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Some Criteria for Transitivity of Semigroup Actions

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Abstract. In this paper, we study transitivity on the semigroup actions. Indeed, we introduce two criteria, "entropy minimality" and "nonuniformly expanding property", to obtain transitivity.

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1. Introduction

Semigroup actions are dynamical systems with several generators. Some of mathematicians worked on these systems and obtained varietal results. For instance, Rodrigues and Varandaz [16] introduced several cases of specification property on finitely generated semigroup actions and obtained some thermodynamical results on these systems. Koropeski and Nassiri [10] attained transitivity of generic semigroup action of area preserving surface diffeomorphisms. Sarizadeh studied ergodicity on semigroup actions with minimal hyperspaces [17]. In [5], the authors studied semigroup actions of Ruelle-expanding maps. They explored the relation between intrinsic properties of the semigroup action and the thermodynamic formalism of the associated skew-product. Zamani Bahabadi [19] obtained some results on semigroup actions by means of shadowing and average shadowing properties. Bis and Urbanski in [3] presented some results on topological entropy of semigroup actions.

Our main studies on finitely generated semigroup actions are about transitivity, namely having a dense orbit. We tried to find some tools to obtain transitivity. One of criteria is the entropy minimality. Entropy minimality was first introduced by Coven and Smital [6] as a property between minimality and transitivity on ordinary dynamical systems, the dynamical systems with one generator.

In Sect. 2, we study entropy of a finitely generated semigroup action on its nonwandering set. Then, we use it to establish a connection between transitivity and entropy by entropy minimality property. We present some definitions and notations on entropy minimality in Sect. 2.1. Also the results and their proofs are stated in the Sect. 2.2. In Sect. 3, we proceed the nonuniformly expanding semigroup actions introduced by Rashid and Zamani [14]. We show that if these systems have asymptotic average shadowing property, then they are transitive. This topic was studied on ordinary dynamical systems in [18]. Section 3.1 contains essential definitions and Sect. 3.2 comprises main results about nonuniformly expanding property.

2. Entropy Minimality

2.1. Preliminary

Let (X, d) be a compact metric space and let G be a semigroup action generated by $G_1 = \{f_1, \dots, f_k\}$ of homeomorphisms from X to itself. The members of G_1 are called *generators of* G. While G_1 is a singleton, we call G an ordinary dynamical system.

Notice that

$$G = \bigcup_{i=1}^{\infty} G_i, \quad G_i = \{g_1 \circ \cdots \circ g_i | g_j \in G_1, 1 \leq j \leq i\}.$$

If $id_X \in G_1$, id_X is the identity map from X to itself, then $G_n \subset G_{n+1}$ for all positive integer n.

For $\omega = (\dots,\omega_{-1}, \omega_0, \omega_1, \dots) \in \{1, \dots, k\}^{\mathbb{Z}}$, we set $f_\omega^0 := id_X$ and $f_\omega^n := f_{\omega_{n-1}} o f_{\omega_{n-2}} o \dots o f_{\omega_0}$ for all $n > 0$. Therefore, if $h \in G_i$ for some i, then there exist $w_0, w_1, \dots, w_{i-1} \in \{1, \dots, k\}$ such that $h = f_\omega^i = f_{\omega_{i-1}} o f_{\omega_{i-2}} o \dots o f_{\omega_0}$.

A semigroup action G is called abelian if $hog = goh$ for all $h, g \in G$.

We say that a semigroup action G is transitive provided that for every two nonempty open subsets U and V of X, there exists $h \in G$ such that $h(U) \cap V \neq \emptyset$, equivalently, if for every nonempty open subset U of X, $\overline{\bigcup_{h \in G} h(U)} = X$. This equivalency is shown by [11] for ordinary dynamical systems.

A point $p \in X$ is said to be a nonwandering point of G if for every nonempty neighborhood U of p in X, there exists $h \in G$ such that $h(U) \cap U \neq \emptyset$. Otherwise, it is called a wandering point of G. The set of all nonwandering points of G is called nonwandering set of G and denoted by $\Omega(G)$. It is easy to see that $\Omega(G)$ is a closed subset of X.

We say that a subset A of X is G-invariant if A is f-invariant, namely $f(A) \subseteq A$ for all $f \in G_1$. For more details, see [7].

The group action generated by G_1 is denoted by $< G_1 >$. We say that $< G_1 >$ is transitive provided that for every two nonempty open subsets U and V of X, $h(U) \cap V \neq \emptyset$ for some $h \in < G_1 >$. In other word, $< G_1 >$ is transitive if $\overline{\bigcup_{h \in < G_1 >} h(U)} = X$, for every nonempty open subset U of X.

Remark 2.1. Koropecki and Nassiri [10] proved that if G_1 is a countable family of homeomorphisms from X to itself preserving a finite Borel measure with compact support, then transitivity of semigroup action G is equivalent to transitivity of group action $< G_1 >$.

Let $n \in N$ and consider the following metric on X

$$d^n : X \times X \longrightarrow [0, \infty),$$
$$d^n(x, y) = \mathrm{Sup}_{1 \leq i \leq n} \mathrm{Sup}_{g \in G_i}\ d(g(x), g(y)).$$

Given $\epsilon > 0$, the subset $A \subseteq X$ is called a (n, ϵ, G, X)-separated set, provided that for every two disjoint points $x, y \in A$, $d^n(x, y) \geq \epsilon$. Namely, there exist $i \in \{1, \ldots, n\}$ and $g \in G_i$ such that $d(g(x), g(y)) \geq \epsilon$.
We denote a (n, ϵ, G, X)-separated set with maximal cardinality by E_{sep} (n, ϵ, G, X) and its cardinality by $r_{\mathrm{sep}}(n, \epsilon, G, X)$.

Separated topological entropy of G is defined by

$$h_{\mathrm{sep}}(G, G_1, X) := \lim_{\epsilon \to 0^+} \lim_{n \to \infty} \frac{1}{n} \log r_{\mathrm{sep}}(n, \epsilon, G, X).$$

A subset B of X is called a (n, ϵ, G, X)-spanning set if for every $x \in X$, there is $y \in B$ such that $d^n(x, y) < \epsilon$. $E_{\mathrm{span}}(n, \epsilon, G, X)$ and $r_{\mathrm{span}}(n, \epsilon, G, X)$ are the symbols for a (n, ϵ, G, X)-spanning set with minimal cardinality and its cardinality, respectively.

Remark 2.2. (n, ϵ, G, X)-separated and (n, ϵ, G, X)-spanning sets are finite. It is easily showed by compactness of X.

We define spanning topological entropy of G by

$$h_{\mathrm{span}}(G, G_1, X) := \lim_{\epsilon \to 0^+} \lim_{n \to \infty} \frac{1}{n} \log r_{\mathrm{span}}(n, \epsilon, G, X).$$

In [3], it is proved that

$$h_{\mathrm{span}}(G, G_1, X) = h_{\mathrm{sep}}(G, G_1, X).$$

We denote this quantity by $h(G, G_1, X)$ and call it topological entropy of G. Whenever $G_1 = \{f\}$, for some map f, we denote the topological entropy of G by $h(f, X)$.

Let $A \subseteq X$. We set $G_1\big|_A := \{f_1|_A, \ldots, f_k|_A\}$ and denote the semigroup action generated by $G_1\big|_A$ by G_A.

Remark 2.3. In [4], the authors considered G_1 contains the identity map on X and proved that

1. if A is a closed G-invariant subset of X, then
$$h(G_A, G_1\big|_A, A) \leq h(G, G_1, X);$$
2. If $X = A \cup B$, where A and B are two closed and G-invariant subsets of X, then
$$h(G, G_1, X) = \max\{h(G_A, G_1\big|_A, A), h(G_B, G_1\big|_B, B)\}.$$

Similarly, one can see that these two statements are true for every semigroup action G of continuous functions without identity map.
A semigroup action G is called entropy-minimal provided that for every nonempty closed G-invariant proper subset $A \subset X$,

$$h(G_A, G_1\big|_A, A) < h(G, G_1, X).$$

We say that a semigroup action G is minimal while only closed and G-invariant subsets of X are X and the empty set.

2.2. Results

First in the following we introduce a known set of semigroup actions that they are entropy minimal.

Proposition 2.4. *Every minimal finitely generated semigroup action is entropy minimal.*

Proof. It is obvious by definition of minimality. □

Example 2.5. By [9], any boundaryless compact manifold admits a pair of diffeomorphisms that it generates a minimal semigroup action. Then Proposition 2.4 implies that it is an entropy-minimal semigroup action.

Example 2.6. Assume that f_0 and f_1 are two continuous maps from $\{0,1\}^N$ to itself such that $f_0(s_0, s_1, s_2, \ldots) = (0, s_0, s_1, s_2, \ldots)$ and $f_1(s_0, s_1, s_2, \ldots) = (1, s_0, s_1, s_2, \cdots)$.
By [19], the semigroup action generated by $\{f_0, f_1\}$ is minimal. Proposition 2.4 says that it is entropy-minimal.

Eberlein in [7] showed that the entropy of an "abelian" finitely generated semigroup action is equal to the entropy of its restriction to its nonwandering set. Here we prove it for every semigroup action (not necessarily abelian) with invariant nonwandering set, and by another technique. It is important to note that if a semigroup action is abelian, then its nonwandering set is invariant but its reverse is not true. The following example confirms this point.

Example 2.7. Consider the maps f_0 and f_1 as in Example 2.6. It is obvious that the semigroup action G generated by f_0 and f_1 is not abelian. We show that the nonwandering set of G, $\Omega(G)$, is equal to the set $\{0,1\}^N$ and so it is G-invariant.
Take $s = (s_0, s_1, \ldots) \in \{0,1\}^N$. Let U be an arbitrary neighborhood of s. There exists a basis open set $C_{s_0,\ldots,s_n} = \{s_0\} \times \cdots \times \{s_n\} \times \{0,1\} \times \{0,1\} \times \cdots \times \{0,1\} \times \cdots \subseteq U$ such that $s \in C_{s_0,\ldots,s_n}$.
We have $f_{s_0} \circ f_{s_1} \circ \ldots \circ f_{s_n}(s) \in C_{s_0,\ldots,s_n} \subseteq U$ and $f_{s_0} \circ f_{s_1} \circ \ldots \circ f_{s_n}(C_{s_0,\ldots,s_n}) \subseteq C_{s_0,\ldots,s_n}$. Hence $f_{s_0} \circ f_{s_1} \circ \ldots \circ f_{s_n}(U) \cap U \neq \emptyset$ and so $s \in \Omega(G)$.

For the proof of the following theorem (Theorem 2.8), we need some notations stating here.

Take $w \in \{1, \ldots, k\}^{\mathbb{Z}}$. We say that a subset $A \subseteq X$ is a (n, ε, w, X)-separated set if for every two disjoint points $x, y \in A$, $d(f_w^i(x), f_w^i(y)) \geq \varepsilon$, for some $i \in \{1, \ldots, n\}$. A (n, ε, w, X)-separated set with maximal cardinality denoted by $E_{\text{sep}}(n, \varepsilon, w, X)$ and its cardinality by $r_{\text{sep}}(n, \varepsilon, w, X)$.

A subset B of X is called a (n, ε, w, X)-spanning set whenever for every $x \in X$ there is $y \in B$ such that $d(f_w^i(x), f_w^i(y)) < \varepsilon$, for every $i \in \{1, \ldots, n\}$. We denote a (n, ε, w, X)-spanning set with minimal cardinality by $E_{\text{span}}(n, \varepsilon, w, X)$ and its cardinality by $r_{\text{span}}(n, \varepsilon, w, X)$.

Theorem 2.8. *Let G be a semigroup action generated by G_1 of continuous functions and let its nonwandering set, $\Omega(G)$, be G-invariant. Then*

$$h(G, G_1, X) = h(G_{\Omega(G)}, G_1\big|_{\Omega(G)}, \Omega(G)).$$

Proof. $\Omega(G)$ is a closed G-invariant subset of X. By Remark 2.3 $h(G_{\Omega(G)}, G_1|_{\Omega(G)}, \Omega(G)) \leq h(G, G_1, X)$. It is enough to prove the reverse inequality. Take $m \in \mathbb{N}$ and $\varepsilon > 0$. The set

$$U = \{x \in X \mid d^m(x, y) < \varepsilon \text{ for some } y \in E_{\text{span}}(m, \varepsilon, G, \Omega(G))\}$$

is an open neighborhood of $\Omega(G)$ in X. Since $U^c = X \backslash U$ is a compact wandering set, there is α, $0 < \alpha \leq \varepsilon$, such that $h(B(y, \alpha)) \cap B(y, \alpha) = \emptyset$ for every $h \in G$ and $y \in U^c$, where $B(y, \alpha)$ is a ball of radius α about y. Given $w \in \{1, \ldots, k\}^{\mathbb{Z}}$ and $l \in \mathbb{N}$, define the map

$$\varphi_{l,w} : X \to (E_{\text{span}}(m, \varepsilon, G, \Omega(G)) \cup E_{\text{span}}(m, \alpha, G, U^c))^l,$$

by

$$\varphi_{l,w}(x) = (y_0, \ldots, y_{l-1}),$$

1. if $f_w^{im}(x) \in U$, then $y_i \in E_{\text{span}}(m, \varepsilon, G, \Omega(G))$ and $d^m(f_w^{im}(x), y_i) < \varepsilon$; and
2. if $f_w^{im}(x) \in U^c$, then $y_i \in E_{\text{span}}(m, \alpha, G, U^c)$ and $d^m(f_w^{im}(x), y_i) < \alpha$.

Now, take $n > m r_{\text{span}}(m, \alpha, G, U^c)$ and l so that $(l-1)m < n \leq lm$.
We show that $\varphi_{l,w}$ is one-to-one on $E_{\text{sep}}(n, 2\varepsilon, w, X)$.

Let $x, y \in E_{\text{sep}}(n, 2\varepsilon, w, X)$, $\varphi_{l,w}(x) = \varphi_{l,w}(y) = (y_0, \ldots, y_{l-1})$, $0 \leq t < m$ and $0 \leq i < l$.

$$\begin{aligned} d(f_w^{im+t}(x), f_w^{im+t}(y)) &\leq d^m(f_w^{im}(x), f_w^{im}(y)) \\ &\leq d^m(f_w^{im}(x), y_i) + d^m(f_w^{im}(y), y_i) \\ &< 2\varepsilon. \end{aligned}$$

Since $x, y \in E_{\text{sep}}(n, 2\varepsilon, w, X)$, $x = y$ and so $\sharp\varphi_{l,w}(E_{\text{sep}}(n, 2\varepsilon, w, X)) = r_{\text{sep}}(n, 2\varepsilon, w, X)$.

We claim that if $x, y \in E_{\text{sep}}(n, 2\varepsilon, G, X)$, then there is $w \in \{1, \ldots, k\}^{\mathbb{Z}}$ such that x and y are in a $(n, 2\varepsilon, w, X)$-separated set. To prove this, let $x, y \in E_{\text{sep}}(n, 2\varepsilon, G, X)$. Then, there is $s \in \{1, \ldots, n\}$ and $h \in G_s$ such that $d(h(x), h(y)) \geq 2\varepsilon$. Obviously, there exists $w \in \{1, \ldots, k\}^{\mathbb{Z}}$ such that $h = f_w^s$ and so $d(f_w^s(x), f_w^s(y)) \geq 2\varepsilon$. Hence x and y are in a $(n, 2\varepsilon, w, X)$-separated set. Since $E_{\text{sep}}(n, 2\varepsilon, G, X)$ is a finite set, there are $(n, 2\varepsilon, w^i, X)$-separated set A_i, $i = 1, \cdots, r$, such that

$$E_{\text{sep}}(n, 2\varepsilon, G, X) \subseteq \cup_{i=1}^r A_i.$$

Therefore

$$\sharp E_{\text{sep}}(n, 2\varepsilon, G, X) \leq r \max_{1 \leq i \leq r} \sharp A_i.$$

Now it is enough that for an arbitrary $w \in \{1, \ldots, k\}^{\mathbb{Z}}$, we compute $r_{\text{sep}}(n, 2\varepsilon, w, X)$ which is equal to $\sharp\varphi_{l,w}(E_{\text{sep}}(n, 2\varepsilon, w, X))$. First we note that if $(y_0, \ldots, y_{l-1}) = \varphi_{l,w}(x)$ and $y_s \in E_{\text{span}}(m, \alpha, G, U^c)$ for some s, $0 \leq s \leq l-1$, then y_s does not repeat in this l-tuple. Because for $y_s = y_{s'} \in E_{\text{span}}(m, \alpha, G, U^c)$, $0 \leq s < s' \leq l-1$, $d^m(f_w^{sm}(x), y_s) < \alpha$ and $d^m(f_w^{s'm}(x), y_s) < \alpha$. So $d(f_w^{sm}(x), y_s) < \alpha$ and $d(f_w^{s'm}(x), y_s) < \alpha$, namely, $f_w^{sm}(x), f_w^{s'm}(x) \in B(y_s, \alpha)$. Hence $f_{w_{s'm-1}} \circ \cdots \circ f_{w_{sm}}(B(y, \alpha)) \cap B(y, \alpha) \neq \emptyset$, which is a contradiction with that $B(y, \alpha)$ is a wandering set. Assume that A_s has maximal cardinality among A_i's, $i = 1, \ldots, r$. There exists $w^s \in$

$\{1,\ldots,k\}^{\mathbb{Z}}$ such that $A_s = E_{\text{sep}}(n, 2\varepsilon, w^s, X)$. Let $q = r_{\text{span}}(m, \alpha, w^s, U^c)$ and $p = r_{\text{span}}(m, \varepsilon, w^s, \Omega(G))$. By an argument similar to proof of Theorem 1.4 in section 9.1.1 of [15], we imply that

$$\sharp\varphi_{l,w}(E_{\text{sep}}(n, 2\varepsilon, w^s, X) \leq (q+1)! l^q p^l.$$

So $r_{\text{sep}}(n, 2\varepsilon, G, X) \leq r(q+1)! l^q p^l$. Moreover, we have $\sharp A_s \leq r_{\text{span}}(m, \varepsilon, w^s, \Omega(G)) \leq r_{\text{sep}}(m, \varepsilon, w^s, \Omega(G)) \leq r_{\text{sep}}(m, \varepsilon, G, \Omega(G))$.

Therefore,

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{n} \log r_{\text{sep}}(n, 2\varepsilon, G, X)) \\
&\quad\leq \lim_{n\to\infty} \frac{1}{n} \log(r(q+1)! l^q p^l) \\
&\quad\leq \lim_{l\to\infty} \frac{1}{(l-1)m} \log(r(q+1)! l^q p^l) \\
&\quad\leq \frac{\log(p)}{m} = \frac{\log r_{\text{span}}(m, \varepsilon, w^s, \Omega(G))}{m} \\
&\quad\leq \frac{\log r_{\text{sep}}(m, \varepsilon, G, \Omega(G))}{m}.
\end{aligned}$$

m and ε are arbitrary. As $m \to \infty$ and $\varepsilon \to 0^+$, we have

$$h(G, G_1, X) \leq h(G_{\Omega(G)}, G_1|_{\Omega(G)}, \Omega(G)).$$

□

Corollary 2.9. *If G is an abelian entropy-minimal semigroup action on X, then $\Omega(G) = X$.*

Proof. It is easy to see that if the semigroup action G is abelian, then $\Omega(G)$ is a G-invariant subset of X. On the other hand, $\Omega(G)$ is a closed subset of X. Theorem 2.8 and the definition of entropy minimality imply that $\Omega(G) = X$. □

Example 2.10. Let $f_0(x) = \frac{x}{2}$ and $f_1(x) = \frac{x}{3}$, $x \in [0,1]$ and let G be the semigroup action generated by $\{f_0, f_1\}$. It is clear that G is abelian and $\Omega(G) \neq [0,1]$. By Corollary 2.9, this semigroup action is not entropy-minimal.

In [6], the authors showed that entropy-minimal ordinary dynamical systems are transitive. In sequel, we prove that the entropy minimality property is a criterion to obtain transitivity for semigroup actions of measure preserving homeomorphisms.

Theorem 2.11. *Let G be an entropy-minimal semigroup action of homeomorphisms from X to itself preserving a finite Borel measure with compact support, and let $\Omega(G)$ be invariant. Then G is transitive.*

Proof. On the contrary assume that the semigroup action G is not transitive. By Remark 2.2, there is a nonempty open subset $U \subseteq X$ such that $\overline{\bigcup_{h\in<G_1>} h(U)}$ is not equal to X. Put $V := X \setminus \overline{\bigcup_{h\in<G_1>} h(U)}$. Then V is open and $X = \overline{\bigcup_{h\in<G_1>} h(U)} \cup \overline{\bigcup_{h\in<G_1>} h(V)}$. Set $U_1 := \overline{\bigcup_{h\in<G_1>} h(U)}$

and $V_1 := \overline{\bigcup_{h\in<G_1>} h(V)}$. Since U_1 and V_1 are G-invariant compact subsets of X, by Remark 2.3,

$$h(G, G_1, X) = \max\{h(G_{U_1}, G_1|_{U_1}, U_1), h(G_{V_1}, G_1|_{V_1}, V_1)\}.$$

But U_1 is a proper subset of X and entropy minimality of G implies

$$h(G, G_1, X) > h(G_{U_1}, G_1|_{U_1}, U_1).$$

So $h(G, G_1, X) = h(G_{V_1}, G_1|_{V_1}, V_1)$. On the other hand, G is entropy-minimal, hence $V_1 = X$. Since $V_1 = \overline{\bigcup_{h\in<G_1>} h(V)} = X$, so there exists $h \in< G_1 >$ such that $h(V) \cap U \neq \emptyset$.

Let W be a nonempty open subset of V such that $h(W) \subseteq U$. Then $W \subseteq h^{-1}(U)$ and for all $k \in G$, $kW \subseteq kh^{-1}(U) \subseteq U_1$. Hence $kW \cap W \subseteq U_1 \cap V = \emptyset$. This implies that W is a wandering subset of V_1. By Theorem 2.8 and entropy minimality of G, we have $h(G, G_1, X) = h(G_{\Omega(G)}, G_1|_{\Omega(G)}, \Omega(G))$ and so $\Omega(G) = X$. Hence $V_1 = \Omega(G) = X$ which contradicts the wandering subset W of V_1. □

Corollary 2.12. *Let G be an abelian entropy-minimal finitely generated semigroup action of homeomorphisms from X to itself preserving a finite Borel measure with compact support.Then it is transitive.*

Proof. G is abelian so its nonwandering set is G-invariant. By Theorem 2.11 we obtain transitivity of G. □

Proposition 2.13. *Let f be an entropy-minimal map from a compact metric space X to itself and let g be a constant self-map of X. Then the semigroup action generated by $\{f, g\}$ is entropy minimal.*

Proof. Let G be the semigroup action generated by $\{f, g\}$ and let $\{b\}$ be the image of g. It is easy to see that a subset B of X is G-invariant if and only if it is f-invariant and contains b. Assume that B is a closed G-invariant proper subset of X. f is entropy-minimal so $h(f\big|_B, B) < h(f, X)$. We can see that every (n, ϵ, f, X)-separated set of maximal cardinality is a (n, ϵ, G, X)-separated set of maximal cardinality and conversely. Also every $(n, \epsilon, f\big|_B, B)$-separated set of maximal cardinality is a (n, ϵ, G_B, B)-separated set of maximal cardinality and conversely. Hence $h(f, X) = h(G, X)$ and $h(f\big|_B, B) = h(G_B, B)$. We have $h(G_B, B) = h(f\big|_B, B) < h(f, X) = h(G, X)$ that it concludes entropy minimality of G. □

Corollary 2.14. *Let f be a topollogically transitive, piecewise monotonic self-map of the interval $[c, d]$ and let g be a constant map of $[c, d]$ to itself. Then the semigroup action generated by $\{f, g\}$ is entropy minimal.*

Proof. Let G be the semigroup action generated by $\{f, g\}$. By Theorem 2 in [6] and Proposition 2.13, f and so G are entropy minimal. □

In the following example, we show that the inverse of Theorem 2.11 is not true.

Example 2.15. Let $\{x_i\}_{i\in\mathbb{Z}}$ be an increasing sequence of $[0,1]$ such that $x_i \to 1$ as $n \to +\infty$ and $x_i \to 0$ as $n \to -\infty$. Consider the map f from $[0,1]$ to itself such that $f(0)=0, f(1)=1$ and $f(x_i)=x_i$ for all integer i, moreover f maps $[x_i, x_{i+1}]$ onto $[x_{i-1}, x_{i+2}]$ linearly piecewise with three linear pieces. Certainly f is not piecewise monotonic map. Barge and Martin [2] showed that f is topologically transitive. In [6], the authors found a closed $f-$invariant proper subset X of $[0,1]$ such that $h(f,[0,1]) = h(f\big|_X, X)$. So f is not entropy minimal. Choose $b \in X$ and define the constant map $g(x)=b$ on $[0,1]$. The semigroup action G generated by $\{f,g\}$ is transitive because f is topologically transitive. But G is not entropy-minimal since by Proposition 2.13, $h(G,[0,1]) = h(f,[0,1])$ and $h(G_X, X) = h(f\big|_X, X)$.

3. Nonuniformly Expanding

3.1. Preliminary

Among this section, G is a semigroup action generated by C^1 local diffeomorphisms $f_1, f_2, \ldots, f_k$ of a compact manifold X to itself and σ is the shift map from $\{1,\ldots,k\}^{\mathbb{N}}$ to $\{1,\ldots,k\}^{\mathbb{N}}$ such that $\sigma(\omega_0,\omega_1,\omega_2,\ldots) = (\omega_1,\omega_2,\ldots)$, for every $(\omega_0,\omega_1,\ldots) \in \{1,\ldots,k\}^{\mathbb{N}}$.

We say that a semigroup action G is nonuniformly expanding provided that there exist $\lambda > 0$ and a full Lebesgue measure subset A of X such that for every $x \in A$ there is $\omega = (\omega_0,\omega_1,\ldots) \in \{1,\ldots,k\}^{\mathbb{N}}$ such that

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} log \parallel Df_{\omega_i}(f_\omega^i(x))^{-1} \parallel < -\lambda.$$

Given $0 < \eta < 1$ and $\omega \in \{1,\ldots,k\}^{\mathbb{N}}$, we say that $n \geq 1$ is a $(\eta\,,\omega)$-hyperbolic time for a point $x \in X$ if

$$\prod_{j=n-k}^{n-1} \parallel Df_{\omega_j}(f_\omega^j(x))^{-1} \parallel \leq \eta^k \qquad \text{for all } 1 \leq k \leq n.$$

If for large $n \in \mathrm{N}$ there are $k \geq \theta n$ for some θ and integers $1 \leq n_1 < n_2 < \cdots < n_k \leq n$ which are $(\eta\,,\omega)$-hyperbolic times for x we say that the frequency of $(\eta\,,\omega)$-hyperbolic times for $x \in X$ is bigger than θ.

Let $\delta > 0$ and $0 < \eta < 1$. Given $n \geq 1$, $\omega \in \{1,\ldots,k\}^{\mathbb{N}}$ and $x \in X$. A neighborhood $V_{n\,,\omega}(x)$ of x is called a (η,δ)-hyperbolic preball if

1. f_ω^n sends $V_{n\,,\omega}(x)$ diffeomorphically onto $B(f_\omega^n(x),\delta)$;
2. for every $y \in V_{n\,,\omega}(x)$ and $1 \leq k \leq n$
$$\parallel Df^k_{\sigma^{n-k}(\omega)}(f_\omega^{n-k}(y))^{-1} \parallel \leq \eta^k.$$

Remark 3.1. One can see these definitions for ordinary dynamical systems in [1].

Now we introduce a property that we need to obtain transitivity of nonuniformly expanding semigroup actions.

A sequence $\{x_i\}_{i\in\mathbb{N}} \subseteq X$ is called an asymptotic average pseudo orbit for the semigroup action G provided that there exists $\omega = (\omega_0, \omega_1, \ldots) \in \{1, \ldots, k\}^{\mathbb{N}}$ such that

$$\lim_{n\to\infty} \sup \frac{1}{n} \sum_{i=0}^{n-1} d(f_{\omega_i}(x_i), x_{i+1}) = 0.$$

We say that a sequence $\{x_i\}_{i\in\mathbb{N}} \subseteq X$ is asymptotic average shadowed by some point of X if there exist $\varphi = (\varphi_0, \varphi_1, \ldots) \in \{1, \ldots, k\}^{\mathbb{N}}$ and $z \in X$ such that

$$\lim_{n\to\infty} \sup \frac{1}{n} \sum_{i=0}^{n-1} d(f_{\varphi}^{i}(z), x_i) = 0.$$

A semigroup action G has asymptotic average shadowing property provided that every asymptotic average pseudo orbit in X is asymptotic average shadowed by some point of X.

Gu [8] defines these properties for dynamical systems with one generator.

3.2. Results

We present some propositions and lemma of [14] listed in the following as Propositions 3.2, 3.3 and 3.4. We need them to prove the main result of this part, that is Theorem 3.5.

Proposition 3.2. *Let semigroup action G be nonuniformly expanding. Then there are $0 < \eta < 1$ and $\theta > 0$ such that the frequency of $(\eta\,,\,\omega)$-hyperbolic times for Lebesgue almost every point $x \in X$ is bigger than θ.*

Proposition 3.3. *Let n be a $(\eta\,,\,\omega)$-hyperbolic time for $x \in X$. Then there exist a $(\sqrt{\eta}, \delta)$-hyperbolic preball $V_{n\,,\,\omega}(x)$.*

Proposition 3.4. *Let $V_{n\,,\,\omega}(x)$ be a (η, δ)-hyperbolic preball. Then for every $y, z \in V_{n\,,\,\omega}(x)$ and $1 \le k \le n$ we have*

$$d(f_{\omega}^{n-k}(y), f_{\omega}^{n-k}(z)) \le \eta^k d(f_{\omega}^{n}(y), f_{\omega}^{n}(z)).$$

Here we obtain transitivity by means of asymptotic average shadowing property.

Theorem 3.5. *If G is a nonuniformly expanding semigroup action with asymptotic average shadowing property, then $\langle G\rangle$ is transitive.*

Proof. Let U and V be arbitrary nonempty open subsets of X. Choose $x \in U$, $y \in V$ and $\epsilon > 0$ such that Proposition 3.2 is true for x and y, also $B(x, \epsilon) \subseteq U$ and $B(x, \epsilon) \subseteq V$.

Let D be the diameter of X. Continuity of $f_i^{-1}, i = 1, 2, \ldots, k$, implies that there exists $0 < \xi < D$ such that if $d(p, q) < \xi$ then $d(f_i^{-1}(p), f_i^{-1}(q)) < \epsilon$, $i = 1, 2, \ldots, k$.

By Proposition 3.2, there are $0 < \eta < \frac{\xi}{D}$ and $\theta > 0$ such that the frequencies of η-hyperbolic times of x and y are greater than θ. Also by definition of nonuniformly expanding semigroup action and propositions 3.2

and 3.3, there exist $\delta > 0, \omega \in \{1, \ldots, k\}^{\mathbb{N}}$ and $\phi \in \{1, \ldots, k\}^{\mathbb{N}}$ such that corresponding to n_x as a (η, ω)-hyperbolic time for x and corresponding to n_y as a (η, ϕ)-hyperbolic time for y, there exist $(\sqrt{\eta}, \delta)$-hyperbolic preballs $V_{(n_x,\omega)}(x)$ of x and $V_{(n_y,\phi)}(y)$ of y, respectively, such that $f_\omega^{n_x}$ maps $V_{(n_x,\omega)}(x)$ diffeomorphically on to the ball of radius δ around $f_\omega^{n_x}(x)$ and $f_\phi^{n_y}$ maps $V_{(n_y,\phi)}(y)$ diffeomorphically on to the ball of radius δ around $f_\phi^{n_y}(y)$.

Consider the sequence $\{\alpha_i\} = \{x, y, x, y, x, f_{\omega_0}(x), y, f_{\phi_0}(y), \ldots, x, f_{\omega_0}(x), \ldots, f_\omega^{2^{l-1}-1}(x), y, f_{\phi_0}(y), \ldots, f_\phi^{2^{l-1}-2}(y), f_\phi^{2^{l-1}-1}(y), \ldots\}$.

It is easy to see that for some l and $2^l \leq n < 2^{l+1}$,

$$\frac{1}{n}\sum_{i=0}^{i=n} d(f_{\gamma_i}(\alpha_i), \alpha_{i+1}) < \frac{2(l+1)D}{n},$$

whenever $\gamma = (*, *, *, *, \omega_0, *, \phi_0, *, \omega_0, \omega_1, \omega_2, *, \phi_0, \phi_1, \phi_2, *, \ldots)$ and "$*$" is any arbitrary element of $\{1, 2, \ldots, k\}$. So

$$\lim_{n \longrightarrow \infty} \frac{1}{n}\sum_{i=0}^{i=n} d(f_{\gamma_i}(\alpha_i), \alpha_{i+1}) = 0.$$

The sequence $\{\alpha_i\}$ is an asymptotic average pseudo orbit for G. Hence it can be asymptotically shadowed in average by some point of X, that is, there exist $z \in X$ and $\tau \in \{1, \ldots, k\}^{\mathbb{N}}$ such that

$$\lim_{n \mapsto \infty} \frac{1}{n}\sum_{i=0}^{n-1} d(f_\tau^i(z), \alpha_i) = 0.$$

□

Claim. There exist infinitely many η-hyperbolic times n_x such that corresponding to every n_x there is a positive integer m_x such that $d(f_\tau^{m_x}(z), f_\omega^{n_x}(x)) < \delta$. Also there exist infinitely many η-hyperbolic times n_y such that corresponding to every n_y there is a positive integer m_y such that $d(f_\tau^{m_y}(z), f_\phi^{n_y}(y)) < \delta$.

Proof of Claim. On the contrary suppose that there is a positive integer N such that for all η-hyperbolic time $t > N$,

$$d(f_\tau^i(z), f_\omega^t(x)) > \delta$$

and

$$d(f_\tau^i(z), f_\phi^t(y)) > \delta$$

for any $i > 0$. Then for large n, it would be obtained that

$$\frac{1}{n}\sum_{i=0}^{n-1} d(f_\tau^i(z), \alpha_i) \geq \frac{\delta}{n}\#\{N < t < n : t \textit{ is hyperbolic time for } x \textit{ or } y\}.$$

So

$$\liminf_{n \mapsto \infty} \frac{1}{n}\sum_{i=0}^{n-1} d(f_\tau^i(z), \alpha_i) \geq \delta\theta,$$

which contradicts with asymptotic average shadowing property.

Using the claim there are η-hyperbolic times n_x and n_y for x and y, respectively, such that

$$d(f_\tau^{m_x}(z), f_\omega^{n_x}(x)) < \delta$$

and

$$d(f_\tau^{m_y}(z), f_\phi^{n_y}(y)) < \delta,$$

for some positive integers m_x, m_y. By Propositions 3.3 and 3.4, for all $p, q \in V_{(n_x,\omega)}(x)$,

$$d(f_{\omega_0}(p), f_{\omega_0}(q)) \leq \eta^{\frac{n-1}{2}} d(f_\omega^n(p), f_\omega^n(q)) < \eta^{\frac{n-1}{2}} D < \eta D < \xi,$$

and for all $p, q \in V_{(n_y,\phi)}(y)$,

$$d(f_{\phi_0}(p), f_{\phi_0}(q)) \leq \eta^{\frac{n-1}{2}} d(f_\phi^n(p), f_\phi^n(q)) < \eta^{\frac{n-1}{2}} D < \eta D < \xi.$$

So $d(p,q) < \epsilon$. This show that $V_{(n_x,\omega)}(x) \subset U$ and $V_{(n_y,\phi)}(y) \subset V$. Since $f_\omega^{n_x}$ maps $V_{(n_x,\omega)}(x)$ diffeomorphically on to the ball of radius δ around $f_\omega^n(x)$ and $f_\phi^{n_y}$ maps $V_{(n_y,\phi)}(y)$ diffeomorphically on to the ball of radius δ around $f_\phi^n(y)$, we have

$$f_\tau^{m_x}(z) \in f_\omega^{n_x}(V_{(n_x,\omega)}(x))$$

and

$$f_\tau^{m_y}(z) \in f_\phi^{n_y}(V_{(n_y,\phi)}(y)).$$

Therefore there exists $h \in \langle G \rangle$ such that $h(U) \cap V \neq \phi$ and so the group action $\langle G \rangle$ is transitive. □

Corollary 3.6. *Let G be a nonuniformly expanding semigroup action of C^1 local diffeomorphisms $f_1, f_2, \ldots, f_k$ from X to itself preserving a finite Borel measure with compact support. If G has the asymptotic average shadowing property, then it is transitive.*

Proof. By Remark 2.1 and Theorem 3.5 it is obvious. □

Remark 3.7. One can see other criteria for transitivity of semigroup actions in [12] and [13] that we have introduced and studied.

Remark 3.8. In [14], the authors studied on ergodicity of semigroup actions as a stronger property than transitivity. They proved that every transitive nonuniformly expanding semigroup action of conformal C^1 local diffeomorphisms is ergodic. For definition of ergodicity on semigroup actions, see [14].

Theorem 3.5, Corollary 3.6 and Remark 3.8 eventuate the following corollary.

Corollary 3.9. *Let G be a nonuniformly expanding semigroup action generated by conformal C^1 local diffeomorphisms from a compact manifold to itself preserving a finite Borel measure with compact support. If G has the asymptotic average shadowing property then it is ergodic.*

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