Mediterr. J. Math. (2018) 15:96
https://doi.org/10.1007/s00009-018-1146-4
1660-5446/18/030001-10
published online April 26, 2018

Mediterranean Journal of Mathematics

Smoothness and Shape Preserving Properties of Bernstein Semigroup

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Abstract. This paper is concerned with the strongly continuous semigroup $(T(t))_{t\geq 0}$ of operators on $C[0,1]$ which can be represented as a limit of suitable iterates of the Bernstein operators B_n. We present some new smoothness and shape preserving properties of the operators $T(t)$ and B_n. The asymptotic behavior and simultaneous approximation results are also presented.

Mathematics Subject Classification. 47D07, 41A36, 26A51.

Keywords. Markov semigroup, Bernstein operators, Shape preserving properties, Simultaneous approximation.

1. Introduction

Let (C[0,1],$||\cdot||$) be the Banach space of all real-valued, continuous functions on [0,1], endowed with the supremum norm. For $0 \leq j \leq n$, we consider the functions

$$b_{n,j}(x) := \binom{n}{j} x^j (1-x)^{n-j}, \quad x \in [0,1].$$

The classical Bernstein operators $B_n : C[0,1] \to C[0,1]$ are defined by

$$B_n f(x) := \sum_{j=0}^{n} b_{nj}(x) f\left(\frac{j}{n}\right), \quad n \geq 1,\ f \in C[0,1],\ x \in [0,1].$$

Let $t \geq 0$ and $(k(n))_{n\geq 1}$ be an arbitrary sequence of positive integers such that $\lim\limits_{n\to\infty} \frac{k(n)}{n} = t$. A remarkable feature of the Bernstein operators is the existence of the limit:

$$T(t)f := \lim_{n\to\infty} B_n^{k(n)} f, \quad f \in C[0,1].$$

B_n^k denotes the iterate of order k of B_n.

Moreover, $T(t) : C[0,1] \to C[0,1]$ is a Markov operator (i.e., a positive linear operator transforming the constant function 1 into itself) and $(T(t))_{t\geq 0}$ is a Markov semigroup of operators on $C[0,1]$.

This semigroup and the sequence of Bernstein operators are deeply investigated in the literature: see [1,2,5–7,11,12] and the references therein.

In this paper, we present some new smoothness and shape preserving properties of the operators B_n and $T(t)$.

In Sect. 2, we start using some known properties of Bernestein operators to prove that $T(t)$ preserves the smoothness of f. This result enables us to obtain a Lipschitz-type property of the family $(T(t))_{t\geq 0}$: see Theorem 2.3 and Remark 2.4 (i).

Let $T : C[0,1] \rightarrow C[0,1]$ be the operator B_1, that is

$$Tf(x) = (1-x)f(0) + xf(1), \quad f \in C[0,1], x \in [0,1].$$

It is well known (see, e.g., [3, Rem. 3.11.1], [18, Th. 3.1]) that

$$\lim_{t\to\infty} T(t)f = Tf, \quad f \in C[0,1].$$

Theorems 2.5 and 2.6 are simultaneous approximation type results for $T(t)$; they show that if $f \in C^m[0,1]$, then $T(t)f \in C^m[0,1]$ for every $t \geq 0$ and $(T(t)f)^{(i)} \rightarrow f^{(i)}$ as $t \rightarrow 0$, respectively $(T(t)f)^{(i)} \rightarrow (Tf)^{(i)}$ as $t \rightarrow \infty$ for all $i = 0, 1, \ldots, m$.

Section 2 ends with a result concerning the rate of convergence of $T(t)f$ towards Tf.

Section 3 is devoted to shape preserving properties. We consider the family of strongly m-convex functions with modulus c (see [9]) and the family of approximately m-concave functions with modulus c (see [14]).

The behavior of B_n and $T(t)$ with respect to these families is investigated.

Throughout this paper, we use the notation $e_j(t) = t^j$, $t \in [0,1], j = 0, 1, \ldots$. We need also a well-known result (see, e.g., [1, Cor. 6.3.8]):

If $f \in C[0,1]$ is convex, then $B_n f$ and $T(t)f$ are convex; moreover, $B_n f \geq f$ and $T(t)f \geq f, n \geq 1, t \geq 0$.

2. Smoothness Preservation Properties

Let $(T(t))_{t\geq 0}$ be the semigroup on $C[0,1]$ represented in terms of iterates of the Bernstein operators.

Theorem 2.1. *If* $f \in C^m[0,1]$, *then* $T(t)f \in C^m[0,1], t \geq 0$, *and*

$$||(T(t)f)^{(i)}|| \leq ||f^{(i)}|| \exp\left(-\frac{(i-1)i}{2}t\right), \quad t \geq 0,\ i = 0, 1, \ldots, m. \tag{2.1}$$

Proof. (a) The assertion is true if f is a polynomial. In this case, $T(t)f$ is a polynomial of the same degree, and (2.1) is satisfied; see [10].

(b) Let $f \in C^m[0,1]$ be given, and set $p_n := B_n f, n \geq 1$.

Then (see [5, Sect. 4.6]),

$$\lim_{n\to\infty} p_n^{(i)} = f^{(i)}, \quad i = 0, 1, \ldots, m. \tag{2.2}$$

Let $k, j \geq 1$. According to (a), we have

$$||(T(t)p_k)^{(i)} - (T(t)p_j)^{(i)}|| = ||(T(t)(p_k - p_j))^{(i)}|| \leq ||(p_k - p_j)^{(i)}|| \exp\left(-\frac{(i-1)i}{2}\right), \quad t \geq 0.$$

Combined with (2.2), this shows that for fixed $t \geq 0$ and $i \in \{0, 1, \ldots, m\}$, $((T(t)p_n)^{(i)})_{n\geq 1}$ is a Cauchy sequence in $C[0,1]$; consequently, there exists $\varphi_{t,i} \in C[0,1]$, such that

$$\lim_{n\to\infty} (T(t)p_n)^{(i)} = \varphi_{t,i}. \tag{2.3}$$

In particular, we have $\lim_{n\to\infty} T(t)p_n = T(t)f$ and $\lim_{n\to\infty} (T(t)p_n)^{(1)} = \varphi_{t,1}$; it follows that $T(t)f \in C^1[0,1]$ and $(T(t)f)^{(1)} = \varphi_{t,1}$.

Now, $\lim_{n\to\infty} (T(t)p_n)^{(1)} = \varphi_{t,1} = (T(t)f)^{(1)}$ and $\lim_{n\to\infty} (T(t)p_n)^{(2)} = \varphi_{t,2}$ imply $(T(t)f)^{(1)} \in C^1[0,1]$ and $(T(t)f)^{(2)} = \varphi_{t,2}$; moreover, $\lim_{n\to\infty} (T(t)p_n)^{(2)} = (T(t)f)^{(2)}$.

Repeating these arguments, we find that $(T(t)f)^{(m)} = \varphi_{t,m} \in C[0,1]$, i.e., $T(t)f \in C^m[0,1]$ and

$$\lim_{n\to\infty} (T(t)p_n)^{(i)} = (T(t)f)^{(i)}, \quad t \geq 0, \ i = 0, 1, \ldots, m. \tag{2.4}$$

According to (a), (2.1) is true for the polynomials p_n. Consequently, (2.4) and (2.2) yield

$$\begin{aligned} ||(T(t)f)^{(i)}|| &= \lim_{n\to\infty} ||(T(t)p_n)^{(i)}|| \leq \lim_{n\to\infty} ||p_n^{(i)}|| \exp\left(-\frac{(i-1)i}{2}t\right) \\ &= ||f^{(i)}|| \exp\left(-\frac{(i-1)i}{2}t\right), \end{aligned}$$

and this concludes the proof. □

Remark 2.2. Results of this type, in a more general context, can be found in [8].

Theorem 2.3. *Let* $x \in [0,1], t, s \geq 0, f \in C^2[0,1]$. *Then*

$$|T(s)f(x) - T(t)f(x)| \leq \frac{x(1-x)}{2}|e^{-s} - e^{-t}|||f''||. \tag{2.5}$$

Proof. Let $u \geq 0$ and $g \in C^2[0,1]$. Then $\frac{1}{2}||g''||e_2 \pm g$ are convex functions, and so

$$T(u)\Big(\frac{1}{2}||g''||e_2 \pm g\Big) \geq \frac{1}{2}||g''||e_2 \pm g.$$

We know (see, e.g., [4,6,9]) that $T(u)e_2 = e^{-u}e_2 + (1 - e^{-u})e_1$.

Therefore, $\frac{1}{2}||g''||(e^{-u}e_2 + (1 - e^{-u})e_1) \pm T(u)g \geq \frac{1}{2}||g''||e_2 \pm g$, which implies

$$\frac{1}{2}||g''||(1 - e^{-u})x(1-x) \geq |T(u)g(x) - g(x)|, \ x \in [0,1]. \tag{2.6}$$

Now, let $f \in C^2[0,1]$ and $g := T(t)f$. According to Theorem 2.1, $g \in C^2[0,1]$; consequently, from (2.6), we derive

$$|T(u)(T(t)f)(x) - T(t)f(x)| \leq \frac{1}{2}x(1-x)(1-e^{-u})||(T(t)f)''||.$$

However, (2.1) shows that $||(T(t)f)''|| \leq e^{-t}||f''||$, so that

$$|T(u+t)f(x) - T(t)f(x)| \leq \frac{1}{2}x(1-x)(1-e^{-u})e^{-t}||f''||.$$

□

Setting $s := u + t$, we get (2.5).

Remark 2.4. (i) (2.5) can be extended to functions from $C[0,1]$ by passing to K-functionals and moduli of smoothness, in the spirit of [10].
(ii) With $s = 0$, respectively $s \to \infty$, we get from (2.5)

$$|T(t)f(x) - f(x)| \leq \frac{1}{2}x(1-x)(1-e^{-t})||f''||, \tag{2.7}$$

$$|T(t)f(x) - Tf(x)| \leq \frac{1}{2}x(1-x)e^{-t}||f''||, \tag{2.8}$$

for all $t \geq 0,\ x \in [0,1],\ f \in C^2[0,1]$.

(2.7) and (2.8) were proved in [16]; see also [17].

The next two theorems contain *simultaneous approximation* results.

Theorem 2.5. *Let $f \in C^m[0,1]$. Then*

$$\lim_{t\to 0}(T(t)f)^{(i)} = f^{(i)}, \quad i = 0,1,\ldots,m. \tag{2.9}$$

Proof. It is known (see, e.g., [12, (3.12)], [13, Th. 2.1]) that

$$T(t)e_j = e_j \exp\left(-\frac{(j-1)j}{2}t\right) + e_{j-1}a_{j-1}(t) + \cdots + e_0 a_0(t), \tag{2.10}$$

for $j \geq 0, t \geq 0$ and certain continuous functions $a_0, \ldots, a_{j-1}$.

Since $\lim_{t\to 0} T(t)e_j = e_j$, it follows that

$$\lim_{t\to 0} a_k(t) = 0, \quad k = 0,\ldots,j-1. \tag{2.11}$$

Now, (2.10) and (2.11) imply

$$\lim_{t\to 0}(T(t)e_j)^{(i)} = (e_j)^{(i)}, \quad i,j \geq 0,$$

and therefore

$$\lim_{t\to 0}(T(t)p)^{(i)} = p^{(i)}, \quad i \geq 0, \tag{2.12}$$

for each polynomial function p.

Let $f \in C^m[0,1]$ and $p_n := B_n f, n \geq 1$. Then

$$||(T(t)f)^{(i)} - f^{(i)}|| \leq ||(T(t)f)^{(i)} - (T(t)p_n)^{(i)}|| + ||(p_n)^{(i)} - f^{(i)}|| + ||(T(t)p_n)^{(i)} - (p_n)^{(i)}||.$$

Using (2.1), we get

$$||(T(t)f)^{(i)} - f^{(i)}|| \leq 2||f^{(i)} - (p_n)^{(i)}|| + ||(T(t)p_n)^{(i)} - p_n^{(i)}||.$$

Let i be fixed and $\varepsilon > 0$. According to (2.2), there exists $n \geq 1$, such that $||f^{(i)} - (p_n)^{(i)}|| \leq \frac{\varepsilon}{4}$. (2.12) shows that there exists $\delta > 0$, such that

$$||(T(t)p_n)^{(i)} - (p_n)^{(i)}|| \leq \frac{\varepsilon}{2}, \quad t \in [0, \delta].$$

We conclude that

$$||(T(t)f)^{(i)} - f^{(i)}|| \leq \varepsilon, \quad t \in [0, \delta],$$

and so (2.9) is proved. □

Theorem 2.6. *Let $f \in C^m[0,1]$. Then*

$$\lim_{t\to\infty} (T(t)f)^{(i)} = (Tf)^{(i)}, \quad i = 0, 1, \ldots, m. \tag{2.13}$$

Proof. For $i = 0$, (2.13) is trivially true; for $i = 2, \ldots, m$, it is a consequence of (2.1), because Tf is a polynomial function of degree ≤ 1.

Therefore, let $f \in C^1[0,1]$; we have to prove that

$$\lim_{t\to\infty} (T(t)f)^{'} = (Tf)^{'}. \tag{2.14}$$

Let $j \geq 2$. Then, $\lim\limits_{t\to\infty} T(t)e_j = Te_j = e_1$, so that from (2.10), we get

$$\lim_{t\to\infty} a_1(t) = 1, \ \lim_{t\to\infty} a_k(t) = 0, \quad k \in \{0, 2, 3, \ldots, j-1\}.$$

Combined with (2.10), this yields

$$\lim_{t\to\infty} (T(t)e_j)^{'} = e_1^{'} = e_0 = (Te_j)^{'}, \ i.e.,$$
$$\lim_{t\to\infty} (T(t)e_j)^{'} = (Te_j)^{'}, \quad j \geq 2.$$

This is true also for $j \in \{0, 1\}$, and so

$$\lim_{t\to\infty} (T(t)p)^{'} = (Tp)^{'} \tag{2.15}$$

for each polynomial function p.
Now, let $p_n := B_n f, \ n \geq 1$. Then

$$\begin{aligned} ||(T(t)f)^{'} - (Tf)^{'}|| &\leq ||(T(t)f)^{'} - (T(t)p_n)^{'}|| + ||(T(t)p_n)^{'} - (Tp_n)^{'}|| \\ &\quad + ||(Tp_n)^{'} - (Tf)^{'}|| \\ &= ||(T(t)(f - p_n))^{'}|| + ||(T(t)p_n)^{'} - (Tp_n)^{'}|| \\ &\leq ||(f - p_n)^{'}|| + ||(T(t)p_n)^{'} - (Tp_n)^{'}||. \end{aligned}$$

Let $\varepsilon > 0$. According to (2.2) and (2.15), there exists n, such that $||(f - p_n)^{'}|| \leq \dfrac{\varepsilon}{2}$, and there exists A, such that $||(T(t)p_n)^{'} - (Tp_n)^{'}|| \leq \dfrac{\varepsilon}{2}$ for all $t \geq A$.
Therefore

$$||(T(t)f)^{'} - (Tf)^{'}|| \leq \varepsilon, \quad t \geq A,$$

and this proves (2.14). □

Concerning the rate of convergence $T(t)f \to Tf \ (t \to \infty)$, we have the following result.

Theorem 2.7. *Let $f \in C[0,1]$ be differentiable at 0 and 1. Then*

$$\lim_{t\to\infty} e^t(T(t)f(x) - Tf(x)) = 6x(1-x)\left(\int_0^1 f(s)ds - \frac{f(0)+f(1)}{2}\right), \tag{2.16}$$

uniformly for $x \in [0,1]$.

Proof. Define the function $g \in C[0,1]$ by

$$g(0) := f'(0) + f(0) - f(1),$$
$$g(1) := -f'(1) - f(0) + f(1),$$
$$g(x) := \frac{f(x) - (1-x)f(0) - xf(1)}{x(1-x)}, \quad 0 < x < 1.$$

Let $\varepsilon > 0$; take a polynomial function q, such that $||g - q|| \le \frac{\varepsilon}{2}$. Setting $p(x) := (1-x)f(0) + xf(1) + x(1-x)q(x)$, we get

$$|f(x) - p(x)| \le \frac{\varepsilon}{2} x(1-x), \quad x \in [0,1]. \tag{2.17}$$

For $h \in C[0,1],\ t \ge 0$ and $x \in [0,1]$ denote

$$V(t)h(x) := e^t(T(t)h(x) - Th(x)),\ Lh(x)$$
$$:= 6x(1-x)\Big(\int_0^1 h(s)ds - \frac{h(0)+h(1)}{2}\Big).$$

Using (2.17) and [16, Th. 3.1] (see also [4, (11)]), we get

$$|V(t)(f-p)(x)| \le \varepsilon x(1-x),\ x \in [0,1]. \tag{2.18}$$

Let $\left(J_m^{(1,1)}(s)\right)_{m\ge 0}$ be the Jacobi monic polynomials on $[0,1]$, orthogonal with respect to the weight $s(1-s)$. Let $u_0(s) = 1,\ u_1(s) = s,\ u_j(s) = s(1-s)J_{j-2}^{(1,1)}(s),\ j \ge 2$.
Then (see, e.g., [13])

$$T(t)u_j = u_j exp\Big(-\frac{(j-1)j}{2}t\Big), \quad t \ge 0,\ j \ge 0. \tag{2.19}$$

From (2.19), it follows easily that

$$\lim_{t\to\infty} V(t)u_j = Lu_j = \begin{cases} 0\ , & j \ne 2, \\ u_2\ , & j = 2, \end{cases}$$

and so $\lim_{t\to\infty} V(t)p = Lp$. Consequently, there exists A, such that $||V(t)p - Lp|| \le \frac{5\varepsilon}{8}$ for all $t \ge A$.
Now, for $t \ge A$, we have (see (2.17) and (2.18)):

$$\begin{aligned} |V(t)f(x) - Lf(x)| &\le |V(t)(f-p)(x)| + |V(t)p(x) - Lp(x)| \\ &\quad + |Lp(x) - Lf(x)| \\ &\le \varepsilon x(1-x) + \frac{5\varepsilon}{8} + |L(p-f)(x)| \\ &\le \frac{\varepsilon}{4} + \frac{5\varepsilon}{8} + \frac{\varepsilon}{8} = \varepsilon, \quad x \in [0,1]. \end{aligned}$$

Thus, $\lim_{t\to\infty} V(t)f(x) = Lf(x)$, uniformly on $[0,1]$, which proves (2.16). □

Remark 2.8. Concerning Theorem 2.7, see also [15, Th. 3].

3. Shape Preserving Properties

Let $f \in C[0,1]$ and $0 \le i \le n$. We shall use the notation:

$$\lambda_{ni} := \frac{n!}{n^i(n-i)!}.$$

The divided difference of f on the distinct nodes $a_0, a_1, \ldots, a_i \in [0,1]$ will be denoted by $[a_0, a_1, \ldots, a_i; f]$.
Then (see e.g., [1, p. 460]), for each $x \in [0,1]$,

$$(B_n f)^{(i)}(x) = i!\lambda_{ni} \sum_{h=0}^{n-i} \left[\frac{h}{n}, \frac{h+1}{n}, \ldots, \frac{h+i}{n}; f\right] b_{n-i,h}(x). \tag{3.1}$$

Now, let us denote
$l(f;i) := \inf\{[a_0, a_1, \ldots, a_i; f] : 0 \le a_0 < a_1 < \cdots < a_i \le 1\}$,
$u(f;i) := \sup\{[a_0, a_1, \ldots, a_i; f] : 0 \le a_0 < a_1 < \cdots < a_i \le 1\}$.

Theorem 3.1. *Let $f \in C[0,1], 0 \le i \le n$, and $t \ge 0$. Then*

$$\lambda_{ni} l(f;i) \le l(B_n f;i) \le u(B_n f;i) \le \lambda_{ni} u(f;i), \tag{3.2}$$

$$l(f;i)\exp\left(-\frac{(i-1)i}{2}t\right) \le l(T(t)f;i) \le u(T(t)f;i) \le u(f;i) \exp\left(-\frac{(i-1)i}{2}t\right). \tag{3.3}$$

Proof. Using (3.1), we see that

$$i!\lambda_{ni} l(f;i) \le (B_n f)^{(i)}(x) \le i!\lambda_{ni} u(f;i), \quad x \in [0,1]. \tag{3.4}$$

Let $0 \le a_0 < a_1 < \cdots < a_i \le 1$. Then, according to the mean value theorem for divided differences, there exists $t \in [0,1]$, such that

$$[a_0, a_1, \ldots, a_i; B_n f] = \frac{1}{i!}(B_n f)^{(i)}(t).$$

Now, (3.4) shows that

$$\lambda_{ni} l(f;i) \le [a_0, a_1, \ldots, a_i; B_n f] \le \lambda_{ni} u(f;i),$$

which implies (3.2).
From (3.2), we deduce

$$\begin{aligned}\lambda_{ni}^2 l(f;i) &\le \lambda_{ni} l(B_n f;i) \le l(B_n^2 f;i) \le [a_0, a_1, \ldots, a_i; B_n^2 f] \\ &\le u(B_n^2 f;i) \le \lambda_{ni} u(B_n f;i) \le \lambda_{ni}^2 u(f;i).\end{aligned}$$

Briefly

$$\lambda_{ni}^2 l(f;i) \le [a_0, a_1, \ldots, a_i; B_n^2 f] \le \lambda_{ni}^2 u(f;i).$$

By induction

$$\lambda_{ni}^k l(f;i) \le [a_0, a_1, \dots, a_i; B_n^k f] \le \lambda_{ni}^k u(f;i), k \ge 1. \tag{3.5}$$

Let $t \ge 0$. Choose a sequence $(k(n))_{n\ge1}$, such that $\lim_{n\to\infty} \frac{k(n)}{n} = t$. In (3.5), replace k by $k(n)$ and let $n \to \infty$; it follows that

$$l(f;i)\exp\left(-\frac{(i-1)i}{2}t\right) \le [a_0, a_1, \dots, a_i; T(t)f] \le u(f;i)\exp\left(-\frac{(i-1)i}{2}t\right),$$

for all $0 \le a_0 < a_1 < \cdots < a_i \le 1$. This proves (3.3). □

Definition 3.2. ([9]) *Let $c > 0$. The function $f \in C([0,1])$ is called strongly m-convex with modulus c if*

$$[a_0, a_1, \dots, a_{m+1}; f] \ge c, \tag{3.6}$$

for all $0 \le a_o < a_1 < \cdots < a_{m+1} \le 1$. Equivalently, $l(f; m+1) \ge c$.

Now, we are in a position to prove

Corollary 3.3. *Let $f \in C[0,1]$ be strongly m-convex with modulus c. Then*
(a) $B_n f$ is strongly m-convex with modulus $c\lambda_{n,m+1}$;
(b) $T(t)f$ is strongly m-convex with modulus $c\exp(-\frac{m(m+1)}{2}t)$.

Proof. If f is strongly m-convex with modulus c, then (3.6) shows that $l(f; m+1) \ge c$. From (3.2) and (3.3), we get
$l(B_n f; m+1) \ge \lambda_{n,m+1} l(f; m+1) \ge c\lambda_{n,m+1}$;

$$l(T(t)f; m+1) \ge l(f; m+1)\exp\left(-\frac{m(m+1)}{2}t\right) \ge c\exp\left(-\frac{m(m+1)}{2}t\right),$$

and the proof is finished. □

The following is an extension of a definition from [14].

Definition 3.4. *Let $c > 0$. The function $f \in C[0,1]$ is called approximately m-concave with modulus c if*

$$[a_0, a_1, \dots, a_{m+1}; f] \le c, \tag{3.7}$$

for all $0 \le a_0 < a_1 < \cdots < a_{m+1} \le 1$.
Equivalently, $u(f; m+1) \le c$.

Corollary 3.5. *Let $f \in C[0,1]$ be approximately m-concave with modulus c. Then*
(a) $B_n f$ is approximately m-concave with modulus $c\lambda_{n,m+1}$;
(b) $T(t)f$ is approximately m-concave with modulus $c\exp(-\frac{m(m+1)}{2}t)$.

Proof. According to (3.7), $u(f; m+1) \le c$. To conclude the proof, it suffices to combine this inequality with (3.2) and (3.3). □

Remark 3.6. The limiting case $c = 0$ in Corollaries 3.3 and 3.5 is also investigated in [2, Prop. A.2.5].

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Received: December 28, 2017.
Revised: April 1, 2018.
Accepted: April 20, 2018.