4.9}$$

where $D_z^{\alpha} E_\ell^{\xi,\eta}(z)$ is defined in (3.3). The initial conditions can also be approximated by

$$\sum_{\ell=0}^{n} a_\ell D_z^{i} E_\ell^{\xi,\eta}(z_0) = \varphi_i, i = 0, 1, \ldots, \lceil\alpha_1\rceil. \tag{4.10}$$

So, to obtain the unknown values $\{a_\ell\}_{\ell=0}^{n}$, we construct that the following nonlinear programming problem minimize

$$\bar{R} = \Bigg[\sum_{\ell=0}^{n} a_\ell D_z^{\alpha_1} E_\ell^{\xi,\eta}(z) - f\Bigg(\sum_{\ell=0}^{n} a_\ell D_z^{\alpha_2} E_\ell^{\xi,\eta}(z), \sum_{\ell=0}^{n} a_\ell D_z^{\alpha_3} E_\ell^{\xi,\eta}(z), \ldots, \sum_{\ell=0}^{n} a_\ell D_z^{\alpha_\nu} E_\ell^{\xi,\eta}(z), \sum_{\ell=0}^{n} a_\ell E_\ell^{\xi,\eta}(z), z\Bigg)\Bigg]^2 + \Bigg[\sum_{i=0}^{\lceil\alpha_1\rceil}\sum_{\ell=0}^{n} a_\ell D_z^{i} E_\ell^{\xi,\eta}(z_0) - \varphi_i\Bigg]^2. \tag{4.11}$$

We shall use Leap frog optimization procedure [17] with the cost function R to obtain the unknown values $\{a_\ell\}_{\ell=0}^{n}$ and then the approximate solution (4.3) of the problem.

5. Error Analysis

Theorem 5.1. *Let $u(z) \in C^\infty[0,1]$ be approximated by (4.3), then for every $z \in [0,1]$, there exists $\varpi \in [0,1]$, such that*

$$u(z) - u_n(z) = \frac{\Gamma\big(\xi(n+1)+\eta\big)}{(n+1)!} E_{n+1}^{\xi,\eta}(z) u^{(n+1)}(\varpi), \tag{5.1}$$

and the absolute estimated error,

$$||u(z)-u_n(z)|| \leq \frac{\Gamma\big(\xi(n+1)+\eta\big)}{(n+1)!} E_{n+1}^{\xi,\eta}(z)\boldsymbol{Max}_{\varpi\in[0,1]}||u^{(n+1)}(\varpi)||. \quad (5.2)$$

Proof. Let $u(z)\in C^{\infty}[0,1]$ be approximated by $u_n(z)=\sum_{\ell=0}^{n} a_\ell(z)E_\ell^{\xi,\eta}(z)$. Define the function: $T(z)=u(z)-u_n(z)-\theta E_{n+1}^{\xi,\eta}(z)$. We can choose the parameter θ such that the equation $T(z)=0$ has a solution z_0 with the property $E_{n+1}^{\xi,\eta}(z)\neq 0$. In this case, we can write

$u(z_0)-u_n(z_0)-\theta E_{n+1}^{\xi,\eta}(z_0)=0$, so

$$\theta=\frac{u(z_0)-u_n(z_0)}{E_{n+1}^{\xi,\eta}(z_0)}. \quad (5.3)$$

Since $u(z)\in C^{\infty}[0,1]$, $E_n^{\xi,\eta}(z_0)\in C^n[0,\infty]$ and $E_{n+1}^{\xi,\eta}(z_0)\in C^{n+1}[0,\infty]$, thus, $T(z)\in C^{n+1}[0,1]$ and so its $(n+1)$ $\underline{\text{th}}$ order derivative, namely $T^{n+1}(z)$, has at least one root, that is

$$T^{n+1}(\varpi)=u^{n+1}(\varpi)-\theta[E_{n+1}^{\xi,\eta}(\varpi)]^{n+1}-[E_n^{\xi,\eta}(\varpi)]^n=0, \quad (5.4)$$

by (3.2) , the last term of (5.4), $[E_n^{\xi,\eta}(\varpi)]^n=0$. Also from (3.2), we have

$E_{n+1}^{\xi,\eta}(\varpi)=\frac{\varpi^{n+1}}{\Gamma(\xi(n+1)+\eta)}+$ polynomial of lower degrees, so
$[E_{n+1}^{\xi,\eta}(\varpi)]^{n+1}=\frac{(n+1)n(n-1)3(2)1}{\Gamma(\xi(n+1)+\eta)}=\frac{(n+1)!}{\Gamma((\xi(n+1))+\eta)}$.

Substituting in (5.4), we obtain

$$\theta=\frac{\Gamma((\xi(n+1))+\eta)}{(n+1)!}u^{n+1}(\varpi). \quad (5.5)$$

Equations (5.3)–(5.5) yield

$$u(z_0)-u_n(z_0)=\frac{\Gamma((\xi(n+1))+\eta)}{((n+1)!}E_{n+1}^{\xi,\eta}(z_0)u^{n+1}(\varpi) \quad (5.6)$$

and so $||u(z_0)-u_n(z_0)||=\frac{\Gamma((\xi(n+1))+\eta)}{((n+1)!}||E_{n+1}^{\xi,\eta}(z_0)||||u^{n+1}(\varpi)||$.
Finally, we take the maximum of $||u^{n+1}(\varpi)||$ to obtain (5.2). □

Theorem 5.2. *Let $u(z)\in C^{\infty}[0,1]$ satisfies the linear fractional differential Eq.* (4.1)*and u(z) is approximated by* (4.3) *then for every $z\in[0,1]$, there exists $\varpi\in[0,1]$ such that the residual is estimated by*

$$R(z)\leq\frac{\Gamma\big(\xi(n+1)+\eta\big)}{(n+1)!}E_{n+1}^{\xi,\eta}(z)\left[\sum_{k=1}^{\nu}c_k(z)D_z^{\alpha_k}E_{n+1}^{\xi,\eta}\right]\boldsymbol{Max}_{\varpi\in[0,1]}||u^{(n+1)}(\varpi)||. \quad (5.7)$$

Proof. Let $u_n(z)$ approximates $u(z)$, so by Eq. (4.1) we have

$$\sum_{k=1}^{\nu} c_k(z) D_z^{\alpha_k} u_n(z) = f(z). \tag{5.8}$$

Subtracting (5.8) from (4.1), we obtain
$R_e(z) = \sum_{i=1}^{\nu} c_i(z) D_z^{\alpha_i}[u(z) - u_n(z)] = 0,$
Making use of (5.2),we have (5.7) Where D_z^{α} is defined in (3.3). □

6. Numerical Experiments

It is well known that not all the fractional differential equations have exact solutions, especially if it is nonlinear. If the exact solution $u(z)$ exists, we measure the error by

$$E_u = \frac{1}{n}\left[\sum_{k=0}^{n} \left[u(z_k) - u_n(z_k)\right]^2\right]^{\frac{1}{2}}, \tag{6.1}$$

otherwise, if the exact solution does not exist, we use the optimization error index, that is, the value of the minimized cost function $\bar{R}$ of Eq. (4.6) in linear case or (4.11) in nonlinear case. Some numerical examples are presented below.

Problem 6.1 Consider we have the following problem:

$$D_z^{\frac{1}{2}} u - 2u = f(z), z \in [0, 1], \tag{6.2}$$

where $f(z) = \frac{\Gamma(3)}{\Gamma(5/2)} z^{3/2} - 2z^2$ and $u(0) = 0$. The exact solution of this problem is $u(z) = z^2$. We solve this problem with the proposed method, taking $n = 2$, that is

$$u_2(z) = \sum_{\ell=0}^{2} a_\ell E_\ell^{\xi,\eta}(z) = a_0 E_0^{\xi,\eta}(z) + a_1(z) E_1^{\xi,\eta}(z) + a_2 E_2^{\xi,\eta}(z), \tag{6.3}$$

$$D_z^{\frac{1}{2}} u_2(z) = a_1 D_z^{\frac{1}{2}} E_1^{\xi,\eta}(z) + a_2 D_z^{\frac{1}{2}} E_2^{\xi,\eta}(z). \tag{6.4}$$

Substituting in (6.2):

$$a_1 D_z^{\frac{1}{2}} E_1^{\xi,\eta}(z) + a_2 D_z^{\frac{1}{2}} E_2^{\xi,\eta}(z) - 2\big[a_0 E_0^{\xi,\eta}(z) \\ + a_1 E_1^{\xi,\eta}(z) a_2 E_2^{\xi,\eta}(z)\big] = f(z).$$

Making use of (3.1)–(3.3), we obtain

$$a_1 \frac{D_z^{\frac{1}{2}} z}{\Gamma(\xi+\eta)} + a_2 \left[\frac{D_z^{\frac{1}{2}} z}{\Gamma(\xi+\eta)} + \frac{D_z^{\frac{1}{2}} z^2}{\Gamma(2\xi+\eta)}\right] - 2a_0 \frac{1}{\Gamma(\eta)} \\ -2a_1 \left[\frac{1}{\Gamma(\eta)} + \frac{z}{\Gamma(\xi+\eta)}\right] - 2a_2 \left[\frac{1}{\Gamma(\eta)} + \frac{z}{\Gamma(\xi+\eta)} + \frac{z^2}{\Gamma(2\xi+\eta)}\right] = f(z).$$

Making some arrangements and using (2.3), we obtain

$$\begin{aligned} &\frac{-2a_0}{\Gamma(\eta)} + a_1\left[\frac{-2}{\Gamma(\eta)} + \frac{-2z}{\Gamma(\xi+\eta)} + \frac{1}{\Gamma(\xi+\eta)}\frac{\Gamma(2)}{\Gamma(\frac{3}{2})}z^{\frac{1}{2}}\right] \\ &\quad + a_2\left[\frac{-2}{\Gamma(\eta)} - \frac{2z}{\Gamma(\xi+\eta)} - \frac{2z^2}{\Gamma(2\xi+\eta)} + \frac{1}{\Gamma(\xi+\eta)}\frac{\Gamma(2)}{\Gamma(\frac{1}{2})}z^{\frac{3}{2}} \right. \\ &\quad \left. + \frac{1}{\Gamma(2\xi+\eta)}\frac{\Gamma(3)}{\Gamma(\frac{5}{2})}z^{\frac{3}{2}}\right] = -2z^2 + \frac{\Gamma(3)}{\Gamma(\frac{5}{2})}z^{\frac{3}{2}}. \end{aligned} \tag{6.5}$$

Since $n = 2$, we choose the following set of points: $z_0 = 0, z_1 = \frac{1}{2}, z_2 = 1$, substituting in (6.5) with z_1, z_2 and applying the initial condition in (6.2) for z_0, we have

$$\begin{aligned} &\frac{-2a_0}{\Gamma(\eta)} + a_1\left[\frac{-2}{\Gamma(\eta)} + \frac{-2(\frac{1}{2})}{\Gamma(\xi+\eta)} + \frac{1}{\Gamma(\xi+\eta)}\frac{\Gamma(2)}{\Gamma(\frac{3}{2})}\left(\frac{1}{2}\right)^{\frac{1}{2}} \right. \\ &\quad +a_2\left[\frac{-2}{\Gamma(\eta)} - \frac{2(\frac{1}{2})}{\Gamma(\xi+\eta)} - \frac{2(\frac{1}{2})^2}{\Gamma(2\xi+\eta)} + \frac{1}{\Gamma(\xi+\eta)}\frac{\Gamma(2)}{\Gamma(\frac{3}{2})}\left(\frac{1}{2}\right)^{\frac{1}{2}} \right. \\ &\quad \left. +\frac{1}{\Gamma(2\xi+\eta)}\frac{\Gamma(3)}{\Gamma(\frac{5}{2})}\left(\frac{1}{2}\right)^{\frac{3}{2}}\right] = -2\left(\frac{1}{2}\right)^2 + \frac{\Gamma(3)}{\Gamma(\frac{5}{2})}\left(\frac{1}{2}\right)^{\frac{3}{2}}\right], \end{aligned} \tag{6.6}$$

$$\begin{aligned} &\frac{-2a_0}{\Gamma(\eta)} + a_1\left[\frac{-2}{\Gamma(\eta)} + \frac{-2}{\Gamma(\xi+\eta)} + \frac{1}{\Gamma(\xi+\eta)}\frac{\Gamma(2)}{\Gamma(\frac{3}{2})}\right] + a_2\left[\frac{-2}{\Gamma(\eta)} - \frac{2}{\Gamma(\xi+\eta)} \right. \\ &\quad \left. -\frac{2}{\Gamma(2\xi+\eta)} + \frac{1}{\Gamma(\xi+\eta)}\frac{\Gamma(2)}{\Gamma(\frac{3}{2})} + \frac{1}{\Gamma(2\xi+\eta)}\frac{\Gamma(3)}{\Gamma(\frac{5}{2})}\right] = -2 + \frac{\Gamma(3)}{\Gamma(\frac{5}{2})}, \end{aligned} \tag{6.7}$$

$$a_0 + a_1 + a_2 = 0. \tag{6.8}$$

Selecting $\xi = \eta = 1$, solving the linear system (6.6)–(6.8) for a_0, a_1 and a_2, we obtain

$$\begin{aligned} &-2a_0 + a_1\left[-3 + \frac{1}{\Gamma(\frac{3}{2})}\left(\frac{1}{2}\right)^{\frac{1}{2}}\right] + a_2\left[\frac{-13}{4} + \frac{1}{\Gamma(\frac{3}{2})}\left(\frac{1}{2}\right)^{\frac{1}{2}} \right. \\ &\quad \left. +\frac{1}{\Gamma(\frac{5}{2})}\left(\frac{1}{2}\right)^{\frac{3}{2}}\right] = -\frac{1}{2} + \frac{2}{\Gamma(\frac{5}{2})}\left(\frac{1}{2}\right)^{\frac{3}{2}}, \end{aligned} \tag{6.9}$$

$$\begin{aligned} &-2a_0 + a_1\left[-4 + \frac{1}{\Gamma(\frac{3}{2})}\right] + a_2\left[-5 + \frac{1}{\Gamma(\frac{3}{2})} + \frac{1}{\Gamma(\frac{5}{2})}\right] \\ &\quad = -2 + \frac{2}{\Gamma(\frac{5}{2})}, \end{aligned} \tag{6.10}$$

$$a_0 + a_1 + a_2 = 0. \tag{6.11}$$

Solving this linear system of equations, we have $a_0 = 0, a_1 = -2$ and $a_2 = 2$. Thus, by Eq. (6.3), we have

$$\begin{aligned} u_2(z) &= a_0E_0^{\xi,\eta}(z) + a_1(z)E_1^{\xi,\eta}(z) + a_2E_2^{\xi,\eta}(z) \\ &= -2\left[\frac{1}{\Gamma(1)} + \frac{z}{\Gamma(2)}\right] + 2\left[\frac{1}{\Gamma(1} + \frac{z}{\Gamma(2)} + \frac{z^2}{\Gamma(3)}\right] = 2\left[\frac{z^2}{\Gamma(3)}\right] = z^2, \end{aligned} \tag{6.12}$$

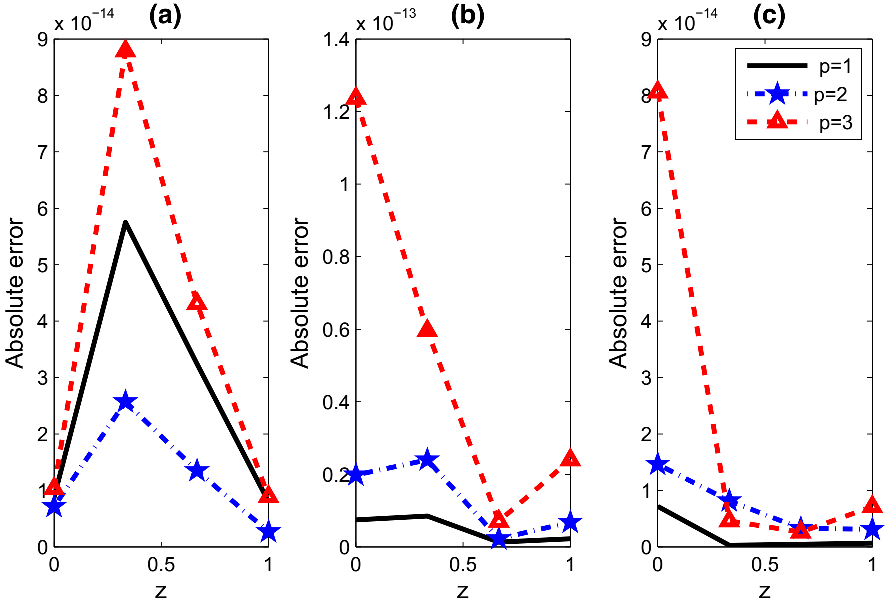

Figure 1. Absolute error of problem 6.2, **a** $\alpha = 1$, **b** $\alpha = 0.8$ and **c** $\alpha = 0.5$, $n = 3$, with some values of power of the exact solution p

which is the exact solution. Any other choices for $0 \le \xi, \eta \le 1$ and will give the same solution.

Problem 6.2 We consider the next problem

$$D_z^\alpha u + u = \frac{\Gamma(p+1)}{\Gamma(p+1-\alpha)} z^{p-\alpha}, z \in [0,1], \alpha \in (0,1], \tag{6.13}$$

with $u(0) = 0$. The exact solution of this problem is $u(z) = z^p$.
A special case of this problem is discussed in [18] with $p = 3.6$. We introduce in Fig. 1 the absolute error of this problem for some values of the fraction order of differentiation α and the power of the exact solution p. In Fig. 2, we plot $log_2 e$ vis $log_2 h$ to illustrate the order of convergence, with $h = \frac{1}{n}$. We take $p = 3.6$ with (a) $\alpha = 0.8$ and (b) $\alpha = 0.25$. We plot also the line $y = 6x - 4$ as a guided line. We see that the guided line is parallel to the approximation errors lines, which confirms that the rate of convergence is of (h^6), which is very good rate. The rate of convergence obtained for the same data in [18] is approximately (h^3).

In Table 1, we introduce the effects of Mittag-Leffler parameters and in the approximation error. The error indices reported in this table are the error E_u measured by Eq. (6.1) and the value of the minimized cost function $\bar{R}$ of Eq. (4.6). We notice that $\bar{R}$ is close to zero. This agrees with the residual estimate R of Eq. (5.7) which is also zero.

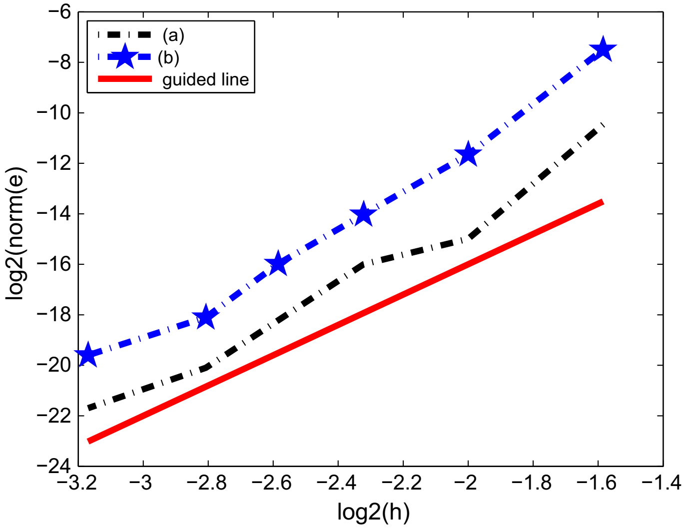

Figure 2. The experimental order of convergence for problem 6.2, **a** $\alpha = 0.8$ and **b** $\alpha = 0.25$

Table 1. Error indices for problem 6.2, $n = p = 3$ and $\alpha = 0.8$

ξ	η	E_u	$\bar{R}$
0.0		9.48e-15	7.13e-27
0.5	0.0	1.69e-14	2.27e-26
1.5		9.16e-12	7.59e-21
2.0		4.55e-12	2.42e-21
	0.5	3.13e-15	6.33e-28
0.5	1.0	7.58e-15	6.13e-27
	1.5	5.77e-14	2.84e-25

Problem 6.3 Consider the problem

$$D_z^\alpha u - z^3 u^2 = \frac{\Gamma(p+1)}{\Gamma(p+1-\alpha)} z^{p-\alpha} - z^{2p+3}, z \in [0,1], \alpha \in (0,1], \tag{6.14}$$

with $u(0) = 0$. The exact solution of this problem is $u(z) = z^p$.
A simple case of this problem is solved in [19] with $p = 1$. Figure 3 illustrates the absolute error for this problem for some values of the fraction order of differentiation α and the power of the exact solution p with $\xi = 1$, and $\eta = 0$. Sakar et al. in [19] obtained (10^{-7}) of absolute error for $p = 1$ and $\alpha = 1$ and $n = 4$, which play agreement of our results.

In Table 2, we introduce the effects of Mittag-Leffler parameters ξ and η in the approximation error. The error indices reported in this table are error E_u measured by Eq. (6.1) and the value of the minimized cost function $\bar{R}$ of Eq. (4.11).

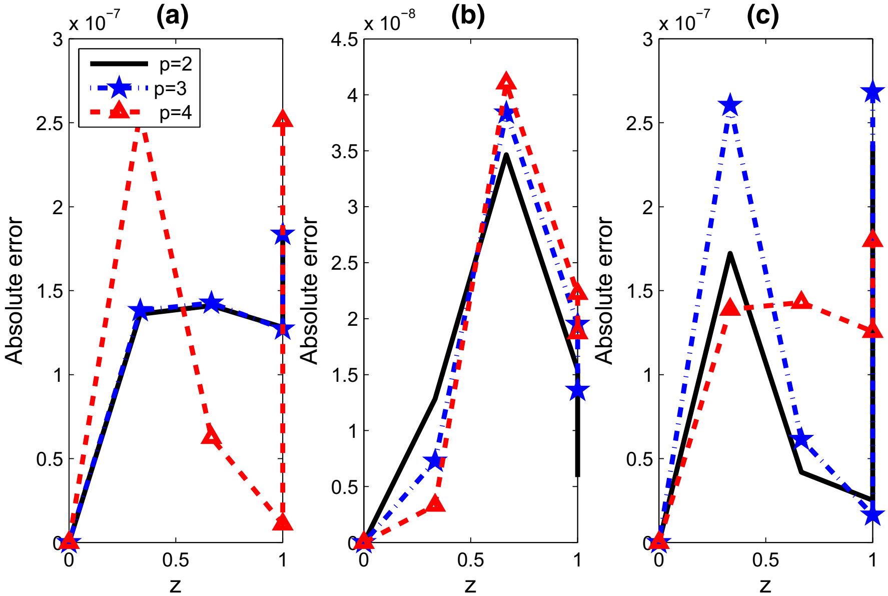

Figure 3. Absolute error of problem 6.3, **a** $\alpha = 1$, **b** $\alpha = 0.8$ and **c** $\alpha = 0.5, n = 4$, with some values of power of the exact solution p

Table 2. Error indices for problem 6.3, $n = p = 3$ and $\alpha = 0.8$

ξ	η	E_u	$\bar{R}$
0.0	1.0	1.78e-07	1.79e-13
	0.5	5.67e-08	1.81e-14
	1.0	1.88e-07	2.04e-13
0.5	0.5	1.89e-07	1.67e-13
	0.0	8.45e-08	4.96e-14
	1.0	1.07e-07	7.76e-14
1.0	0.5	1.32e-07	9.34e-14
	0.0	8.22e-08	5.07e-14

Problem 6.4 Consider the problem

$$D_z^{2.5}u - zu^2 = \frac{\Gamma(p+1)}{\Gamma(p+1-\alpha)} z^{p-\alpha} - z^{2p+3} = 12\sqrt{\frac{z}{\pi}} - z^7, z \in [0, 1], \quad (6.15)$$

with $u(0) = 0$, $u'(0) = 0$ and $u(1) = 1$. The exact solution of this problem is $u(z) = z^3$.

In Table 3, we introduce the effects of Mittag-Leffler parameters ξ and η in the approximation error. We notice that the error E_u is (10^{-9}). These results are compared to (10^{-7}) obtain by Jia et al. [20]. This shows that our results are more accurate.

Table 3. Error indices for problem 6.4, $n = 3$ and $\alpha = 0.8$

ξ	η	E_u	$\bar{R}$
0.0	1.0	5.24854e-09	3.5069e-15
	0.5	1.67066e-09	3.8790e-15
1.0	1.0	3.70904e-09	1.1594e-15
	1.0	4.49112e-09	3.2482e-15
0.5	0.5	4.88756e-09	2.1382e-15
	0.0	5.17359e-10	2.4774e-16
	1.0	3.70904e-09	1.1594e-15
1.0	0.5	1.68007e-09	7.9414e-16
	0.0	5.74088e-10	2.8714e-16

Table 4. The optimization error index $\bar{R}$ for problem 6.5, $n = 3$

ξ	η	$\alpha = 1.3$	$\alpha = 1.1$	$\alpha = 1.0$	$\alpha = 0.8$	$\alpha = 0.5$
0.0	1.0	5.36e-14	9.45e-14	1.21e-13	1.30e-13	1.19e-13
	0.5	5.43e-15	9.56e-15	1.23e-14	1.32e-14	1.21e-14
	1.0	4.54e-14	1.15e-13	1.82e-13	2.33e-13	2.26e-13
0.5	0.5	4.48e-14	7.42e-14	8.77e-14	8.36e-14	7.55e-14
	0.0	8.59e-15	1.75e-14	2.45e-14	2.06e-14	2.91e-14
	1.0	4.25e-14	1.31e-13	2.22e-13	3.10e-13	2.65e-13
1.0	0.5	2.59e-14	8.79e-14	1.51e-13	2.01e-13	1.86e-13
	0.0	4.16e-14	1.32e-13	7.46e-14	5.39e-13	2.05e-14

Problem 6.5 Consider the Riccati equation of fractional order

$$D_z^{\alpha} u = -u^3 z + 1, z \in [0, 1], \alpha \in (0, 2], \tag{6.16}$$

with $u(0) = 0$, $u(1) = \frac{e^2-1}{e^2+1}$ and the exact solution when $\alpha = 1$ is $u(z) = \frac{e^{2z}-1}{e^{2z}+1}$.

A special case of this problem was considered in [21] and [22] restricting the fraction order to $\alpha \in (0, 1]$ with only first boundary condition. Jafari [21] applied the Legendre wavelet, whereas [22] applied Bernoulli wavelet approximations. Since the general exact solution of this problem is not available for any fraction order differentiation α, we introduce in Table 4, the optimization error index of Eq. (4.11) to ensure the efficiency of the proposed method. In Fig. 4, we plot the approximate solution at some selected values of α. From this figure, we notice that these solutions converge to the exact solution as α tends to 1 from right as well as from left. Also, we see that the two boundary conditions are satisfied at each value of α. This means that the proposed method is efficient.

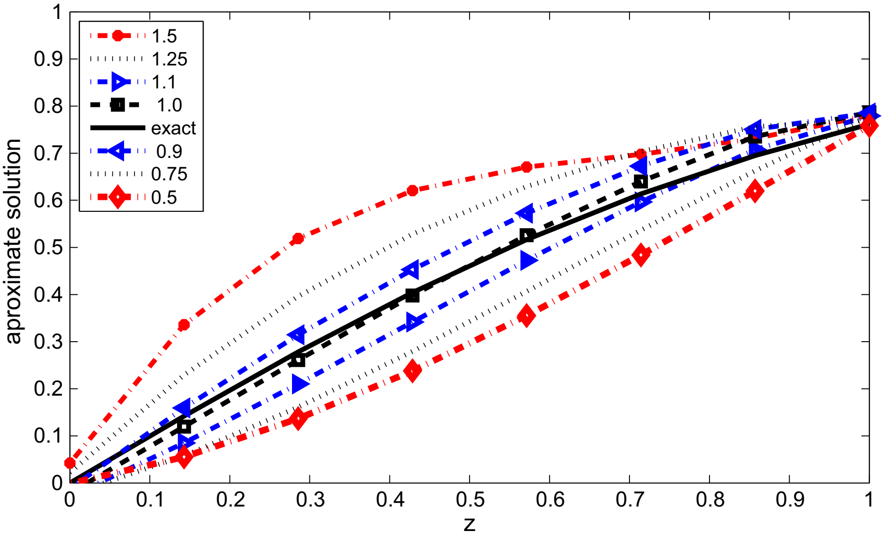

Figure 4. The approximate solution of problem 6.5 for different values of the differentiation order α

7. Conclusion

In this paper, a new approximation for functions based on Mittag-Leffler function is derived and applied together with the collocation method and optimization techniques to numerically solving the variable coefficients nonlinear and linear ordinary fractional differential equations.

The analysis and numerical examples introduced in this work yield that our proposed method gives promising results. In the linear case, the proposed method attains the exact solution when it is applied manually as in Problem 6.1. If the proposed method applied numerical by computer MATLAB program, the error converges to the machine error using small number of unknowns as in Problem 6.2, (see Fig. 2 and Table 1). In the nonlinear case, the proposed method obtained comparable results with previous treatments at small number of unknowns and it is more accurate in most situations (see results of Problem 6.3–6.5).

The proposed method can be used easily to solve other types of fractional differential equations and related problems.

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