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Existence and Uniqueness of Positive Solutions of a Kind of Multi-point Boundary Value Problems for Nonlinear Fractional Differential Equations with *p*-Laplacian Operator

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Abstract. In this paper, we investigate the existence and uniqueness of positive solutions of a kind of multi-point boundary value problems for nonlinear fractional differential equations with p-Laplacian operator using the Banach contraction mapping principle. Furthermore, some examples are given to illustrate our results.

Mathematics Subject Classification. 34A08, 34B15.

Keywords. Fractional differential equation, Multi-point boundary value problems, p-Laplacian operator.

1. Introduction

Fractional calculus has been widely applied to various areas of engineering, mechanics, physics, chemistry, and biology. There are a large number of papers and monographs that deal with many problems in fractional calculus (see [1–6]). Especially, fractional differential equations have been proved to be powerful tools in the modeling of various phenomena in many fields of science and engineering such as physics, fluid mechanics, and heat conduction. For more details of some results on fractional differential equations and their applications, see the monographs of Podlubny [7], Kilbas et al. [8], and Lakshmikantham et al. [9].

For studying the turbulent flow problem in a porous medium, Leibenson [10] introduced the p-Laplacian differential equation as follows:

$$(\varphi_p(u'(t)))' = f(t,\ u(t),\ u'(t)), \quad t \in (0,\ 1), \tag{1.1}$$

where $\varphi_p(s) = |s|^{p-2}\, s,\ \ p > 1$. Motivated by the Leibenson's work, Guo et al. [11] discussed the existence of solution for m-point boundary value problems of p-Laplacian differential equation:

$$\begin{cases} (\varphi_p(u'(t)))' + a(t)f(t,\ u(t)) = 0, \quad t \in (0,\ 1), \\ \varphi_p(u'(0)) = \sum\limits_{i=1}^{m-2} a_i \varphi_p(u'(\xi_i)), \quad u(1) = \sum\limits_{i=1}^{m-2} b_i u(\xi_i), \end{cases} \tag{1.2}$$

$$\begin{cases} (\varphi_p(u'(t)))' + a(t)f(t,\ u(t)) = 0, \quad t \in (0,\ 1), \\ u(0) = \sum\limits_{i=1}^{m-2} a_i u(\xi_i), \quad \varphi_p(u'(1)) = \sum\limits_{i=1}^{m-2} b_i \varphi_p(u'(\xi_i)). \end{cases} \tag{1.3}$$

All of the differential equations of the problems (1.1)–(1.3) include integer order derivatives. Recently many important results relative to boundary value problems of arbitrary noninteger order differential equations with p-Laplacian operator have been obtained (see [12–26]). Especially, Chai [13] used the fixed-point theorem on cones to investigate the existence and multiplicity of positive solutions for fractional differential equations with p-Laplacian operator:

$$\begin{cases} D_{0+}^{\beta}(\varphi_p(D_{0+}^{\alpha}u))(t) + f(t,\ u(t)) = 0, \quad 0 < t < 1, \\ u(0) = 0, \ \ u(1) + \sigma D_{0+}^{\gamma}u(1) = 0, \quad D_{0+}^{\alpha}u(0) = 0, \end{cases} \tag{1.4}$$

where $1 < \alpha \leq 2,\ \ 0 < \beta \leq 1,\ 0 < \ \gamma \leq 1$.

Using the theory of the fixed-point index in a cone Lü [22] studied the existence and multiplicity of positive solutions to m-point boundary value problems of nonlinear fractional differential equations with p-Laplacian operator:

$$\begin{cases} D_{0+}^{\beta}(\varphi_p(D_{0+}^{\alpha}u(t))) + \varphi_p(\lambda)f(t, u(t)) = 0, \quad 0 < t < 1, \\ u(0) = 0, \quad D_{0+}^{\gamma}u(1) = \sum\limits_{i=1}^{m-2} \xi_i D_{0+}^{\gamma}u(\eta_i), \quad D_{0+}^{\alpha}u(0) = 0, \end{cases} \tag{1.5}$$

where $1 < \alpha \leq 2,\ \ 0 < \beta,\ \ \gamma \leq 1$.

However, there are few articles dealing with the existence of solutions to multi-point boundary value problems for fractional differential equations with p-Laplacian operator, where $\beta > 1$.

Li et al. [23] obtained the existence of multiple positive solutions for m-point boundary value problems of the higher order nonlinear Caputo fractional differential equations with p-Laplacian operator:

$$\begin{cases} D_{0+}^{\beta}(\varphi_p(D_{0+}^{\alpha}u(t))) + f(t, u(t)) = 0, \quad 0 < t < 1, \\ \varphi_p(D_{0+}^{\alpha}u(0))^{(i)} = 0, \ \ i = 1, \ \ 2, \ \ \ldots, \ \ l-1, \\ \qquad u^{(j)}(0) = 0, \ \ j = 1, \ \ 2, \ \ \ldots, \ \ n-1, \\ \varphi_p(D_{0+}^{\alpha}u(1)) = \sum\limits_{i=1}^{m-2} b_i[\varphi_p(D_{0+}^{\alpha}u(\xi_i))], \quad u(0) = \sum\limits_{i=1}^{m-2} a_i u^{(j)}(\xi_i), \end{cases} \tag{1.6}$$

where $l-1 < \beta \leq l, \ \ n-1 < \alpha \leq n, \ \ l \geq 1, \ \ n \geq 2$. Their new results are based on the five functionals fixed-point theorem.

No contribution exists, as far as we know, concerning the existence of solutions for multi-point boundary value problems of fractional differential equations with p-Laplacian operator:

$$\begin{cases} D^{\beta}_{0+}(\varphi_p(D^{\alpha}_{0+}u))(t) = f(t,u(t)), \quad 0<t<1, \\ u(0)=0, \;\; D^{\gamma}_{0+}u(1) = \sum\limits_{i=1}^{m-2} \xi_i D^{\gamma}_{0+}u(\eta_i), \\ D^{\alpha}_{0+}u(0)=0, \quad \varphi_p(D^{\alpha}_{0+}u)(1) = \sum\limits_{i=1}^{m-2} \zeta_i \varphi_p(D^{\alpha}_{0+}u)(\eta_i), \end{cases} \tag{1.7}$$

where D^{α}_{0+}, D^{β}_{0+} and D^{γ}_{0+} are the standard Riemann–Liouville derivatives with $1<\alpha,\beta\le 2$, $3<\alpha+\beta\le 4$, $0<\gamma\le 1$, $\alpha-\gamma-1>0$, $0<\xi_i,\eta_i,\zeta_i<1$, $\sum\limits_{i=1}^{m-2}\xi_i\eta_i^{\alpha-\gamma-1}<1$, $\sum\limits_{i=1}^{m-2}\zeta_i\eta_i^{\beta-1}<1$, the p-Laplacian operator is defined as $\varphi_p(s)=|s|^{p-2}s$, $p>1$, and $f\in C([0,1]\times[0,+\infty),\;[0,+\infty))$.

In this paper, we focus on the solvability of the BVP (1.7). By means of the Banach contraction mapping principle, we obtain some new results on the existence and uniqueness of solutions for our problem.

The organization of this article is as follows. In Sect. 2, we give some necessary definitions and preliminary results which will be used to prove our main results. In Sect. 3, we prove the existence and uniqueness of positive solutions for our problem, and in Sect. 4, we give two examples to demonstrate our results.

2. Preliminaries

Definition 2.1. [14] Let $\alpha>0$. The fractional integral operator of a function $f:(0,\;+\infty)\to\mathbf{R}$ is given by

$$I^{\alpha}_{0+}f(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)ds.$$

Definition 2.2. [16] The Riemann–Liouville fractional derivative of order $\alpha>0$ of a function $f:(0,\;+\infty)\to\mathbf{R}$ is given by

$$D^{\alpha}_{0+}f(t)=\frac{1}{\Gamma(n-\alpha)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n\int_0^t (t-s)^{n-\alpha-1}f(s)\mathrm{d}s.$$

Lemma 2.1. [22] *Assume that $u\in C(0,1)\cap L^1(0,1)$ with a fractional derivative of order $\alpha>0$ that belongs to $C(0,1)\cap L^1(0,1)$. Then*

$$I^{\alpha}_{0+}D^{\alpha}_{0+}u(t)=u(t)+C_1t^{\alpha-1}+C_2t^{\alpha-2}+\cdots+C_Nt^{\alpha-N},$$
$$C_i\in\mathbf{R},\quad i=1,2,\ldots,N$$

when N is the smallest integer greater than or equal to α.

Lemma 2.2. [22] *Let $y\in C[0,\;1]$. Then, the fractional differential equation*

$$\begin{cases} D^{\alpha}_{0+}u(t)+y(t)=0, \;\; 0<t<1, \quad 1<\alpha\le 2, \\ u(0)=0, \;\; D^{\gamma}_{0+}u(1)=\sum\limits_{i=1}^{m-2}\xi_i D^{\gamma}_{0+}u(\eta_i), \quad 0<\gamma\le 1, \end{cases}$$

has a unique solution which is given by

$$u(t) = \int_0^1 G(t,s)y(s)\mathrm{d}s,$$

where

$$G(t,s) = G_1(t,s) + G_2(t,s),$$

in which

$$G_1(t,s) = \begin{cases} \frac{t^{\alpha-1}(1-s)^{\alpha-\gamma-1}-(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le s \le t \le 1, \\ \frac{t^{\alpha-1}(1-s)^{\alpha-\gamma-1}}{\Gamma(\alpha)}, & 0 \le t \le s \le 1, \end{cases}$$

$$G_2(t,s) = \begin{cases} \frac{t^{\alpha-1}}{A\Gamma(\alpha)} \sum\limits_{0\le s\le \eta_i} \xi_i[\eta_i^{\alpha-\gamma-1}(1-s)^{\alpha-\gamma-1} - (\eta_i - s)^{\alpha-\gamma-1}], & t \in [0,1], \\ \frac{t^{\alpha-1}}{A\Gamma(\alpha)} \sum\limits_{\eta_i\le s\le 1} \xi_i\eta_i^{\alpha-\gamma-1}(1-s)^{\alpha-\gamma-1}, & t \in [0,1], \end{cases}$$

where

$$A = 1 - \sum_{i=1}^{m-2} \xi_i\eta_i^{\alpha-\gamma-1}.$$

Lemma 2.3. [22] *If* $\sum\limits_{i=1}^{m-2} \xi_i\eta_i^{\alpha-\gamma-1} < 1$, *then the function* $G(t,s)$ *in Lemma* 2.2 *satisfies the following conditions:*

(i) $G(t,s) > 0$, *for* $s,t \in (0,1)$,
(ii) $G(t,s) \le G_*(s,s)$, *for* $s,t \in [0,1]$,

where

$$G_*(s,s) = \frac{1}{\Gamma(\alpha)}(1-s)^{\alpha-\gamma-1} + \frac{1}{A\Gamma(\alpha)} \sum_{i=1}^{m-2} \xi_i\eta_i^{\alpha-\gamma-1}(1-s)^{\alpha-\gamma-1}.$$

We can see that the function $G_*(s,s)$ *in Lemma* 2.3 *satisfies that*

$$\begin{aligned} G_*(s,s) &= \frac{1}{\Gamma(\alpha)}(1-s)^{\alpha-\gamma-1}\left(1 + \frac{1}{A}\sum_{i=1}^{m-2} \xi_i\eta_i^{\alpha-\gamma-1}\right) \\ &= \frac{1}{A\Gamma(\alpha)}(1-s)^{\alpha-\gamma-1}. \end{aligned}$$

Lemma 2.4. *Let* $f \in C([0,1]\times[0,+\infty),[0,+\infty))$. *Then, the BVP* (1.7) *has a unique solution which is given by*

$$u(t) = \int_0^1 G(t,s)\varphi_p^{-1}\left(\int_0^1 H(s,\tau)f(\tau,u(\tau))\mathrm{d}\tau\right)\mathrm{d}s,$$

where $H(t,s) = H_1(t,s) + H_2(t,s)$,

in which

$$H_1(t,s)=\begin{cases}\frac{t^{\beta-1}(1-s)^{\beta-1}-(t-s)^{\beta-1}}{\Gamma(\beta)}, & 0\le s\le t\le 1,\\ \frac{t^{\beta-1}(1-s)^{\beta-1}}{\Gamma(\beta)}, & 0\le t\le s\le 1,\end{cases}$$

$$H_2(t,s)=\begin{cases}\frac{t^{\beta-1}}{B\Gamma(\beta)}\sum\limits_{0\le s\le\eta_i}\zeta_i[\eta_i^{\beta-1}(1-s)^{\beta-1}-(\eta_i-s)^{\beta-1}], & t\in[0,1],\\ \frac{t^{\beta-1}}{B\Gamma(\beta)}\sum\limits_{\eta_i\le s\le 1}\zeta_i\eta_i^{\beta-1}(1-s)^{\beta-1}, & t\in[0,1],\end{cases}$$

where

$$B=1-\sum_{i=1}^{m-2}\zeta_i\eta_i^{\beta-1}.$$

Proof. Let $h\in C[0,1]$. Consider the boundary value problem:

$$\begin{cases}D_{0+}^{\beta}v(t)+h(t)=0,\\ v(0)=0,\ \ v(1)=\sum\limits_{i=1}^{m-2}\zeta_i v(\eta_i).\end{cases}\tag{2.1}$$

Using Lemma 2.1, we have

$$v(t)+c_1t^{\beta-1}+c_2t^{\beta-2}=-I_{0+}^{\beta}h(t).$$

It follows from the condition $v(0)=0$ that $c_2=0$. Thus

$$v(t)=-I_{0+}^{\beta}h(t)-c_1t^{\beta-1}.\tag{2.2}$$

Together with the condition $v(1)=\sum\limits_{i=1}^{m-2}\zeta_i v(\eta_i)$, this yields

$$\frac{1}{\Gamma(\beta)}\int_0^1(1-s)^{\beta-1}h(s)\mathrm{d}s+c_1=\sum_{i=1}^{m-2}\zeta_i\left[\frac{1}{\Gamma(\beta)}\int_0^{\eta_i}(\eta_i-s)^{\beta-1}h(s)\mathrm{d}s+c_1\eta_i^{\beta-1}\right].$$

Then, we can get that

$$c_1=\frac{1}{B\Gamma(\beta)}\left[\sum_{i=1}^{m-2}\zeta_i\int_0^{\eta_i}(\eta_i-s)^{\beta-1}h(s)\mathrm{d}s-\int_0^1(1-s)^{\beta-1}h(s)\mathrm{d}s\right].\tag{2.3}$$

Substituting (2.3) into (2.2) and using the relation $\frac{1}{B\Gamma(\beta)}=\frac{1}{\Gamma(\beta)}+\frac{1-B}{B\Gamma(\beta)}$, we have

$$\begin{aligned}v(t)&=-\frac{1}{\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}h(s)\mathrm{d}s-\frac{t^{\beta-1}}{B\Gamma(\beta)}\sum_{i=1}^{m-2}\zeta_i\int_0^{\eta_i}(\eta_i-s)^{\beta-1}h(s)\mathrm{d}s\\&\quad+\frac{t^{\beta-1}}{B\Gamma(\beta)}\int_0^1(1-s)^{\beta-1}h(s)\mathrm{d}s\\&=-\frac{1}{\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}h(s)\mathrm{d}s-\frac{t^{\beta-1}}{B\Gamma(\beta)}\sum_{i=1}^{m-2}\zeta_i\int_0^{\eta_i}(\eta_i-s)^{\beta-1}h(s)\mathrm{d}s\\&\quad+\frac{t^{\beta-1}}{\Gamma(\beta)}\int_0^1(1-s)^{\beta-1}h(s)\mathrm{d}s+\frac{t^{\beta-1}}{B\Gamma(\beta)}\sum_{i=1}^{m-2}\zeta_i\eta_i^{\beta-1}\int_0^1(1-s)^{\beta-1}h(s)\mathrm{d}s.\end{aligned}$$

Therefore, the solution of the BVP (2.1) is expressed by

$$v(t)=\int_0^1 H(t,s)h(s)\mathrm{d}s. \tag{2.4}$$

Conversely, let $v(t)$ be the function which is given by (2.4). From the continuity of $H(t,s)$ and $h(s)$, we see that $v\in C[0,1]$. In addition, we can easily get $v(0)=0$, since $H(0,s)=0$.

We can rewrite (2.4) as

$$v(t)=-I_{0+}^{\beta}h(t)-\frac{1}{B}\left[\sum_{i=1}^{m-2}\zeta_i\ I_{0+}^{\beta}h(t)\Big|_{t=\eta_i}-I_{0+}^{\beta}h(t)\Big|_{t=1}\right]t^{\beta-1}. \tag{2.5}$$

Applying D_{0+}^{β} on both sides of (2.5), we can obtain

$$\begin{aligned}D_{0+}^{\beta}v(t)&=-h(t)-\frac{1}{B}\left[\sum_{i=1}^{m-2}\zeta_i\ I_{0+}^{\beta}h(t)\Big|_{t=\eta_i}-I_{0+}^{\beta}h(t)\Big|_{t=1}\right]D_{0+}^{\beta}t^{\beta-1}\\&=-h(t).\end{aligned}$$

On the other hand, from the continuity of $h(t)$, we see that $D_{0+}^{\beta}v\in C[0,1]$. We can also have that

$$\begin{aligned}&\sum_{i=1}^{m-2}\zeta_i v(\eta_i)\\&=\sum_{i=1}^{m-2}\zeta_i\left\{-I_{0+}^{\beta}h(t)\Big|_{t=\eta_i}-\frac{1}{B}\left[\sum_{j=1}^{m-2}\zeta_j\ I_{0+}^{\beta}h(t)\Big|_{t=\eta_j}-I_{0+}^{\beta}h(t)\Big|_{t=1}\right]\eta_i^{\beta-1}\right\}\\&=\sum_{i=1}^{m-2}\zeta_i\left\{-I_{0+}^{\beta}h(t)\Big|_{t=\eta_i}-\frac{\eta_i^{\beta-1}}{B}\sum_{j=1}^{m-2}\zeta_j\ I_{0+}^{\beta}h(t)\Big|_{t=\eta_j}+\frac{\eta_i^{\beta-1}}{B}\ I_{0+}^{\beta}h(t)\Big|_{t=1}\right\}\\&=-\sum_{i=1}^{m-2}\zeta_i\ I_{0+}^{\beta}h(t)\Big|_{t=\eta_i}-\sum_{i=1}^{m-2}\zeta_i\frac{\eta_i^{\beta-1}}{B}\sum_{j=1}^{m-2}\zeta_j\ I_{0+}^{\beta}h(t)\Big|_{t=\eta_j}\\&\quad+\sum_{i=1}^{m-2}\zeta_i\frac{\eta_i^{\beta-1}}{B}\ I_{0+}^{\beta}h(t)\Big|_{t=1}\\&=-\sum_{i=1}^{m-2}\zeta_i\ I_{0+}^{\beta}h(t)\Big|_{t=\eta_i}\cdot\left(1+\frac{1-B}{B}\right)+\frac{1-B}{B}\ I_{0+}^{\beta}h(t)\Big|_{t=1}\\&=-\frac{1}{B}\sum_{i=1}^{m-2}\zeta_i\ I_{0+}^{\beta}h(t)\Big|_{t=\eta_i}+\frac{1}{B}\ I_{0+}^{\beta}h(t)\Big|_{t=1}-I_{0+}^{\beta}h(t)\Big|_{t=1}\\&=v(1).\end{aligned}$$

Therefore, we can conclude that $v(t)$ is a solution of the BVP (2.1) and our problem has a unique solution which is given by (2.4).

Now, we prove the main result of this lemma. Let $u(t)$ be the solution of the BVP (1.7) and put $w(t) := D^{\alpha}_{0+}u(t)$. By Lemma 2.2, we know that

$$u(t) = -\int_0^1 G(t,s)w(s)\mathrm{d}s. \tag{2.6}$$

Putting $v(t) = \varphi_p(w(t))$, we have

$$v(t) = -\int_0^1 H(t,s)f(s,u(s))\mathrm{d}s. \tag{2.7}$$

Combining (2.6) and (2.7) yields

$$u(t) = \int_0^1 G(t,s)\varphi_p^{-1}\left(\int_0^1 H(s,\tau)f(\tau,u(\tau))\mathrm{d}\tau\right)\mathrm{d}s. \tag{2.8}$$

The proof is completed. □

Lemma 2.5. *If* $\sum_{i=1}^{m-2}\zeta_i\eta_i^{\beta-1} < 1$, *then the function* $H(t,s)$ *in Lemma* 2.4 *satisfies the following conditions:*

(i) $H(t,s) > 0$, *for* $s,t \in (0,1)$,
(ii) $H(t,s) \le H_*(s,s)$, *for* $s,t \in [0,1]$,

where

$$H_*(s,s) = \frac{1}{\Gamma(\beta)}(1-s)^{\beta-1} + \frac{1}{B\Gamma(\beta)}\sum_{i=1}^{m-2}\zeta_i\eta_i^{\beta-1}(1-s)^{\beta-1}.$$

Proof. The proof is easy, so we omit it. □

We can see that the function $H_*(s,s)$ in Lemma 2.5 satisfies that

$$\begin{aligned} H_*(s,s) &= \frac{1}{\Gamma(\beta)}(1-s)^{\beta-1}\left(1+\frac{1}{B}\sum_{i=1}^{m-2}\zeta_i\eta_i^{\beta-1}\right) \\ &= \frac{1}{B\Gamma(\beta)}(1-s)^{\beta-1}. \end{aligned}$$

The basic properties of the p-Laplacian operator which will be used in the following studies are listed below [14].

(i) If $1 < p < 2,\;\; xy > 0$, and $|x|,\;\; |y| \ge m > 0$, then

$$|\varphi_p(x) - \varphi_p(y)| \le (p-1)m^{p-2}\,|x-y|\,. \tag{2.9}$$

(ii) If $p > 2,\;\; |x|\,,|y| \le M$, then

$$|\varphi_p(x) - \varphi_p(y)| \le (p-1)M^{p-2}\,|x-y|\,. \tag{2.10}$$

3. Main Results

We consider the Banach space $C([0,\ 1])$ endowed with the norm defined by $\|u\| := \max\limits_{0\le t\le 1} |u(t)|$. Denote φ_p^{-1} by φ_q, where $1/p+1/q=1$ and $X := \{x | x \in C[0,1],\ \ D^{\alpha}_{0+}x \in C[0,1],\ \ \varphi_p(D^{\alpha}_{0+}x) \in C[0,1],\quad D^{\beta}_{0+}(\varphi_p(D^{\alpha}_{0+}x)) \in C[0,1]\}$. Define the operator $T : X \to X$ as

$$Tu(t) := \int_0^1 G(t,s)\varphi_q\left(\int_0^1 H(s,\tau)f(\tau,u(\tau))\mathrm{d}\tau\right)\mathrm{d}s.$$

Then, the BVP (1.7) has a solution if and only if the operator T has a fixed point.
Put $\bar{M} := (\alpha-\gamma)^{p-1}A^{p-1}B\Gamma(\alpha)^{p-1}\Gamma(\beta)$.
In this article, the following hypotheses will be used.

(H1) There exist nonnegative functions $g,\quad h \in L[0,1]$ and $M_g := \int_0^1 g(t)\mathrm{d}t > 0$, $0 < M_h := \int_0^1 h(t)\mathrm{d}t < \overline{M}$, such that $f(t,x) \le g(t) + h(t)x^{p-1}$ for any $(t,x) \in [0,\ 1]\times[0,\ r]$, where $r = \left(\frac{M_g}{\overline{M}-M_h}\right)^{q-1}$.
(H2) $|f(t,x) - f(t,y)| < L\,|x-y|$ for any $t \in [0,\ 1]$ and any $x,\ y \in [0,\ r]$.
(H3) There exist $\bar{m},\delta > 0$, such that $f(t,x) \ge \bar{m}t^{\delta-1}$ for any $(t,x) \in [0,\ 1]\times[0,\ r]$.

Lemma 3.1. *If (H1) holds, then*

$$T(E) \subset E,$$

where

$$E := \{u \in X \mid\ ||u|| \le r\}.$$

Proof. It is easy to see that $T(E) \subset X$. Now, we prove that $\|Tu\| \le r$ for any $u \in E$. By Lemma 2.5 and (H1), we can see that for any $t \in [0,1]$

$$\begin{aligned}
|Tu(t)| &= \left|\int_0^1 G(t,s)\varphi_q\left(\int_0^1 H(s,\tau)f(\tau,u(\tau))\mathrm{d}\tau\right)\mathrm{d}s\right|\\
&\le \left|\int_0^1 G(t,s)\varphi_q\left(\max_{s\in[0,1]} H_*(s,s)\int_0^1 f(\tau,u(\tau))\mathrm{d}\tau\right)\mathrm{d}s\right|\\
&\le \left|\int_0^1 G(t,s)\varphi_q\left(\frac{1}{B\Gamma(\beta)}\int_0^1 (g(\tau)+h(\tau)u(\tau)^{p-1})\mathrm{d}\tau\right)\mathrm{d}s\right|\\
&\le \left|\int_0^1 G(t,s)\varphi_q\left(\frac{1}{B\Gamma(\beta)}\left(\int_0^1 g(\tau)\mathrm{d}\tau + r^{p-1}\int_0^1 h(\tau)\mathrm{d}\tau\right)\right)\mathrm{d}s\right|\\
&\le \left|\int_0^1 G(t,s)\varphi_q\left(\frac{1}{B\Gamma(\beta)}(M_g + r^{p-1}M_h)\right)\mathrm{d}s\right|.
\end{aligned}$$

By considering the properties of the function $G(t,s)$ in Lemma 2.3, we can get

$$
\begin{aligned}
|Tu(t)| &\le \left(\frac{1}{B\Gamma(\beta)}(M_g + r^{p-1}M_h)\right)^{q-1} \int_0^1 G_*(s,s)\mathrm{d}s \\
&= \left(\frac{1}{B\Gamma(\beta)}(M_g + r^{p-1}M_h)\right)^{q-1} \cdot \frac{1}{A(\alpha-\gamma)\Gamma(\alpha)} \\
&= \frac{(M_g + r^{p-1}M_h)^{q-1}}{B^{q-1}\Gamma(\beta)^{q-1}A(\alpha-\gamma)\Gamma(\alpha)}.
\end{aligned}
$$

Since $1/p + 1/q = 1$, $p-1$ is the inverse number of $q-1$. So by simple calculation, we know that

$$
|Tu(t)| \le \left(\frac{M_g + r^{p-1}M_h}{\overline{M}}\right)^{q-1}.
$$

It follows from the notation $r = \left(\frac{M_g}{\overline{M}-M_h}\right)^{q-1}$ that $(\overline{M} - M_h)r^{p-1} = M_g$.

Therefore, we can get

$$
\begin{aligned}
|Tu(t)| &\le \left(\frac{(\overline{M} - M_h)r^{p-1} + r^{p-1}M_h}{\overline{M}}\right)^{q-1} \\
&= (r^{p-1})^{q-1} \\
&= r.
\end{aligned}
$$

This yields $\|Tu\| \le r$. The proof is completed. □

Lemma 3.2. *The followings hold:*

(i) If (H3) holds, then there exists $K_0 := \frac{\bar{m}\Gamma(\delta)}{B\Gamma(\beta+\delta)} \sum_{i=1}^{m-2} \zeta_i(\eta_i^{\beta-1} - \eta_i^{\beta-1+\delta})$, *such that*

$$
\int_0^1 H(t,s)f(s,u(s))\mathrm{d}s \ge K_0 t^{\beta-1} \tag{3.1}
$$

for any $t \in [0,1]$.

(ii) There exists $K_1 := \frac{1}{\Gamma(\beta+1)}\left[1 + \frac{1}{B}\sum_{i=1}^{m-2}\zeta_i(\eta_i^{\beta-1} - \eta_i^{\beta})\right]$, *such that*

$$
\int_0^1 H(t,s)\mathrm{d}s \le K_1 t^{\beta-1} \tag{3.2}
$$

for any $t \in [0,1]$.

Proof. (i) Since (H3) holds, we obtain

$$
\begin{aligned}
\int_0^1 H(t,s)f(s,u(s))\mathrm{d}s &\ge \int_0^1 H(t,s)\bar{m}s^{\delta-1}\mathrm{d}s \\
&= \int_0^1 H_1(t,s)\bar{m}s^{\delta-1}\mathrm{d}s + \int_0^1 H_2(t,s)\bar{m}s^{\delta-1}\mathrm{d}s.
\end{aligned} \tag{3.3}
$$

Evaluating two parts of the right-hand side in (3.3), respectively, we have

$$\begin{aligned}\int_0^1 H_1(t,s)\bar{m}s^{\delta-1}\mathrm{d}s &= \frac{\bar{m}}{\Gamma(\beta)}\left[t^{\beta-1}\int_0^1 (1-s)^{\beta-1}s^{\delta-1}\mathrm{d}s - \int_0^t (t-s)^{\beta-1}s^{\delta-1}\mathrm{d}s\right]\\ &= \frac{\bar{m}}{\Gamma(\beta)}[t^{\beta-1}B(\delta,\beta) - t^{\beta-1+\delta}B(\delta,\beta)]\\ &= \frac{\bar{m}\Gamma(\delta)}{\Gamma(\beta+\delta)}[t^{\beta-1} - t^{\beta-1+\delta}]\\ &\geq 0,\\ \int_0^1 H_2(t,s)\bar{m}s^{\delta-1}\mathrm{d}s &= \frac{\bar{m}t^{\beta-1}}{B\Gamma(\beta)}\sum_{i=1}^{m-2}\zeta_i\left[\eta_i^{\beta-1}\int_0^1 (1-s)^{\beta-1}s^{\delta-1}\mathrm{d}s\right.\\ &\quad \left.-\int_0^{\eta_i}(\eta_i-s)^{\beta-1}s^{\delta-1}\mathrm{d}s\right]\\ &= \frac{\bar{m}t^{\beta-1}}{B\Gamma(\beta)}\sum_{i=1}^{m-2}\zeta_i\left[\eta_i^{\beta-1}B(\delta,\beta) - \eta_i^{\beta-1+\delta}B(\delta,\beta)\right]\\ &= \frac{\bar{m}\Gamma(\delta)}{B\Gamma(\beta+\delta)}\sum_{i=1}^{m-2}\zeta_i(\eta_i^{\beta-1} - \eta_i^{\beta-1+\delta})t^{\beta-1}.\end{aligned}$$

Therefore, we know that (3.1) is satisfied.

(ii) From the definition of $H(t,s)$, we see that

$$\int_0^1 H(t,s)\mathrm{d}s = \int_0^1 H_1(t,s)\mathrm{d}s + \int_0^1 H_2(t,s)\mathrm{d}s.$$

In addition, we can get

$$\begin{aligned}\int_0^1 H_1(t,s)\mathrm{d}s &= \frac{1}{\Gamma(\beta)}\left[t^{\beta-1}\int_0^1 (1-s)^{\beta-1}\mathrm{d}s - \int_0^t (t-s)^{\beta-1}\mathrm{d}s\right]\\ &= \frac{1}{\Gamma(\beta+1)}(t^{\beta-1} - t^{\beta})\\ &\leq \frac{1}{\Gamma(\beta+1)}t^{\beta-1},\\ \int_0^1 H_2(t,s)\mathrm{d}s &= \frac{t^{\beta-1}}{B\Gamma(\beta)}\sum_{i=1}^{m-2}\zeta_i\left[\eta_i^{\beta-1}\int_0^1 (1-s)^{\beta-1}\mathrm{d}s - \int_0^{\eta_i}(\eta_i-s)^{\beta-1}\mathrm{d}s\right]\\ &= \frac{1}{B\Gamma(\beta+1)}\sum_{i=1}^{m-2}\zeta_i(\eta_i^{\beta-1} - \eta_i^{\beta})t^{\beta-1}.\end{aligned}$$

Therefore, we conclude that (3.2) holds. □

Theorem 3.1. *Suppose that the assumptions (H1)–(H3) and*

$$\frac{(q-1)K_0^{q-2}K_1L}{A(\alpha-\gamma)\Gamma(\alpha)} < 1 \tag{3.4}$$

hold and $p > 2$*. Then, the BVP* (1.7) *has a unique positive solution.*

Proof. Define the operators $T_0,\ T_1$ as follows.

$$T_0u(t) := \varphi_q\left(\int_0^1 H(t,s)f(s,u(s))\mathrm{d}\tau\right),$$
$$T_1u(t) := \int_0^1 G(t,s)u(s)\mathrm{d}s, \quad Tu(t) = (T_1 \circ T_0 u)(t).$$

Since $G(0,s) = 0$, we have that for any $x \in X$,

$$Tx(0) = \int_0^1 G(0,s)T_0x(s)\mathrm{d}s = 0.$$

In the case $p > 2$, due to $1/p + 1/q = 1$, we can get $1 < q < 2$. So from (3.1) and the basic properties of p-Laplacian operator, we can see that for any $x, y \in X$ and any $t \in (0,1]$:

$$\begin{aligned}|T_0x(t) - T_0y(t)| &= \left|\varphi_q\left(\int_0^1 H(t,s)f(s,x(s))\mathrm{d}s\right) - \varphi_q\left(\int_0^1 H(t,s)f(s,y(s))\mathrm{d}s\right)\right| \\ &\le (q-1)(K_0t^{\beta-1})^{q-2}\left|\int_0^1 H(t,s)f(s,x(s))\mathrm{d}s\right. \\ &\quad \left.-\int_0^1 H(t,s)f(s,y(s))\mathrm{d}s\right| \\ &\le (q-1)(K_0t^{\beta-1})^{q-2}\int_0^1 H(t,s)\left|f(s,x(s)) - f(s,y(s))\right|\mathrm{d}s \\ &\le (q-1)(K_0t^{\beta-1})^{q-2}\int_0^1 H(t,s)L\left|x(s) - y(s)\right|\mathrm{d}s \\ &\le (q-1)(K_0t^{\beta-1})^{q-2}L\left\|x-y\right\|\int_0^1 H(t,s)\mathrm{d}s. \end{aligned} \tag{3.5}$$

Applying (3.2) to (3.5), we have

$$\begin{aligned}|T_0x(t) - T_0y(t)| &\le (q-1)(K_0t^{\beta-1})^{q-2}L\left\|x-y\right\|K_1t^{\beta-1} \\ &\le (q-1)K_0^{q-2}K_1L\left\|x-y\right\|t^{(\beta-1)(q-1)}.\end{aligned}$$

Therefore, we can obtain

$$\begin{aligned}|Tx(t) - Ty(t)| &= |T_1 \circ T_0x(t) - T_1 \circ T_0y(t)| \\ &= \left|\int_0^1 G(t,s)(T_0x)(s)\mathrm{d}s - \int_0^1 G(t,s)(T_0y)(s)\mathrm{d}s\right| \\ &\le \int_0^1 G(t,s)\left|T_0x(s) - T_0y(s)\right|\mathrm{d}s \\ &\le (q-1)K_0^{q-2}K_1L\left\|x-y\right\|\int_0^1 G(t,s)s^{(\beta-1)(q-1)}\mathrm{d}s.\end{aligned}$$

On the other hand, since $(\beta-1)(q-1)>0$, by Lemma 2.3, we get

$$\begin{aligned}\int_0^1 G(t,s)s^{(\beta-1)(q-1)}\mathrm{d}s &\le \int_0^1 G(t,s)\mathrm{d}s\\ &\le \int_0^1 G_*(s,s)\mathrm{d}s\\ &= \frac{1}{A(\alpha-\gamma)\Gamma(\alpha)}.\end{aligned}$$

This yields

$$\|Tx-Ty\| \le \frac{(q-1)K_0^{q-2}K_1L}{A(\alpha-\gamma)\Gamma(\alpha)}\|x-y\|. \tag{3.6}$$

Combining Lemma 3.1, (3.4) and (3.6) implies that $T:E\to E$ is a contraction mapping. By means of the Banach contraction mapping principle, we can see that T has a unique fixed point in E, that is to say, the BVP (1.7) has a unique positive solution. □

Lemma 3.3. *If (H1) holds, there exists $M_0 := \frac{M_g + r^{p-1}M_h}{B\Gamma(\beta)}$, such that for any $u\in E$ and any $t\in[0,1]$,*

$$\int_0^1 H(t,s)f(s,u(s))\mathrm{d}s \le M_0. \tag{3.7}$$

Proof. In a similar way to the proof of Lemma 3.1, we can see that

$$\begin{aligned}\int_0^1 H(t,s)f(s,u(s))\mathrm{d}s &\le \int_0^1 H_*(s,s)f(s,u(s))\mathrm{d}s\\ &\le \max_{s\in[0,1]} H_*(s,s)\int_0^1 f(s,u(s))\mathrm{d}s\\ &\le \frac{1}{B\Gamma(\beta)}\int_0^1 (g(s)+h(s)u(s)^{p-1})\mathrm{d}s\\ &\le \frac{1}{B\Gamma(\beta)}\left(\int_0^1 g(s)\mathrm{d}\tau + r^{p-1}\int_0^1 h(s)\mathrm{d}\tau\right)\\ &= \frac{M_g+r^{p-1}M_h}{B\Gamma(\beta)}.\end{aligned}$$

The proof is completed. □

Theorem 3.2. *Suppose that the assumptions (H1), (H2,) and*

$$\frac{(q-1)Lr^{2-p}}{\beta\overline{M}} < 1 \tag{3.8}$$

hold and $1<p<2$. Then, the BVP (1.7) has a unique positive solution.

Proof. It is easy to see that $q > 2$. Using Lemmas 2.5 and 3.3 and the basic properties of p-Laplacian operator, we can get that for any $x, y \in X$ and any $t \in [0,1]$:

$$\begin{aligned}
|T_0x(t) - T_0y(t)| &= \left|\varphi_q\left(\int_0^1 H(t,s)f(s,x(s))\mathrm{d}s\right) - \varphi_q\left(\int_0^1 H(t,s)f(s,y(s))\mathrm{d}s\right)\right| \\
&\leq (q-1)M_0^{q-2}\left|\int_0^1 H(t,s)f(s,x(s))\mathrm{d}s - \int_0^1 H(t,s)f(s,y(s))\mathrm{d}s\right| \\
&\leq (q-1)M_0^{q-2}\int_0^1 H(t,s)\left|f(s,x(s)) - f(s,y(s))\right|\mathrm{d}s \\
&\leq (q-1)M_0^{q-2}\int_0^1 H(t,s)L\left|x(s) - y(s)\right|\mathrm{d}s \\
&\leq (q-1)M_0^{q-2}L\left\|x-y\right\|\int_0^1 H(t,s)\mathrm{d}s \\
&\leq (q-1)M_0^{q-2}L\left\|x-y\right\|\int_0^1 H_*(s,s)\mathrm{d}s \\
&= \frac{(q-1)M_0^{q-2}L}{B\Gamma(\beta+1)}\left\|x-y\right\|.
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
|Tx(t) - Ty(t)| &= |T_1 \circ T_0x(t) - T_1 \circ T_0y(t)| \\
&= \left|\int_0^1 G(t,s)(T_0x)(s)\mathrm{d}s - \int_0^1 G(t,s)(T_0y)(s)\mathrm{d}s\right| \\
&\leq \int_0^1 G(t,s)\left|T_0x(s) - T_0y(s)\right|\mathrm{d}s \\
&\leq \frac{(q-1)M_0^{q-2}L\left\|x-y\right\|}{B\Gamma(\beta+1)}\int_0^1 G(t,s)\mathrm{d}s.
\end{aligned}$$

In a similar way to the proof of Theorem 3.1, we can evaluate

$$\|Tx - Ty\| \leq \frac{(q-1)M_0^{q-2}L}{(\alpha-\gamma)AB\Gamma(\alpha)\Gamma(\beta+1)}\left\|x-y\right\|.$$

From the definition of M_0, we have

$$\begin{aligned}
\frac{(q-1)M_0^{q-2}L}{(\alpha-\gamma)AB\Gamma(\alpha)\Gamma(\beta+1)} &= \frac{(q-1)L}{\beta(\alpha-\gamma)AB\Gamma(\alpha)\Gamma(\beta)}\cdot\left(\frac{M_g + r^{p-1}M_h}{B\Gamma(\beta)}\right)^{q-2} \\
&= \frac{(q-1)L}{\beta\overline{M}^{q-1}}\cdot(M_g + r^{p-1}M_h)^{q-2} \\
&= \frac{(q-1)L}{\beta\overline{M}}\cdot\left(\frac{M_g}{\overline{M} - M_h}\right)^{q-2} \\
&= \frac{(q-1)Lr^{2-p}}{\beta\overline{M}}. \qquad (3.9)
\end{aligned}$$

Therefore, combining Lemma 3.1, (3.8), and (3.9) implies that $T : E \to E$ is a contraction mapping. By means of the Banach contraction mapping principle, we can see that T has a unique fixed point in E, that is to say, the BVP (1.7) has a unique positive solution. □

4. Examples

To demonstrate our main results, we present the following examples.

Example 4.1. Consider the boundary value problem:

$$\begin{cases} D_{0+}^{1.5}(\varphi_3(D_{0+}^{1.5}u))(t) = 10\sqrt{t} + 0.01[u(t)]^2, \quad 0 < t < 1, \\ u(0) = 0, \quad D_{0+}^{1.5}u(0) = 0, \\ D_{0+}^{0.25}u(1) = 0.1D_{0+}^{0.25}u(0.25) + 0.1D_{0+}^{0.25}u(0.5) + 0.1D_{0+}^{0.25}u(0.75), \\ \varphi_3(D_{0+}^{1.5}u)(1) = 0.1\varphi_3(D_{0+}^{1.5}u)(0.25) + 0.1\varphi_3(D_{0+}^{1.5}u)(0.5) + 0.1\varphi_3(D_{0+}^{1.5}u)(0.75). \end{cases} \tag{4.1}$$

Then, the BVP (4.1) has a unique positive solution.

Proof. The BVP (4.1) can be regarded as the boundary value problem (1.7), where $f(t,\ x) = 10\sqrt{t} + 0.01x^2,\ \ p = 3,\ \ \alpha = 1.5,\ \ \beta = 1.5,\ \ \gamma = 0.25,\ \ \eta_1 = 0.25,\ \ \eta_2 = 0.5,\ \ \eta_3 = 0.75,\ \ \xi_1 = 0.1,\ \ \xi_2 = 0.1, \xi_3 = 0.1,\ \ \zeta_1 = 0.1,\ \ \zeta_2 = 0.1,\ \ \zeta_3 = 0.1$.

Then, we can get

$$\begin{aligned} &q = 1.5 < 2, \\ &A = 1 - \sum_{i=1}^{m-2} \xi_i \eta_i^{\alpha-\gamma-1} = 0.7521\ldots, \\ &B = 1 - \sum_{i=1}^{m-2} \zeta_i \eta_i^{\beta-1} = 0.7927\ldots, \\ &\bar{M} = (\alpha-\gamma)^{p-1}A^{p-1}B\Gamma(\alpha)^{p-1}\Gamma(\beta) = 0.4877\ldots, \\ &f \in C([0,1]\times[0,+\infty),\ \ [0,+\infty)). \end{aligned}$$

Taking $\delta = 1.5,\ \ \bar{m} = 10,\ \ g(t) = 10,\ \ h(t) = 0.01$, we have

$$M_g = \int_0^1 g(t)\mathrm{d}t = 10,\ \ M_h = \int_0^1 h(t)\mathrm{d}t = 0.01,$$

and for any $t \in [0,\ 1], f(t,\ x) \geq \bar{m}t^{\delta-1},\ \ f(t,\ x) \leq g(t) + h(t)x^{p-1}$.

Therefore, we obtain that

$$r = \left(\frac{M_g}{\overline{M} - M_h}\right)^{q-1} = 4.5753\ldots,$$

and for any $x,\ y \in [0,\ \ r], |f(t,\ x) - f(t,\ y)| = 0.01|x^2 - y^2| \leq 0.01\cdot 2r\cdot|x-y| < L|x-y|$, where $L = 0.1$.

By computation we deduce that

$$K_0 = \frac{\bar{m}\Gamma(\delta)}{B\Gamma(\beta+\delta)} \sum_{i=1}^{m-2} \zeta_i(\eta_i^{\beta-1} - \eta_i^{\beta-1+\delta}) = 0.6698\ldots,$$

$$K_1 = \frac{1}{\Gamma(\beta+1)} \left[1 + \frac{1}{B} \sum_{i=1}^{m-2} \zeta_i(\eta_i^{\beta-1} - \eta_i^{\beta})\right] = 0.8419\ldots,$$

$$K = \frac{(q-1)K_0^{q-2}K_1 L}{A(\alpha-\gamma)\Gamma(\alpha)} = 0.0617\ldots.$$

Therefore, by Theorem 3.1, the BVP (4.1) has a unique positive solution. □

Example 4.2. Consider the boundary value problem:

$$\begin{cases} D_{0+}^{1.5}(\varphi_{1.5}(D_{0+}^{1.5}u))(t) = 0.5t + 0.1u(t), \quad 0 < t < 1, \\ u(0) = 0, \quad D_{0+}^{1.5}u(0) = 0, \\ D_{0+}^{0.25}u(1) = 0.1D_{0+}^{0.25}u(0.25) + 0.1D_{0+}^{0.25}u(0.5) + 0.1D_{0+}^{0.25}u(0.75), \\ \varphi_{1.5}(D_{0+}^{1.5}u)(1) = 0.1\varphi_{1.5}(D_{0+}^{1.5}u)(0.25) + 0.1\varphi_{1.5}(D_{0+}^{1.5}u)(0.5) + 0.1\varphi_{1.5}(D_{0+}^{1.5}u)(0.75). \end{cases} \tag{4.2}$$

Then, the BVP (4.2) has a unique positive solution.

Proof. The BVP (4.2) can be regarded as the boundary value problem (1.7), where $f(t,\ x) = 0.5t + 0.1x,\ \ p = 1.5,\ \ \alpha = 1.5,\ \ \beta = 1.5,\ \ \gamma = 0.25,\ \ \eta_1 = 0.25,\ \ \eta_2 = 0.5,\ \ \eta_3 = 0.75,\ \ \xi_1 = 0.1,\ \ \xi_2 = 0.1,$

$$\xi_3 = 0.1, \quad \zeta_1 = 0.1, \quad \zeta_2 = 0.1, \quad \zeta_3 = 0.1.$$

Then, we can get

$$q = 3 > 2,$$

$$A = 1 - \sum_{i=1}^{m-2} \xi_i \eta_i^{\alpha-\gamma-1} = 0.7521\ldots,$$

$$B = 1 - \sum_{i=1}^{m-2} \zeta_i \eta_i^{\beta-1} = 0.7927\ldots,$$

$$\bar{M} = (\alpha-\gamma)^{p-1} A^{p-1} B\Gamma(\alpha)^{p-1}\Gamma(\beta) = 0.6412\ldots,$$

$$f \in C([0,1] \times [0,+\infty), \quad [0,+\infty)).$$

Taking $g(t) = 0.5,\ \ h(t) = 0.1$, we have

$$M_g = \int_0^1 g(t)\mathrm{d}t = 0.5,$$

$$M_h = \int_0^1 h(t)\mathrm{d}t = 0.1,$$

$$r = \left(\frac{M_g}{\overline{M} - M_h}\right)^{q-1} = 0.8534\ldots.$$

So we obtain that for any $t \in [0,\ 1]$ and any $x,\ y \in [0,\ \ r]$,

$$\begin{aligned} &|f(t,\ x) - f(t,\ y)| \leq L\,|x-y|\,, \\ &f(t,\ x) \leq g(t) + h(t)x^{p-1}, \end{aligned}$$

where $L = 0.1$.

By computation we deduce that

$$K' = \frac{(q-1)Lr^{2-p}}{\beta \overline{M}} = 0.1921\ldots.$$

By Theorem 3.2, we can prove that the BVP (4.2) has a unique positive solution. □

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