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Modified Proximal Point Algorithms for Solving Constrained Minimization and Fixed Point Problems in Complete CAT(0) Spaces

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Abstract. In this paper, we propose a new modified proximal point algorithm for a countably infinite family of nonexpansive mappings in complete CAT(0) spaces and prove strong convergence theorems for the proposed process under suitable conditions. We also apply our results to solving linear inverse problems and minimization problems. Several numerical examples are given to show the efficiency of the presented method.

Mathematics Subject Classification. 47H09, 47H10, 65K10, 65K15.

Keywords. Convex minimization problem, Fixed point problems, Proximal point algorithm, Nonexpansive mappings, CAT(0) spaces.

1. Introduction

Let (X, d) be a geodesic metric space and let $f : X \to (-\infty, \infty]$ be a proper and convex function. One of the major problems in optimization theory is to find a point $x \in X$ such that

$$f(x) = \min_{y \in X} f(y).$$

We denote the set of all minimizers of f on X by $\operatorname{argmin}_{y \in X} f(y)$.

The proximal point algorithm is an important tool in solving optimization problem which was initiated by Martinet [28] in 1970. Later, Rockafellar [34] studied the convergence of the proximal point algorithm for finding a solution of the unconstrained convex minimization problem in a Hilbert space H as follows. Let f be a proper, convex and lower semi-continuous function on H. The proximal point algorithm is defined by $x_1 \in H$ and

$$x_{n+1} = \operatorname*{argmin}_{y \in H} \left[f(y) + \frac{1}{2\lambda_n} \|y - x_n\|^2 \right], \quad \forall n \in \mathbb{N},$$

where $\lambda_n > 0$ for all $n \in \mathbb{N}$. It was shown that if f has a minimizer and $\inf_n \lambda_n > 0$ then the sequence $\{x_n\}$ converges weakly to a minimizer of f; see also [9]. However, the proximal point algorithm does not necessarily converge strongly in general; see [6,17]. Recently several authors proposed modifications of Rochafellar's proximal point algorithm to have strong convergence, for example, in 2000, Kamimura and Takahashi [21] combined the proximal point algorithm with Halpern's iteration process [19] so that the strong convergence is guaranteed.

In recent times, many convergence results by the proximal point algorithm for solving unconstrained convex minimization problems have been extended from the classical linear spaces such as Hilbert spaces and Banach spaces to the setting of manifolds; see [3–5,7,17,21,27,28,30–32,34]. For example, in 2013, Bačák [4] introduced the proximal point algorithm in a complete CAT(0) space (X, d) as follows: $x_1 \in X$ and

$$x_{n+1} = \underset{y \in X}{\operatorname{argmin}} \left[f(y) + \frac{1}{2\lambda_n} d(y, x_n)^2 \right], \quad \forall n \in \mathbb{N},$$

where $\lambda_n > 0$ for all $n \in \mathbb{N}$. It was shown that if f has a minimizer and $\sum_{n=1}^{\infty} \lambda_n = \infty$, then the sequence $\{x_n\}$ Δ-converges to its minimizer. Later in 2015, Cholamjiak et al. [12] established the strong convergence of the sequence to minimizers of a convex and lower semi-continuous function and to a common fixed point of two nonexpansive mappings in complete CAT(0) spaces. Recently, Suparatulatorn et al. [37] introduced the modified proximal point algorithm combined with Halpern's iteration process for a nonexpansive mapping T in a complete CAT(0) space (X, d) as follows. Let f be a proper, convex and lower semi-continuous function on X. The modified proximal point algorithm is defined by $u, x_1 \in X$ and

$$\begin{cases} y_n = \underset{y \in X}{\operatorname{argmin}} \left[f(y) + \frac{1}{2\lambda_n} d(y, x_n)^2 \right], \\ x_{n+1} = \alpha_n u \oplus (1 - \alpha_n) T y_n, \quad \forall n \in \mathbb{N}, \end{cases} \tag{1.1}$$

where $\lambda_n > 0$ for all $n \in \mathbb{N}$. It was proved that if f has a minimizer and $\{\lambda_n\}, \{\alpha_n\}$ satisfy some mild conditions, then the sequence $\{x_n\}$ generated by (1.1) converges strongly to its minimizer.

Motivated by research in this direction, the following question arises:
Question I: Can we construct an iterative process for finding minimizers of a convex and lower semi-continuous function and common fixed points of a countably infinite family of nonexpansive mappings in complete CAT(0) spaces?

The aim of this paper is to propose a new modified proximal point algorithm for a countably infinite family of nonexpansive mappings in complete CAT(0) spaces and to prove strong convergence theorems for the proposed algorithm under suitable conditions. In the last section, we apply our results to solve the linear inverse problem and the minimization problem including numerical examples. Our results not only give an affirmative answer to the above question but also generalize the corresponding results of Bačák [4], Cholamjiak et al. [12], Suparatulatorn et al. [37], Dhompongsa and Panyanak [15], and many others.

2. Preliminaries and Useful Lemmas

Let (X, d) be a metric space and $x, y \in X$. A *geodesic* joining x to y is a map γ from a closed interval $[0, k] \subset \mathbb{R}$ to X such that $\gamma(0) = x, \gamma(k) = y$ and $d(\gamma(s_1), \gamma(s_2)) = |s_1 - s_2|$ for all $s_1, s_2 \in [0, k]$. So γ is an isometry and $d(x, y) = k$. The image of γ is called a *geodesic segment* joining x and y. When it is unique, this geodesic is denoted by $[x, y]$ and we write $\alpha x \oplus (1 - \alpha)y$ for the unique point z in the geodesic segment joining from x to y such that $d(x, z) = (1 - \alpha)d(x, y)$ and $d(y, z) = \alpha d(x, y)$ for $\alpha \in [0, 1]$. The space X is said to be a *geodesic metric space* if every two points of X are joined by a geodesic, and X is said to be *uniquely geodesic* if there is exactly one geodesic joining x and y for each $x, y \in X$. A nonempty subset C of X is said to be *convex* if C includes every geodesic segment joining any two of its points.

A *geodesic triangle* $\triangle(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points x_1, x_2, x_3 in X and a geodesic segment between each pair of vertices. A *comparison triangle* for the geodesic triangle $\triangle(x_1, x_2, x_3)$ in (X, d) is a triangle $\overline{\triangle}(x_1, x_2, x_3) := \triangle(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in the Euclidean plane $\mathbb{E}^2$ such that $d_{\mathbb{E}^2}(\bar{x}_i, \bar{x}_j) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$.

A geodesic metric space is said to be a *CAT(0) space* if all geodesic triangles satisfy the following comparison axiom: Let $\triangle$ be a geodesic triangle in X and let $\overline{\triangle}$ be a comparison triangle for $\triangle$. Then $\triangle$ is said to satisfy the *CAT(0) inequality* if for all $x, y \in \triangle$ and all comparison points $\bar{x}, \bar{y} \in \overline{\triangle}$,

$$d(x, y) \leq d_{\mathbb{E}^2}(\bar{x}, \bar{y}).$$

Following [8], a metric space X is said to be a *CAT(0) space* if it is geodesically connected and if every geodesic triangle in X is at least as thin as its comparison triangle in the Euclidean plane $\mathbb{E}^2$.

The following examples are CAT(0) spaces:

(1) Hadamard manifolds;
(2) Euclidean spaces $\mathbb{R}^n$;
(3) hyperbolic spaces $\mathbb{H}^n$;
(4) Hilbert spaces;
(5) products of CAT(0) spaces;
(6) when endowed with the induced metric, a convex subset of Euclidean space $\mathbb{R}^n$ is CAT(0);
(7) Hilbert ball with the hyperbolic metric;
(8) trees.

There are several equivalent conditions for a geodesic metric space (X, d) to be a CAT(0) space, one of them is the following inequality (see [15]), which is to be satisfied for any $x, y, z \in X$ and $\alpha \in [0, 1]$:

$$d(z, \alpha x \oplus (1 - \alpha)y)^2 \leq \alpha d(z, x)^2 + (1 - \alpha)d(z, y)^2 - \alpha(1 - \alpha)d(x, y)^2. \tag{2.1}$$

In particular, if x, y, z are points in X and $\alpha \in [0,1]$, then we have

$$d(z, \alpha x \oplus (1-\alpha)y) \leq \alpha d(z,x) + (1-\alpha)d(z,y).$$

We next collect some elementary facts about CAT(0) spaces.

Lemma 2.1. *Let (X,d) be a CAT(0) space. Then, the following statements hold:*

(i) *(X,d) is uniquely geodesic; (see [8]).*
(ii) *Let $x, y \in X$. For each $\alpha \in [0,1]$, there exists a unique point $z = \alpha x \oplus (1-\alpha)y$ such that $d(x,z) = (1-\alpha)d(x,y)$ and $d(y,z) = \alpha d(x,y)$; (see [23]).*
(iii) *Let $p, x, y \in X$ and $\alpha \in [0,1]$. Then, we have*

$$d(\alpha p \oplus (1-\alpha)x, \alpha p \oplus (1-\alpha)y) \leq (1-\alpha)d(x,y);$$

(see [15]).
(iv) *Let $g : [0,1] \to [x,y]$ be a function defined by $g(\alpha) = \alpha x \oplus (1-\alpha)y$. Then, g is bijective and continuous; (see [15]).*

Let C be a nonempty subset of a metric space (X,d) and let $T : C \to C$ be a mapping. A point $x \in X$ is said to be a *fixed point* of T if $x = Tx$. The set of all fixed points of T is denoted by $F(T)$. Recall that a mapping T is said to be *nonexpansive* if

$$d(Tx, Ty) \leq d(x,y)$$

for all $x, y \in C$. Fixed point theory for a CAT(0) space was first studied by Kirk [24] in 2003. Since then, fixed point theory for nonexpansive mappings in CAT(0) spaces has been investigated rapidly. Some interesting results concerning the solution of fixed point problems for nonexpansive mappings in the framework of CAT(0) spaces can also be found, for examples, in [12, 14, 25, 26, 35].

A function $f : C \to (-\infty, \infty]$ defined on a nonempty convex subset C of a CAT(0) space is *convex* if, for any geodesic $\gamma : [0,1] \to C$, the composite function $f \circ \gamma$ is convex. We say that a function f defined on C is *lower semi-continuous* at a point $x \in C$ if

$$f(x) \leq \liminf_{n\to\infty} f(x_n)$$

for each sequence $\{x_n\}$ such that $\lim_{n\to\infty} x_n = x$. A function f is said to be *lower semi-continuous* on C if it is lower semi-continuous at any point in C. For any $\lambda > 0$, define the *Moreau–Yosida resolvent* of f in complete CAT(0) spaces as follows:

$$J_\lambda(x) = \operatorname*{argmin}_{y\in C} \left[f(y) + \frac{1}{2\lambda} d(y,x)^2 \right]$$

for all $x \in C$. If f is a proper, convex, and lower semi-continuous function, then the set of fixed points of the resolvent associated with f coincides with the set of minimizers of f; see [3]. Also, the resolvent J_λ of f is nonexpansive for all $\lambda > 0$; see [20].

Lemma 2.2. *Let (X, d) be a complete CAT(0) space and let $f : X \to (-\infty, \infty]$ be a proper, convex and lower semi-continuous function. Then, the following statements hold.*

(i) *(see [1]) for each $x \in X$ and $\lambda > \mu > 0$,*

$$J_\lambda x = J_\mu \left(\frac{\lambda - \mu}{\lambda} J_\lambda x \oplus \frac{\mu}{\lambda} x \right);$$

(ii) *(see [20,29]) for each $x, y \in X$ and $\lambda > 0$,*

$$\frac{1}{2\lambda} d(J_\lambda x, y)^2 - \frac{1}{2\lambda} d(x, y)^2 + \frac{1}{2\lambda} d(J_\lambda x, x)^2 \leq f(y) - f(J_\lambda x).$$

Proposition 2.3. ([37]) *Let (X, d) be a complete CAT(0) space and let $f : X \to (-\infty, \infty]$ be a proper, convex and lower semi-continuous function. Let T be a nonexpansive mapping on X such that $F(T) \cap F(J_\lambda) \neq \emptyset$ for all λ. Then, for any $\lambda > 0$, $F(T \circ J_\lambda) = F(T) \cap F(J_\lambda)$.*

The following two lemmas are useful for our main results; see [33,35].

Lemma 2.4. *Let (X, d) be a complete CAT(0) space, let C be a nonempty closed convex subset of X, and $T : C \to C$ be a nonexpansive mapping. Let $u \in C$ be fixed. For each $t \in (0, 1)$, the mapping $V_t : C \to C$ defined by $V_t x = tu \oplus (1 - t)Tx$ for $x \in C$ has a unique fixed point $y_t \in C$, that is, $y_t = V_t y_t = tu \oplus (1 - t)Ty_t$.*

Lemma 2.5. *Let C and T be as the preceding lemma. Then $\{y_t\}$ remains bounded as $t \to 0$ if and only if $F(T) \neq \emptyset$. In this case, the following statements hold:*

(i) *$\{y_t\}$ converges to the unique fixed point z of T which is nearest to u;*
(ii) *$d(u, z)^2 \leq \mu_n d(u, x_n)^2$ for all Banach limit μ and all bounded sequences $\{x_n\}$ with $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.*

The following condition was first introduced by Aoyama et al. [2] in 2007. Let C be a nonempty subset of a complete CAT(0) space (X, d) and $\{T_n\}$ be a countably infinite family of mappings from C into itself. We say that $\{T_n\}$ satisfies the *AKTT condition* (The letters A, K, T, and T stand for Aoyama, Kimura, Takahashi, and Toyoda) if

$$\sum_{n=1}^{\infty} \sup_{z \in D} \{d(T_{n+1}z, T_n z)\} < \infty$$

for each bounded subset D of C. If C is a closed subset of X and $\{T_n\}$ satisfies the AKTT condition, then we can define a mapping $T : C \to C$ by $Tx = \lim_{n\to\infty} T_n x$ for all $x \in C$. In this case, we also say that $(\{T_n\}, T)$ satisfies the AKTT condition. By using the same argument as Lemma 3.2 in [2], we get the following result.

Lemma 2.6. *If $(\{T_n\}, T)$ satisfies the AKTT condition, then $\lim_{n\to\infty} \sup_{z \in D}\{d(Tz, T_n z)\} = 0$ for all bounded subsets D of C.*

Let μ be a continuous linear functional on l^∞, the Banach space of bounded real sequences, and $(c_1, c_2, ...) \in l^\infty$. We write $\mu_n(c_n)$ instead of $\mu((c_1, c_2, ...))$. We call μ a *Banach limit* if μ satisfies $\|\mu\| = \mu(1, 1, ...) = 1$ and $\mu_n(c_n) = \mu_n(c_{n+1})$ for each $(c_1, c_2, ...) \in l^\infty$. For a Banach limit μ, we know that $\liminf_{n\to\infty} c_n \leq \mu_n(c_n) \leq \limsup_{n\to\infty} c_n$ for all $(c_1, c_2, ...) \in l^\infty$. Thus, if $(c_1, c_2, ...) \in l^\infty$ with $\lim_{n\to\infty} c_n = c^*$, then $\mu_n(c_n) = c^*$; see [10,16,38] for more details.

Lemma 2.7. ([36]) *Let $(c_1, c_2, ...) \in l^\infty$ be such that $\mu_n(c_n) \leq 0$ for all Banach limit μ. If $\limsup_{n\to\infty}(c_{n+1} - c_n) \leq 0$, then $\limsup_{n\to\infty} c_n \leq 0$.*

Lemma 2.8. ([2]) *Let $\{w_n\}$ be a sequence of nonnegative real numbers, $\{\alpha_n\}$ be a sequence of real numbers in $[0, 1]$ with $\sum_{n=1}^\infty \alpha_n = \infty$, $\{\gamma_n\}$ be a sequence of nonnegative real numbers with $\sum_{n=1}^\infty \gamma_n < \infty$, and $\{\beta_n\}$ be a sequence of real numbers with $\limsup_{n\to\infty} \beta_n \leq 0$. Suppose that*

$$w_{n+1} \leq (1 - \alpha_n)w_n + \alpha_n\beta_n + \gamma_n, \quad \forall n \in \mathbb{N}.$$

Then $\lim_{n\to\infty} w_n = 0$.

3. Main Results

In this section, we study the strong convergence theorem of a modified proximal point algorithm for a countably infinite family of nonexpansive mappings.

Theorem 3.1. *Let C be a nonempty closed convex subset of a complete CAT(0) space (X, d) and $f : C \to (-\infty, \infty]$ be a proper, convex and lower semi-continuous function. Let $\{T_n\}$ be a countably infinite family of nonexpansive mappings of C into itself with $\Omega = \bigcap_{n=1}^\infty F(T_n) \cap \operatorname{argmin}_{y\in C} f(y) \neq \emptyset$. Suppose that $u, x_1 \in C$ are arbitrarily chosen and $\{x_n\}$ is a sequence of C generated by*

$$\begin{cases} y_n = \underset{y\in C}{\operatorname{argmin}} \left[f(y) + \dfrac{1}{2\lambda_n} d(y, x_n)^2 \right], \\ z_n = \alpha_n u \oplus (1 - \alpha_n)T_n y_n, \\ x_{n+1} = \beta_n z_n \oplus (1 - \beta_n)T_n z_n, \quad \forall n \in \mathbb{N}, \end{cases} \tag{3.1}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\lambda_n\}$ are sequences which satisfy the conditions:

(C1) $0 < \alpha_n < 1$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^\infty \alpha_n = \infty$, *and* $\sum_{n=1}^\infty |\alpha_{n+1} - \alpha_n| < \infty$*;*
(C2) $\beta_n \in (b, 1]$ *for some* $b \in (0, 1)$ *and* $\sum_{n=1}^\infty |\beta_{n+1} - \beta_n| < \infty$*;*
(C3) $\lambda_n \geq \lambda > 0$ *for some* $\lambda \in (0, \infty)$ *and* $\sum_{n=1}^\infty |\lambda_{n+1} - \lambda_n| < \infty$*.*

Suppose that $(\{T_n\}, T)$ satisfies the AKTT condition and $F(T) = \bigcap_{n=1}^\infty F(T_n)$. Then the sequence $\{x_n\}$ converges strongly to a point in Ω which is nearest to u.

Proof. Let $q \in \Omega$ and $y_n = J_{\lambda_n} x_n$ for all $n \in \mathbb{N}$. Then, by Lemma 2.2(i), we obtain

$$\begin{aligned} d(x_{n+1}, q) &\leq \beta_n d(z_n, q) + (1-\beta_n) d(T_n z_n, q) \\ &\leq \beta_n d(z_n, q) + (1-\beta_n) d(z_n, q) \\ &= d(z_n, q) \\ &\leq \alpha_n d(u, q) + (1-\alpha_n) d(T_n y_n, q) \\ &\leq \alpha_n d(u, q) + (1-\alpha_n) d(y_n, q) \\ &= \alpha_n d(u, q) + (1-\alpha_n) d(J_{\lambda_n} x_n, q) \\ &\leq \alpha_n d(u, q) + (1-\alpha_n) d(x_n, q) \\ &\leq \max\{d(u, q), d(x_n, q)\}. \end{aligned}$$

By mathematical induction, we have

$$d(x_{n+1}, q) \leq \max\{d(u, q), d(x_1, q)\}, \quad \forall n \in \mathbb{N}.$$

This implies that $\{x_n\}$ is bounded and so are $\{y_n\}$, $\{z_n\}$, $\{J_\lambda x_n\}$, $\{T_n x_n\}$, $\{T_n y_n\}$, $\{T_n z_n\}$, and $\{T x_n\}$. Without loss of generality, we assume that $\lambda_n > \lambda_{n-1}$. By Proposition 2.3 and condition $(C3)$, we have

$$\begin{aligned} d(y_n, y_{n-1}) &\leq d(y_n, J_{\lambda_n} x_{n-1}) + d(J_{\lambda_n} x_{n-1}, y_{n-1}) \\ &= d(J_{\lambda_n} x_n, J_{\lambda_n} x_{n-1}) + d(J_{\lambda_n} x_{n-1}, J_{\lambda_{n-1}} x_{n-1}) \\ &\leq d(x_n, x_{n-1}) \\ &+ d\Big(J_{\lambda_{n-1}} \left(\frac{\lambda_n - \lambda_{n-1}}{\lambda_n} J_{\lambda_n} x_{n-1} \oplus \frac{\lambda_{n-1}}{\lambda_n} x_{n-1}\right), J_{\lambda_{n-1}} x_{n-1}\Big) \\ &\leq d(x_n, x_{n-1}) + d\Big(\frac{\lambda_n - \lambda_{n-1}}{\lambda_n} J_{\lambda_n} x_{n-1} \oplus \frac{\lambda_{n-1}}{\lambda_n} x_{n-1}, x_{n-1}\Big) \\ &= d(x_n, x_{n-1}) + \frac{|\lambda_n - \lambda_{n-1}|}{\lambda_n} d(J_{\lambda_n} x_{n-1}, x_{n-1}) \\ &\leq d(x_n, x_{n-1}) + \frac{|\lambda_n - \lambda_{n-1}|}{\lambda} d(J_{\lambda_n} x_{n-1}, x_{n-1}). \end{aligned}$$

By the definition of $\{x_n\}$, $\{z_n\}$, and Lemma 2.1(ii)-(iii), we have

$$\begin{aligned} d(z_n, z_{n-1}) &= d(\alpha_n u \oplus (1-\alpha_n) T_n y_n, \alpha_{n-1} u \oplus (1-\alpha_{n-1}) T_{n-1} y_{n-1}) \\ &\leq d(\alpha_n u \oplus (1-\alpha_n) T_n y_n, \alpha_n u \oplus (1-\alpha_n) T_n y_{n-1}) \\ &\quad + d(\alpha_n u \oplus (1-\alpha_n) T_n y_{n-1}, \alpha_n u \oplus (1-\alpha_n) T_{n-1} y_{n-1}) \\ &\quad + d(\alpha_n u \oplus (1-\alpha_n) T_{n-1} y_{n-1}, \alpha_{n-1} u \oplus (1-\alpha_{n-1}) T_{n-1} y_{n-1}) \\ &\leq (1-\alpha_n) d(T_n y_n, T_n y_{n-1}) + (1-\alpha_n) d(T_n y_{n-1}, T_{n-1} y_{n-1}) \\ &\quad + |\alpha_n - \alpha_{n-1}| d(u, T_{n-1} y_{n-1}) \\ &\leq (1-\alpha_n) d(y_n, y_{n-1}) + d(T_n y_{n-1}, T_{n-1} y_{n-1}) \\ &\quad + |\alpha_n - \alpha_{n-1}| d(u, T_{n-1} y_{n-1}) \\ &\leq (1-\alpha_n)\Big[d(x_n, x_{n-1}) + \frac{|\lambda_n - \lambda_{n-1}|}{\lambda} d(J_{\lambda_n} x_{n-1}, x_{n-1})\Big] \end{aligned}$$

$$
\begin{aligned}
&\quad + d(T_n y_{n-1}, T_{n-1} y_{n-1}) + |\alpha_n - \alpha_{n-1}| d(u, T_{n-1} y_{n-1}) \\
&\le (1-\alpha_n) d(x_n, x_{n-1}) + (1-\alpha_n) \frac{|\lambda_n - \lambda_{n-1}|}{\lambda} d(J_{\lambda_n} x_{n-1}, x_{n-1}) \\
&\quad + d(T_n y_{n-1}, T_{n-1} y_{n-1}) + |\alpha_n - \alpha_{n-1}| d(u, T_{n-1} y_{n-1}),
\end{aligned}
$$

and thus

$$
\begin{aligned}
d(x_{n+1}, x_n) &= d(\beta_n z_n \oplus (1-\beta_n) T_n z_n, \beta_{n-1} z_{n-1} \oplus (1-\beta_{n-1}) T_{n-1} z_{n-1}) \\
&\le d(\beta_n z_n \oplus (1-\beta_n) T_n z_n, \beta_n z_n \oplus (1-\beta_n) T_{n-1} z_{n-1}) \\
&\quad + d(\beta_n z_n \oplus (1-\beta_n) T_{n-1} z_{n-1}, \beta_n z_{n-1} \oplus (1-\beta_n) T_{n-1} z_{n-1}) \\
&\quad + d(\beta_n z_{n-1} \oplus (1-\beta_n) T_{n-1} z_{n-1}, \beta_{n-1} z_{n-1} \oplus (1-\beta_{n-1}) T_{n-1} z_{n-1}) \\
&\le \beta_n d(z_n, z_{n-1}) + (1-\beta_n) d(T_n z_n, T_{n-1} z_{n-1}) \\
&\quad + |\beta_n - \beta_{n-1}| d(z_{n-1}, T_{n-1} z_{n-1}) \\
&\le \beta_n d(z_n, z_{n-1}) + (1-\beta_n) \big(d(T_n z_n, T_n z_{n-1}) + d(T_n z_{n-1}, T_{n-1} z_{n-1}) \big) \\
&\quad + |\beta_n - \beta_{n-1}| d(z_{n-1}, T_{n-1} z_{n-1}) \\
&\le \beta_n d(z_n, z_{n-1}) + (1-\beta_n) \big(d(z_n, z_{n-1}) + d(T_n z_{n-1}, T_{n-1} z_{n-1}) \big) \\
&\quad + \ |\beta_n - \beta_{n-1}| d(z_{n-1}, T_{n-1} z_{n-1}) \\
&\le d(z_n, z_{n-1}) + d(T_n z_{n-1}, T_{n-1} z_{n-1}) + |\beta_n - \beta_{n-1}| d(z_{n-1}, T_{n-1} z_{n-1}) \\
&\le (1-\alpha_n) d(x_n, x_{n-1}) + (1-\alpha_n) \frac{|\lambda_n - \lambda_{n-1}|}{\lambda} d(J_{\lambda_n} x_{n-1}, x_{n-1}) \\
&\quad + d(T_n y_{n-1}, T_{n-1} y_{n-1}) + |\alpha_n - \alpha_{n-1}| d(u, T_{n-1} y_{n-1}) \\
&\quad + d(T_n z_{n-1}, T_{n-1} z_{n-1}) + |\beta_n - \beta_{n-1}| d(z_{n-1}, T_{n-1} z_{n-1}) \\
&\le (1-\alpha_n) d(x_n, x_{n-1}) + d(T_n y_{n-1}, T_{n-1} y_{n-1}) + d(T_n z_{n-1}, T_{n-1} z_{n-1}) \\
&\quad + \left((1-\alpha_n) \frac{|\lambda_n - \lambda_{n-1}|}{\lambda} + |\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}| \right) M,
\end{aligned}
$$

where $M = \max\{\sup_n d(J_{\lambda_n} x_{n-1}, x_{n-1}), \sup_n d(u, T_{n-1} y_{n-1}), \sup_n d(z_{n-1}, T_{n-1} z_{n-1})\}$.

Putting $\delta_n = ((1-\alpha_n)\frac{|\lambda_n - \lambda_{n-1}|}{\lambda} + |\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}|)M + d(T_n y_{n-1}, T_{n-1} y_{n-1}) + d(T_n z_{n-1}, T_{n-1} z_{n-1})$. We have that $\sum_{n=2}^{\infty} \delta_n \le M \sum_{n=2}^{\infty} ((1-\alpha_n)\frac{|\lambda_n - \lambda_{n-1}|}{\lambda} + |\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}|) + \sum_{n=2}^{\infty} \sup\{d(T_n z, T_{n-1} z) : z \in \{x_k\}\} + \sum_{n=2}^{\infty} \sup\{d(T_n z, T_{n-1} z) : z \in \{y_k\}\}$. Hence, it follows from conditions $(C1)$-$(C3)$, AKTT condition, and Lemma 2.6 that

$$
\lim_{n \to \infty} d(x_{n+1}, x_n) = 0. \tag{3.2}
$$

By Lemma 2.2(ii), we see that

$$
\frac{1}{2\lambda_n} d(y_n, q)^2 - \frac{1}{2\lambda_n} d(x_n, q)^2 + \frac{1}{2\lambda_n} d(x_n, y_n)^2 \le f(q) - f(y_n).
$$

Then, by $f(q) \le f(y_n)$ for all $n \in \mathbb{N}$, it follows that

$$
d(y_n, q)^2 \le d(x_n, q)^2 - d(x_n, y_n)^2. \tag{3.3}
$$

From the definition of $\{x_n\}$ and (3.3), we have

$$\begin{aligned} d(x_{n+1},q)^2 &= d(\beta_n z_n \oplus (1-\beta_n)T_n z_n, q)^2 \\ &\le \beta_n d(z_n,q)^2 + (1-\beta_n)d(T_n z_n,q)^2 - \beta_n(1-\beta_n)d(z_n,T_n z_n)^2 \\ &\le d(z_n,q)^2 \\ &= d(\alpha_n u \oplus (1-\alpha_n)T_n y_n, q)^2 \\ &\le \alpha_n d(u,q)^2 + (1-\alpha_n)d(T_n y_n,q)^2 - \alpha_n(1-\alpha_n)d(u,T_n y_n)^2 \\ &\le (1-\alpha_n)d(y_n,q)^2 + \alpha_n(d(u,q)^2 - (1-\alpha_n)d(u,T_n y_n)^2) \\ &\le (1-\alpha_n)(d(x_n,q)^2 - d(x_n,y_n)^2) + \alpha_n(d(u,q)^2 \\ &\quad - (1-\alpha_n)d(u,T_n y_n)^2); \end{aligned}$$

thus

$$\begin{aligned} (1-\alpha_n)d(x_n,y_n)^2 &\le d(x_n,q)^2 - d(x_{n+1},q)^2 \\ &\quad + \alpha_n\left[d(u,q)^2 - d(x_n,q)^2 - (1-\alpha_n)d(u,T_n y_n)^2\right] \\ &\le |d(x_n,q) - d(x_{n+1},q)|\,(d(x_n,q) + d(x_{n+1},q)) \\ &\quad + \alpha_n\left[d(u,q)^2 - d(x_n,q)^2 - (1-\alpha_n)d(u,T_n y_n)^2\right] \\ &\le d(x_{n+1},x_n)\,(d(x_n,q) + d(x_{n+1},q)) + \alpha_n d(u,q)^2. \end{aligned}$$

This implies that

$$d(x_n,y_n)^2 \le \frac{1}{1-\alpha_n}\left\{d(x_{n+1},x_n)(d(x_n,q)+d(x_{n+1},q)) + \alpha_n d(u,q)^2\right\}.$$

By condition $(C1)$ and (3.2), we conclude that

$$\lim_{n\to\infty} d(x_n,y_n) = 0. \tag{3.4}$$

Since $\lambda_n \ge \lambda > 0$, by Lemma 2.2(i), we have

$$\begin{aligned} d(J_\lambda x_n, y_n) &= d\left(J_\lambda x_n, J_\lambda\left(\frac{\lambda_n-\lambda}{\lambda_n}J_{\lambda_n}x_n \oplus \frac{\lambda}{\lambda_n}x_n\right)\right) \\ &\le d\left(x_n, \left(\frac{\lambda_n-\lambda}{\lambda_n}\right)J_{\lambda_n}x_n \oplus \frac{\lambda}{\lambda_n}x_n\right) \\ &= \left(1-\frac{\lambda}{\lambda_n}\right)d(x_n,y_n). \end{aligned}$$

Thus, by (3.4), we get

$$\lim_{n\to\infty} d(J_\lambda x_n, y_n) = 0. \tag{3.5}$$

It follows by condition $(C1)$ that

$$d(z_n,T_n y_n) = d(\alpha_n u \oplus (1-\alpha_n)T_n y_n, T_n y_n) = \alpha_n d(u,T_n y_n) \to 0. \tag{3.6}$$

Now, we have that

$$\begin{aligned} d(x_{n+1}, z_n) &= d(\beta_n z_n \oplus (1-\beta_n)T_n z_n, z_n) \\ &= (1-\beta_n) d(z_n, T_n z_n) \\ &\leq (1-b)(d(z_n, T_n y_n) + d(T_n y_n, T_n x_n) \\ &\quad + d(T_n x_n, T_n x_{n+1}) + d(T_n x_{n+1}, T_n z_n)) \\ &\leq (1-b)(d(z_n, T_n y_n) + d(y_n, x_n) + d(x_n, x_{n+1}) + d(x_{n+1}, z_n)). \end{aligned}$$

So, we have

$$d(x_{n+1}, z_n) \leq \frac{1-b}{b}(d(z_n, T_n y_n) + d(y_n, x_n) + d(x_n, x_{n+1})).$$

This implies by (3.2), (3.4) and (3.6) that

$$\lim_{n\to\infty} d(x_{n+1}, z_n) = 0. \tag{3.7}$$

Consequently, by (3.2), (3.4), (3.6) and (3.7), we have

$$\begin{aligned} d(T_n x_n, x_n) &\leq d(T_n x_n, T_n y_n) + d(T_n y_n, z_n) + d(z_n, x_{n+1}) + d(x_{n+1}, x_n) \\ &\leq d(x_n, y_n) + d(T_n y_n, z_n) + d(z_n, x_{n+1}) + d(x_{n+1}, x_n) \to 0. \end{aligned} \tag{3.8}$$

We see that

$$\begin{aligned} d(T_n \circ J_\lambda x_n, x_n) &\leq d(T_n \circ J_\lambda x_n, T_n y_n) + d(T_n y_n, T_n x_n) + d(T_n x_n, x_n) \\ &\leq d(J_\lambda x_n, y_n) + d(y_n, x_n) + d(T_n x_n, x_n). \end{aligned}$$

Thus, by (3.4), (3.5), and (3.8), we have

$$\lim_{n\to\infty} d(T_n \circ J_\lambda x_n, x_n) = 0. \tag{3.9}$$

By Lemma 2.6 and (3.9), we get that

$$\begin{aligned} d(T \circ J_\lambda x_n, x_n) &\leq d(T \circ J_\lambda x_n, T_n \circ J_\lambda x_n) + d(T_n \circ J_\lambda x_n, x_n) \\ &\leq \sup\{d(Tq, T_n q) : q \in \{J_\lambda x_k\}\} + d(T_n \circ J_\lambda x_n, x_n) \to 0. \end{aligned} \tag{3.10}$$

For each $t \in (0,1)$, let z_t be a unique point of C such that $z_t = tu \oplus (1-t)V z_t$ with $V = T \circ J_\lambda$. It follows from Lemma 2.5, Proposition 2.3 and (3.10) that $\{z_t\}$ converges to a point $z \in F(V) = F(T \circ J_\lambda) = F(T) \cap F(J_\lambda) = \bigcap_{n=1}^{\infty} F(T_n) \cap F(J_\lambda) = \Omega$ which is nearest to u, and

$$d(u,z)^2 \leq \mu_n d(u, x_n)^2 \text{ for all Banach limits } \mu,$$

that is $\mu_n \left(d(u,z)^2 - d(u,x_n)^2\right) \leq 0$. Moreover, by $\lim_{n\to\infty} d(x_{n+1}, x_n) = 0$, we have

$$\limsup_{n\to\infty} \left(\left(d(u,z)^2 - d(u,x_{n+1})^2\right) - \left(d(u,z)^2 - d(u,x_n)^2\right)\right) = 0. \tag{3.11}$$

By (3.4) and (3.8), we obtain that

$$\begin{aligned} d(x_n, T_n y_n) &\leq d(x_n, T_n x_n) + d(T_n x_n, T_n y_n) \\ &\leq d(x_n, T_n x_n) + d(x_n, y_n) \to 0. \end{aligned}$$

This implies by (3.11) and Lemma 2.7 that

$$\limsup_{n\to\infty}\left(d(u,z)^2-(1-\alpha_n)d(u,T_ny_n)^2\right)=\limsup_{n\to\infty}\left(d(u,z)^2-d(u,x_n)^2\right)\le 0. \tag{3.12}$$

Finally, we show that $\lim_{n\to\infty} d(x_n,z)=0$. By the definition of $\{x_n\}$, we have

$$\begin{aligned}
d(x_{n+1},z)^2 &= d(\beta_n z_n\oplus(1-\beta_n)T_nz_n,z)^2\\
&\le (\beta_n d(z_n,z)+(1-\beta_n)d(T_nz_n,z))^2\\
&\le d(z_n,z)^2\\
&= d(\alpha_n u\oplus(1-\alpha_n)T_ny_n,z)^2\\
&\le \alpha_n d(u,z)^2+(1-\alpha_n)d(T_ny_n,z)^2-\alpha_n(1-\alpha_n)d(u,T_ny_n)^2\\
&\le \alpha_n d(u,z)^2+(1-\alpha_n)d(y_n,z)^2-\alpha_n(1-\alpha_n)d(u,T_ny_n)^2\\
&= \alpha_n d(u,z)^2+(1-\alpha_n)d(J_{\lambda_n}x_n,z)^2-\alpha_n(1-\alpha_n)d(u,T_ny_n)^2\\
&\le (1-\alpha_n)d(x_n,z)^2+\alpha_n\left(d(u,z)^2-(1-\alpha_n)d(u,T_ny_n)^2\right).
\end{aligned}$$

This implies by $\sum_{n=1}^{\infty}\alpha_n=\infty$, inequality (3.12) and Lemma 2.8 that $\lim_{n\to\infty} d(x_n,z)^2=0$. Hence $\{x_n\}$ converges strongly to $z\in\Omega$ which is nearest to u. □

Since every real Hilbert space is a complete CAT(0) space, the following result can be obtained from Theorem 3.1 immediately.

Theorem 3.2. *Let C be a nonempty closed convex subset of a Hilbert space H and $f: C\to(-\infty,\infty]$ be a proper convex and lower semi-continuous function. Let $\{T_n\}$ be a countably infinite family of nonexpansive mappings of C into itself with $\Omega=\bigcap_{n=1}^{\infty}F(T_n)\cap \operatorname{argmin}_{y\in C}f(y)\ne\emptyset$. Suppose that $u,x_1\in C$ are arbitrarily chosen and $\{x_n\}$ is a sequence of C generated by*

$$\begin{cases} y_n=\underset{y\in C}{\operatorname{argmin}}\left[f(y)+\dfrac{1}{2\lambda_n}\|y-x_n\|^2\right],\\ z_n=\alpha_n u+(1-\alpha_n)T_ny_n,\\ x_{n+1}=\beta_n z_n+(1-\beta_n)T_nz_n,\quad \forall n\in\mathbb{N},\end{cases} \tag{3.13}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\lambda_n\}$ are sequences which satisfy the conditions:

(C1) $0<\alpha_n<1$, $\lim_{n\to\infty}\alpha_n=0$, $\sum_{n=1}^{\infty}\alpha_n=\infty$, *and* $\sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n|<\infty$*;*

(C2) $\beta_n\in(b,1]$ *for some* $b\in(0,1)$ *and* $\sum_{n=1}^{\infty}|\beta_{n+1}-\beta_n|<\infty$*;*

(C3) $\lambda_n\ge\lambda>0$ *for some* $\lambda\in(0,\infty)$ *and* $\sum_{n=1}^{\infty}|\lambda_{n+1}-\lambda_n|<\infty$.

Suppose that $(\{T_n\},T)$ satisfies the AKTT condition and $F(T)=\bigcap_{n=1}^{\infty}F(T_n)$. Then the sequence $\{x_n\}$ converges strongly to a point in Ω which is nearest to u.

Remark 3.3. (i) Theorem 3.1 generalizes the results of Cholamjiak [11], Suparatulatorn et al. [37] and Khan and Abbas [22] to a countably infinite family of nonexpansive mappings involving the convex and lower semi-continuous function in complete CAT(0) spaces.

(ii) Theorem 3.1 extends the main result in Bačák [4], and the corresponding results in Ariza-Ruiz et al. [3] and Cholamjiak et al. [12]. In fact, we present a new modified proximal point algorithm for solving the constrained convex minimization problem as well as the fixed point problem of a countably infinite family of nonexpansive mappings in complete CAT(0) spaces.

(iii) Theorem 3.2 is an improvement and generalization of the main result in Rockafellar [34] and Güler [17].

4. Applications and Numerical Examples

In this section, we discuss some concrete examples as well as the numerical results for supporting Theorems 3.1 and 3.2. All codes were written in Scilab.

Example 4.1. Let $X = \mathbb{R}^4$ with the Euclidean norm and $C = \{\mathbf{x} = (w_1, w_2, w_3, w_4)^t \in \mathbb{R}^4 : 0 \leq w_1, w_2, w_3, w_4 \leq 50\}$. For each $\mathbf{x} = (w_1, w_2, w_3, w_4)^t \in C$, we define mappings T_n on C as follows:

$$T_n\mathbf{x} = \left(\frac{w_1 + 2n - 1}{2n}, \frac{w_2 + 6n - 2}{3n}, \frac{w_3}{5n}, \frac{w_4 + 56n - 7}{8n}\right)^t, \quad \forall n \in \mathbb{N}.$$

For each $\mathbf{x} \in C$, we define $f : C \to (-\infty, \infty]$ by

$$f(\mathbf{x}) = \frac{1}{2}\|A\mathbf{x} - b\|^2,$$

where $A = \begin{pmatrix} 5 & -4 & 1 & 2 \\ 7 & 2 & -4 & 1 \\ 1 & 4 & 9 & 4 \\ 4 & 6 & 5 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} 11 \\ 18 \\ 37 \\ 23 \end{pmatrix}$.

We can check that T_n is nonexpansive for each $n \in \mathbb{N}$ and f is proper convex and lower semi-continuous. From [13] we know that

$$\begin{aligned} J_1\mathbf{x} &= \operatorname*{argmin}_{\mathbf{y} \in C}\left[f(\mathbf{y}) + \frac{1}{2}\|\mathbf{y} - \mathbf{x}\|^2\right] \\ &= \operatorname{prox}_f \mathbf{x} \\ &= (I + A^tA)^{-1}(\mathbf{x} + A^tb). \end{aligned}$$

Then the algorithm (3.1) becomes:

$$\begin{cases} \mathbf{y}_n = J_1\mathbf{x}_n = (I + A^tA)^{-1}(\mathbf{x}_n + A^tb), \\ \mathbf{z}_n = \alpha_n\mathbf{u} + (1 - \alpha_n)T_n\mathbf{y}_n, \\ \mathbf{x}_{n+1} = \beta_n\mathbf{z}_n + (1 - \beta_n)T_n\mathbf{z}_n, \quad \forall n \in \mathbb{N}. \end{cases} \tag{4.1}$$

We choose $\alpha_n = \frac{1}{50n}, \beta_n = \frac{5n}{150n+1}, \mathbf{u} = (4, 4, 4, 6)^t$. It can be observed that all the assumptions of Theorem 3.1 are satisfied. So, we rewrite the algorithm (4.1) as follows:

$$\mathbf{y}_n = \begin{pmatrix} 92 & 22 & 6 & 25 \\ 22 & 73 & 54 & 16 \\ 6 & 54 & 124 & 39 \\ 25 & 16 & 39 & 23 \end{pmatrix}^{-1} \begin{pmatrix} w_1^{(n)} + 310 \\ w_2^{(n)} + 278 \\ w_3^{(n)} + 387 \\ w_4^{(n)} + 211 \end{pmatrix},$$

$$\mathbf{z}_n = \begin{pmatrix} \dfrac{(50n-1)y_1^{(n)} + 100n^2 - 44n + 1}{100n^2} \\ \dfrac{(50n-1)y_2^{(n)} + 300n^2 - 94n + 2}{150n^2} \\ \dfrac{(50n-1)y_3^{(n)} + 20n}{250n^2} \\ \dfrac{(50n-1)y_4^{(n)} + 2800n^2 - 358n + 7}{400n^2} \end{pmatrix},$$

$$\mathbf{x}_{n+1} = \begin{pmatrix} \dfrac{(10n^2+145n+1)z_1^{(n)} + 290n^2 - 143n - 1}{300n^2 + 2n} \\ \dfrac{(15n^2+145n+1)z_2^{(n)} + 870n^2 - 284n - 2}{450n^2 + 3n} \\ \dfrac{(25n^2+145n+1)z_3^{(n)}}{750n^2 + 5n} \\ \dfrac{(40n^2+145n+1)z_4^{(n)} + 8120n^2 - 959n - 7}{1200n^2 + 8n} \end{pmatrix}, \quad \forall n \in \mathbb{N}. \tag{4.2}$$

Using the algorithm (4.2) with the initial point $\mathbf{x}_1 = (7, 6, 9, 5)^t$, we have numerical results in Table 1.

Remark 4.2. Table 1 shows that the sequence $\{x_n\}$ converges strongly to a unique point $(1, 2, 0, 7)^t$ which is a solution of the set of common fixed point of a countably infinite family of nonexpansive mappings. Such a solution $(1, 2, 0, 7)^t$ is also a solution of the following linear system:

$$\begin{aligned} 5w_1 - 4w_2 + w_3 + 2w_4 &= 11 \\ 7w_1 + 2w_2 - 4w_3 + w_4 &= 18 \\ w_1 + 4w_2 + 9w_3 + 4w_4 &= 37 \\ 4w_1 + 6w_2 + 5w_3 + w_4 &= 23. \end{aligned}$$

4.1. Linear Inverse Problems

In this subsection, we apply Theorem 3.2 to solve the constrained linear system:

$$Ax = b \tag{4.3}$$

Table 1. Numerical results of Example 4.1 for the algorithm (4.2)

n	$\boldsymbol{x}_n = (w_1^{(n)}, w_2^{(n)}, w_3^{(n)}, w_4^{(n)})^t$	$\\|\boldsymbol{x}_n - \boldsymbol{x}_{n-1}\\|_2$	$f(\boldsymbol{x}_n)$
1	(7.0000000, 6.0000000, 9.0000000, 5.0000000)	–	8905.5000
2	(1.0893431, 2.0017121, 0.0362800, 6.9788830)	1.1627e+01	0.3976928
3	(0.9964970, 2.0062872, 0.0033685, 7.0012404)	1.0112e−01	0.0034635
4	(0.9981909, 2.0030739, 0.0016536, 7.0006915)	4.0543e−03	0.0008440
5	(0.9989199, 2.0018660, 0.0010352, 7.0004487)	1.5594e−03	0.0003192
⋮	⋮	⋮	⋮
25	(0.9999368, 2.0001279, 0.0000873, 7.0000455)	9.6257e−06	0.0000018
⋮	⋮	⋮	⋮
46	(0.9999722, 2.0000591, 0.0000423, 7.0000228)	2.1444e−06	0.0000004
47	(0.9999729, 2.0000576, 0.0000413, 7.0000223)	2.0386e−06	0.0000004
48	(0.9999737, 2.0000561, 0.0000403, 7.0000217)	1.9403e−06	0.0000004
49	(0.9999743, 2.0000548, 0.0000394, 7.0000213)	1.8488e−06	0.0000004
50	(0.9999750, 2.0000535, 0.0000385, 7.0000208)	1.7636e−06	0.0000003

where A is a bounded linear operator on a subset C of H and $b \in C$. For each $x \in C$, we define $f : C \to (-\infty, \infty]$ by

$$f(x) = \frac{1}{2}\|Ax - b\|^2,$$

and consider the constrained convex minimization problem:

$$\min_{x \in C} f(x) = \min_{x \in C} \frac{1}{2}\|Ax - b\|^2. \tag{4.4}$$

Then x^* is a solution of the constrained linear system (4.3) if and only if x^* is a solution of the constrained convex minimization problem (4.4) with the minimizer equal to 0.

Using the proximity operator [13], we obtain the following result:

Theorem 4.3. *Let C be a nonempty closed convex subset of a Hilbert space H, let $A : C \to C$ be a bounded linear operator and $b \in C$. Suppose that $u, x_1 \in C$ are arbitrarily chosen and $\{x_n\}$ is a sequence of C generated by*

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)(I + A^t A)^{-1}(x_n + A^t b), \quad \forall n \in \mathbb{N}, \tag{4.5}$$

where $\{\alpha_n\}$ is a sequence such that $0 < \alpha_n < 1$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, and $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$. If (4.3) is consistent, then the sequence $\{x_n\}$ converges strongly to a solution of a linear system.

Table 2. Numerical results of Example 4.4 for the algorithm (4.5)

n	$\boldsymbol{x}_n = (w_1^{(n)}, w_2^{(n)}, w_3^{(n)}, w_4^{(n)}, w_5^{(n)}, w_6^{(n)})^t$	$\|\boldsymbol{x}_n - \boldsymbol{x}_{n-1}\|_2$
1	(3.1000000, 4.2000000, 5.3000000, 6.4000000, 7.5000000, 8.6000000)	–
2	(−0.3693540, 0.4659444, 0.8146918, 0.5616160, 0.2518216, 1.6647111)	1.3447e+01
3	(0.2850216, 1.1290127, 0.3439065, −0.2080428, 0.2723027, −0.2304299)	2.2965e+00
4	(0.5939141, 1.4465892, 0.1538636, −0.5138597, 0.3736722, −1.0473090)	1.0017e+00
5	(0.7299891, 1.5865225, 0.0705308, −0.6475571, 0.4203410, −1.4059554)	4.4014e−01
⋮	⋮	⋮
93	(0.8367745, 1.6965082, 0.0030090, −0.7545688, 0.4557981, −1.6917780)	3.9539e−06
⋮	⋮	⋮
181	(0.8367770, 1.6965160, 0.0029419, −0.7546366, 0.4557607, −1.6919200)	1.0209e−06
182	(0.8367770, 1.6965161, 0.0029416, −0.7546370, 0.4557605, −1.6919209)	1.0096e−06
183	(0.8367770, 1.6965161, 0.0029412, −0.7546374, 0.4557603, −1.6919217)	9.9846e−07
184	(0.8367770, 1.6965162, 0.0029408, −0.7546378, 0.4557601, −1.6919225)	9.8751e−07
185	(0.8367770, 1.6965162, 0.0029404, −0.7546381, 0.4557598, −1.6919233)	9.7674e−07

Example 4.4. Solve the following linear system:

$$\begin{aligned} 2w_1 + 7w_2 + 3w_3 - w_4 + w_5 + 4w_6 &= 8 \\ 3w_1 - w_2 - 5w_3 + 4w_4 + 2w_5 + w_6 &= -3 \\ w_1 + w_2 + 2w_3 + 9w_4 - 4w_5 - 3w_6 &= -1 \\ 7w_1 - 2w_2 + 6w_3 - w_4 + w_5 + w_6 &= 2 \\ 9w_1 + 6w_2 + w_3 + w_4 - 7w_5 + 4w_6 &= 7 \\ w_1 + 4w_2 + w_3 + 5w_4 + w_5 - w_6 &= 6 \end{aligned} \tag{4.6}$$

subject to $-100 \le w_1, w_2, w_3, w_4, w_5, w_6 \le 100$.

$$\text{Put } A = \begin{pmatrix} 2 & 7 & 3 & -1 & 1 & 4 \\ 3 & -1 & -5 & 4 & 2 & 1 \\ 1 & 1 & 2 & 9 & -4 & -3 \\ 7 & -2 & 6 & -1 & 1 & 1 \\ 9 & 6 & 1 & 1 & -7 & 4 \\ 1 & 4 & 1 & 5 & 1 & -1 \end{pmatrix} \quad \mathbf{x} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \\ w_6 \end{pmatrix} \text{ and } b = \begin{pmatrix} 8 \\ -3 \\ -1 \\ 2 \\ 7 \\ 6 \end{pmatrix}.$$

We choose $\mathbf{u} = (2.5, 2.9, 5.1, 3.7, 4.5, 6.6)^t$ and $\alpha_n = \frac{1}{500n}$. Using the algorithm (4.5) in Theorem 4.3 with the initial point $\mathbf{x}_1 = (3.1, 4.2, 5.3, 6.4, 7.5, 8.6)^t$, we have numerical results in Table 2.

From Table 2, we observe that

$$\boldsymbol{x}_{185} = (0.8367770, 1.6965162, 0.0029404, -0.7546381, 0.4557598, -1.6919233)^t$$

is an approximation of the solution of a linear system (4.6) with accuracy at 6 significant digits.

4.2. Minimization Problem

In this subsection, we apply Theorem 3.2 to solve the constrained convex minimization problem. The following result can be obtained from Theorem 3.2 immediately.

Theorem 4.5. *Let C be a nonempty closed convex subset of a Hilbert space H and let $f : C \to (-\infty, \infty]$ be a proper convex and lower semi-continuous function which f attains a minimizer. Suppose that $u, x_1 \in C$ are arbitrarily chosen and $\{x_n\}$ is a sequence of C generated by*

$$\begin{cases} y_n = \underset{y \in C}{\operatorname{argmin}} \left[f(y) + \frac{1}{2\lambda_n} \|y - x_n\|^2 \right], \\ x_{n+1} = \alpha_n u + (1 - \alpha_n) y_n, \quad \forall n \in \mathbb{N}, \end{cases} \tag{4.7}$$

where $\{\alpha_n\}$ and $\{\lambda_n\}$ are sequences which satisfies the conditions:

(C1) $0 < \alpha_n < 1$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, *and* $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;

(C2) $\lambda_n \geq \lambda > 0$ *for some* $\lambda \in (0, \infty)$ *and* $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$.

Then the sequence $\{x_n\}$ converges strongly to a minimizer of f.

Example 4.6. Solve the following minimization problem:

$$\min_{\mathbf{x} \in \mathbb{R}^5} \|\mathbf{x}\|_1 + \frac{1}{2} \|\mathbf{x}\|_2^2 + (1, -3, -2, 3, 2)^t \mathbf{x} + 9, \tag{4.8}$$

where $\mathbf{x} = (w_1, w_2, w_3, w_4, w_5)^t$ such that $-100 \leq w_1, w_2, w_3, w_4, w_5 \leq 100$.

Let $f(\mathbf{x}) = \|\mathbf{x}\|_1 + \frac{1}{2}\|\mathbf{x}\|_2^2 + (1, -3, -2, 3, 2)^t \mathbf{x} + 9$. We can check that f is proper convex and lower semi-continuous. Using the proximity operator [13] and the soft thresholding operator [18], we get that

$$\begin{aligned} J_1 \mathbf{x} &= \underset{\mathbf{y} \in C}{\operatorname{argmin}} \left[f(\mathbf{y}) + \frac{1}{2} \|\mathbf{y} - \mathbf{x}\|^2 \right] \\ &= \operatorname{prox}_f \mathbf{x} \\ &= \operatorname{prox}_{\frac{\|\cdot\|_1}{2}} \left(\frac{\mathbf{x} - (1, -3, -2, 3, 2)^t}{2} \right) \\ &= \Bigg(\max \left\{ \frac{|w_1 - 1| - 1}{2}, 0 \right\} \operatorname{sgn}(w_1 - 1), \max \left\{ \frac{|w_2 + 3| - 1}{2}, 0 \right\} \operatorname{sgn}(w_2 + 3), \\ &\quad \max \left\{ \frac{|w_3 + 2| - 1}{2}, 0 \right\} \operatorname{sgn}(w_3 + 2), \max \left\{ \frac{|w_4 - 3| - 1}{2}, 0 \right\} \operatorname{sgn}(w_4 - 3), \\ &\quad \max \left\{ \frac{|w_5 - 2| - 1}{2}, 0 \right\} \operatorname{sgn}(w_5 - 2) \Bigg)^t, \end{aligned}$$

where $\operatorname{sgn}(\cdot)$ is the signum function of $\delta \in \mathbb{R}$ defined by

$$\operatorname{sgn}(\delta) = \begin{cases} 1, & \delta > 0 \\ 0 & \delta = 0 \\ -1 & \delta < 0. \end{cases}$$

Table 3. Numerical results of Example 4.6 for the algorithm (4.9)

n	$\boldsymbol{x}_n = (w_1^{(n)}, w_2^{(n)}, w_3^{(n)}, w_4^{(n)}, w_5^{(n)})^t$	$\\|\boldsymbol{x}_n - \boldsymbol{x}_{n-1}\\|_2$	$f(\boldsymbol{x}_n)$
1	(9.0000000, 4.0000000, −5.0000000, 2.0000000, 7.0000000)	–	150.50000
2	(3.4975025, 2.9990010, −0.9950050, 0.0049950, 2.0019980)	8.7338e+00	32.120285
3	(0.7501249, 2.4997501, 0.0019990, −0.9970015, 0.0019990)	3.7142e+00	7.4134737
4	(0.0003332, 2.2499167, 0.5011663, −1.4978341, −0.4985005)	1.1726e+00	4.4081494
5	(0.0002499, 2.1249688, 0.7508123, −1.7483129, −0.7488128)	4.5092e−01	4.1025765
⋮	⋮	⋮	⋮
51	(0.0000200, 2.0000000, 1.0000600, −1.9998600, −0.9999000)	3.7407e−06	4.0000400
⋮	⋮	⋮	⋮
96	(0.0000105, 2.0000000, 1.0000316, −1.9999263, −0.9999474)	1.0263e−06	4.0000211
97	(0.0000104, 2.0000000, 1.0000312, −1.9999271, −0.9999479)	1.0049e−06	4.0000208
98	(0.0000103, 2.0000000, 1.0000309, −1.9999278, −0.9999485)	9.8421e−07	4.0000206
99	(0.0000102, 2.0000000, 1.0000306, −1.9999286, −0.9999490)	9.6412e−07	4.0000204
100	(0.0000101, 2.0000000, 1.0000303, −1.9999293, −0.9999495)	9.4465e−07	4.0000202

We choose $\alpha_n = \frac{1}{1000n+1}$ and $\mathbf{u} = (1, 2, 4, 5, 4)^t$. So, we rewrite the algorithm (4.7) as follows:

$$\mathbf{x}_{n+1} = \begin{pmatrix} \frac{1}{1000n+1} + \frac{1000n}{1000n+1} \max\left\{\frac{|w_1^{(n)}-1|-1}{2}, 0\right\} \operatorname{sgn}(w_1^{(n)} - 1) \\ \frac{1}{1000n+1} + \frac{1000n}{1000n+1} \max\left\{\frac{|w_2^{(n)}+3|-1}{2}, 0\right\} \operatorname{sgn}(w_2^{(n)} + 3) \\ \frac{1}{1000n+1} + \frac{1000n}{1000n+1} \max\left\{\frac{|w_3^{(n)}+2|-1}{2}, 0\right\} \operatorname{sgn}(w_3^{(n)} + 2) \\ \frac{1}{1000n+1} + \frac{1000n}{1000n+1} \max\left\{\frac{|w_4^{(n)}-3|-1}{2}, 0\right\} \operatorname{sgn}(w_4^{(n)} - 3) \\ \frac{1}{1000n+1} + \frac{1000n}{1000n+1} \max\left\{\frac{|w_5^{(n)}-2|-1}{2}, 0\right\} \operatorname{sgn}(w_5^{(n)} - 2) \end{pmatrix}, \quad \forall n \in \mathbb{N}. \tag{4.9}$$

Using the algorithm (4.9) with the initial point $\mathbf{x}_1 = (9, 4, -5, 2, 7)^t$, we have numerical results in Table 3.

From Table 3, we see that the sequence $\{x_n\}$ converges strongly to a unique point $(0, 2, 1, -2, -1)^t$ which is a minimizer of a function f.

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