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On Stable Solutions to Weighted Quasilinear Problems of Gelfand Type

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Abstract. Let $p \geq 2$ and $w, f \in L^1_{loc}(\mathbb{R}^N)$ be nonnegative functions such that $w(x) \leq C_1|x|^a$ and $f(x) \geq C_2|x|^b$ for large $|x|$. We prove the Liouville type theorem for stable $W^{1,p}_{loc}$ solutions of weighted quasilinear problem

$$-\mathrm{div}(w(x)|\nabla u|^{p-2}\nabla u) = f(x)e^u \quad \text{in } \mathbb{R}^N.$$

The result holds true for $N < \dfrac{(p-a)(p+3)+4b}{p-1}$ and is sharp in the case that w and f are Hardy–Hénon potentials. We also prove the full classification of solutions which are stable outside a compact set to Gelfand equation $-\Delta_N u = e^u$ in $\mathbb{R}^N$.

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1. Introduction and Main Results

In this paper we always assume that $p \geq 2$ and $w, f \in L^1_{loc}(\mathbb{R}^N)$ are nonnegative functions. Let us consider the following weighted quasilinear equation

$$-\,\mathrm{div}(w(x)|\nabla u|^{p-2}\nabla u) = f(x)e^u \quad \text{in } \mathbb{R}^N. \tag{1.1}$$

If $w \equiv 1$, the left hand side of (1.1) becomes the well-known p-Laplace operator. The terms $w(x)$ and $f(x)$ are usually regarded as weights while e^u is the Gelfand or Liouville nonlinearity. Due to the degenerate nature of the term $|\nabla u|^{p-2}$ when $p > 2$, solutions to (1.1) must be understood in the weak sense. Moreover, solutions to elliptic equations with Hardy potentials may possess singularities. Therefore, it is natural to study weak solutions of (1.1) in a suitable weighted Sobolev space. For this purpose, let us define

$$\|\varphi\|_w = \left(\int_{\mathbb{R}^N} w(x)|\nabla\varphi|^p \,\mathrm{d}x\right)^{\frac{1}{p}}$$

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for $\varphi \in C_c^\infty(\mathbb{R}^N)$ and denote by $W_0^{1,p}(\mathbb{R}^N, w)$ the closure of $C_c^\infty(\mathbb{R}^N)$ with respect to the $\|\cdot\|_w$–norm. Remark that for $w \in L^1_{loc}(\mathbb{R}^N)$ we have $C_c^1(\mathbb{R}^N) \subset W_0^{1,p}(\mathbb{R}^N, w)$ and $u \in W_{loc}^{1,p}(\mathbb{R}^N, w)$ means that if for any $\varphi \in C_c^\infty(\mathbb{R}^N)$, there holds $u\varphi \in W_0^{1,p}(\mathbb{R}^N, w)$. Let us make also the meaning of weak solution and stable solution more precisely.

Definition 1.1. A function $u \in W_{loc}^{1,p}(\mathbb{R}^N, w)$ is said to be a *weak solution* of (1.1) if $f(x)e^u \in L^1_{loc}(\mathbb{R}^N)$ and

$$\int_{\mathbb{R}^N} w(x)|\nabla u|^{p-2}(\nabla u, \nabla\varphi)\,\mathrm{d}x = \int_{\mathbb{R}^N} f(x)e^u\varphi\,\mathrm{d}x \tag{1.2}$$

for all $\varphi \in C_c^1(\mathbb{R}^N)$.

Definition 1.2. A weak solution u of (1.1) is *stable* if

$$\begin{aligned}\int_{\mathbb{R}^N} w(x)\left[|\nabla u|^{p-2}|\nabla\varphi|^2 + (p-2)|\nabla u|^{p-4}(\nabla u, \nabla\varphi)^2\right]\,\mathrm{d}x\\ \geq \int_{\mathbb{R}^N} f(x)e^u\varphi^2\,\mathrm{d}x\end{aligned} \tag{1.3}$$

for all $\varphi \in C_c^1(\mathbb{R}^N)$. If there exists a compact set $K \subset \mathbb{R}^N$ such that (1.3) holds only for $\varphi \in C_c^1(\mathbb{R}^N \backslash K)$ then we say that u is *stable outside a compact set*.

We recall that the stability condition translates into the fact that the second variation at u of the energy functional

$$E(u) = \int_{\mathbb{R}^N}\left(\frac{w(x)|\nabla u|^p}{p} - f(x)e^u\right)\,\mathrm{d}x$$

is nonnegative. Therefore all the local minima of the functional are stable weak solutions of (1.1).

Proposition 1.3. *If u is a stable solution of* (1.1)*, then*

$$(p-1)\int_{\mathbb{R}^N} w(x)|\nabla u|^{p-2}|\nabla\varphi|^2\mathrm{d}x \geq \int_{\mathbb{R}^N} f(x)e^u\varphi^2\mathrm{d}x \tag{1.4}$$

for every $\varphi \in C_c^1(\mathbb{R}^N)$. If u is stable outside a compact set, then the same inequality holds for every $\varphi \in C_c^1(\mathbb{R}^N \backslash \overline{B_R})$ and for some $R > 0$. Here and in the following B_R denotes the ball of center $0 \in \mathbb{R}^N$ and radius R.

Since $f(x)e^u \geq 0$, remark that (1.2), (1.3) and (1.4) hold for any $\varphi \in W_0^{1,p}(\mathbb{R}^N, w)$ by density arguments.

In this paper we prove a Liouville type theorem for stable solutions of Eq. (1.1). We recall that Liouville type theorems concern about the nonexistence of nontrivial solution in the entire Euclidean space $\mathbb{R}^N$. The most well-known Liouville type theorem for nonlinear problems may be the result in the pioneering article [14], where Gidas and Spruck established the optimal nonexistence result for positive solutions to the equation $-\Delta u = |u|^{q-1}u$ in $\mathbb{R}^N$. They proved that this equation has no positive solution if and only if q is less than the critical exponent $\dfrac{N+2}{N-2}$, which is ∞ if $N = 2$.

In recent years, not only weak and positive solutions but also other types of solutions to Eq. (1.1) such as stable solutions have been studied intensively by several authors. Readers can find physical motivation and recent development on the topic of stable solutions in monograph [7] by Dupaigne and references therein.

We should refer to the work [11] by Farina for Gelfand equation

$$-\Delta u = e^u \quad \text{in } \mathbb{R}^N,$$

where he proved that the equation does not admit stable C^2 solutions for $N \le 9$. Later, this nonexistence result was extended to stable C^1 solutions of quasilinear equation $-\Delta_p u = e^u$ when $N < \dfrac{p(p+3)}{p-1}$ in [17].

The weighted semilinear elliptic equation of Gelfand type

$$-\text{div}(w(x)\nabla u) = f(x)e^u \quad \text{in } \mathbb{R}^N$$

was also studied recently by many authors. In [5] several Liouville type theorems for classical stable solutions of this equation were established under different assumptions on w and f. Paper [20] deals with more specific equation $-\Delta u = |x|^b e^u$ but for stable solutions of class H^1_{loc}, which covers solutions having singularities. Later, the result in [20] was extended to equation $-\text{div}(w(x)u) = f(x)e^u$ in [15] and equation $-\Delta_p u = f(x)e^u$ in [4]. Similar works on singular problems can be found in [12,13,16,19] and references therein.

Liouville type theorems were also established for elliptic equations with other type of nonlinearity, such as Lane-Emden and MEMS. We refer to paper [8,10] for stable C^2 solutions of semilinear equation $-\Delta u = f(u)$ and papers [2,6,18] for stable C^1 solutions of quasilinear equation $-\Delta_p u = f(u)$. In general, Liouville type theorems for stable solutions of nonlinear elliptic equations are usually guaranteed in low dimensional case.

The main purpose of this paper is to obtain a sharp Liouville type theorem for stable solutions of class $W^{1,p}_{loc}$ to Eq. (1.1). Our result therefore directly extends a result in [4], which deals with equation

$$-\Delta_p u = f(x)e^u \quad \text{in } \mathbb{R}^N.$$

Not only handling the weight $w(x)$ in divergence operator, our result also extends [4] in more general setting by three folds.

- Let us emphasize that in [4], the authors considered only $C^1(\mathbb{R}^N)$ solutions, which are locally bounded. This $C^1(\mathbb{R}^N)$ regularity assumption is natural when $w \equiv f \equiv 1$ and $N \le p$. However, if the weights w and f are Hardy potentials or if $N > p$, then solutions of Eq. (1.1) may have singularities and do not belong to class $C^1(\mathbb{R}^N)$ anymore (see Proposition 1.6 for an example). Therefore, the class of $W^{1,p}_{loc}$ solutions is more suitable setting for (1.1) and we will work with this type of solutions in our paper.
- The work [4] requires that $a = 0$, $N > p$ and $b > -q$. It should be notice that the assumption $N > p$ competes with $C^1(\mathbb{R}^N)$ regularity

assumption as we discuss before. All these assumptions are relaxed in our theorem.

- We also construct an example to show the sharpness of our result.

Our main result is the following theorem.

Theorem 1.4. *Assume that* $0 \le w(x) \le C_1|x|^a$ *and* $f(x) \ge C_2|x|^b$ *for a.e.* $x \in \mathbb{R}^N \backslash B_{R_0}$ *and some* $C_1, C_2, R_0 > 0$. *If* $N < \dfrac{(p-a)(p+3)+4b}{p-1}$, *then Eq.* (1.1) *admits no stable solution.*

Remark 1.5. If $a = 0$, we obtain a similar result in [4]. If $a = b = 0$, we have the one in [17]. If $p = 2$, we get the result in [5,15]. If $p = 2$ and $a = 0$, we obtain the result in [20]. Finally, if $p = 2$ and $a = b = 0$ we have the Liouville theorem in pioneering article [11]. Therefore, our conclusion in Theorem 1.4 unifies and extends results in [4,5,11,15,17,20] to stable solutions of class $W^{1,p}_{loc}$.

The assumption on dimension N in Theorem 1.4 is optimal. Indeed, let us consider the Hardy–Hénon problem

$$-\operatorname{div}(|x|^a|\nabla u|^{p-2}\nabla u) = |x|^b e^u \quad \text{in } \mathbb{R}^N. \tag{1.5}$$

We have the following.

Proposition 1.6. *If* $b > a - p$ *and* $N \ge \dfrac{(p-a)(p+3)+4b}{p-1}$, *then*

$$U(x) = \ln \frac{(p-a+b)^{p-1}(N+a-p)}{|x|^{p-a+b}}$$

is a stable solution of Eq. (1.5).

Remark 1.7. The assumption $b > a-p$ in Proposition 1.6 is necessary to ensure that $|x|^a$, $|x|^b$ and $|x|^b e^U$ belong to $L^1_{loc}(\mathbb{R}^N)$. Indeed, since $b > a-p$ and $N \ge \dfrac{(p-a)(p+3)+4b}{p-1}$, we can deduce that $N \ge \dfrac{(p-a)(p+3)+4(a-p)}{p-1}$, which means $a-p > -N$. Therefore, $|x|^a$, $|x|^b$ and $|x|^b e^U$ belong to $L^1_{loc}(\mathbb{R}^N)$ and function U is well-defined.

Regarding solutions which are stable outside a compact set, in the work [11] Farina has classified all such solutions of equation $-\Delta u = e^u$ in $\mathbb{R}^2$. In this paper, we extend Farina's classification result to general dimension $N \ge 2$.

Theorem 1.8. *Let* u *be a weak solution to equation*

$$-\Delta_N u = e^u \quad \text{in } \mathbb{R}^N. \tag{1.6}$$

Then, u *is stable outside a compact set if and only if* u *is of the form*

$$u(x) = \ln \frac{\lambda^N N^{2N-1}}{(N-1)^{N-1}\left(1+\lambda^{\frac{N}{N-1}}|x-x_0|^{\frac{N}{N-1}}\right)^N}, \quad x \in \mathbb{R}^N \tag{1.7}$$

for some $\lambda > 0$ *and* $x_0 \in \mathbb{R}^N$.

2. Liouville Type Theorem

This section is devoted to the proof of Theorem 1.4. For convenience, we always denote by C a generic constant whose concrete values may change from line to line or even in the same line. If this constant depends on an arbitrary small number ε, then we may denote it by C_ε. We also use Young inequality in the form $ab \le \varepsilon a^p + C_\varepsilon b^q$ for $p, q > 1$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$.

We begin with the following a priori estimate for stable solutions of (1.1).

Proposition 2.1. *Suppose that u is a stable solution of Eq. (1.1). Then for any $\alpha \in \left(0, \frac{4}{p-1}\right)$, there exists a constant $C = C(p, \alpha) > 0$ such that for any function $\eta \in C_c^1(\mathbb{R}^N)$ with $0 \le \eta \le 1$ and $\nabla\eta = 0$ in a neighborhood of $\{x \in \mathbb{R}^N : f(x) = 0\}$ we have*

$$\int_{\mathbb{R}^N} f(x) e^{(\alpha+1)u} \eta^{p(\alpha+1)} \, \mathrm{d}x \le C \int_{\mathbb{R}^N} w(x)^{\alpha+1} f(x)^{-\alpha} |\nabla\eta|^{p(\alpha+1)} \, \mathrm{d}x. \tag{2.1}$$

Proof. Our proof is inspired by the techniques used in [4,6,17,20], but we need to pay more attention with $W^{1,p}_{loc}$ solution. As u is not assumed to be bounded, $e^{\beta u}\psi$ is not, a priori, a suitable test function for any $\beta > 0$, even with $\psi \in C_c^\infty(\Omega)$. For each $k \in \mathbb{N}$ we define positive $C^1(\mathbb{R})$ functions

$$a_k(t) = \begin{cases} e^{\frac{\alpha t}{2}}, & t < k, \\ \left(\frac{\alpha}{2}(t-k) + 1\right) e^{\frac{\alpha k}{2}}, & t \ge k, \end{cases}$$

and

$$b_k(t) = \begin{cases} e^{\alpha t}, & t < k, \\ (\alpha(t-k) + 1)\, e^{\alpha k}, & t \ge k. \end{cases}$$

Simple calculations yield

$$a_k^2(t) \ge b_k(t), \quad a_k'(t)^2 = \frac{\alpha}{4} b_k'(t) \quad \text{and} \quad a_k(t)^p a_k'(t)^{2-p} + b_k(t)^p b_k'(t)^{1-p} \le C e^{\alpha t} \tag{2.2}$$

for all $t \in \mathbb{R}$, where C depends only on p and α. Moreover, since $u \in W^{1,p}_{loc}(\mathbb{R}^N, w)$, clearly $a_k(u), b_k(u) \in W^{1,p}_{loc}(\mathbb{R}^N, w)$ for any $k \in \mathbb{N}$. We split the proof into four steps.

Step 1 For any $\varepsilon \in (0, 1)$, any $k \in \mathbb{N}$ and any nonnegative function $\psi \in C_c^1(\mathbb{R}^N)$, there exists a constant $C_\varepsilon = C(p, \varepsilon) > 0$ such that

$$(1-\varepsilon) \int_{\mathbb{R}^N} w(x) |\nabla u|^p b_k'(u) \psi^p \, \mathrm{d}x \le C_\varepsilon \int_{\mathbb{R}^N} w(x) b_k(u)^p b_k'(u)^{1-p} |\nabla\psi|^p \, \mathrm{d}x + \int_{\mathbb{R}^N} f(x) e^u b_k(u) \psi^p \, \mathrm{d}x. \tag{2.3}$$

To prove this, using $\varphi = b_k(u)\psi^p$ as a test function. Since

$$\nabla\varphi = b_k'(u)\psi^p \nabla u + p b_k(u) \psi^{p-1} \nabla\psi,$$

using (1.2) we get

$$\int_{\mathbb{R}^N} w(x)|\nabla u|^p b_k'(u)\psi^p \,\mathrm{d}x + p\int_{\mathbb{R}^N} w(x)|\nabla u|^{p-2} b_k(u)\psi^{p-1}(\nabla u, \nabla\psi)\,\mathrm{d}x$$
$$= \int_{\mathbb{R}^N} f(x)e^u b_k(u)\psi^p \,\mathrm{d}x.$$

Therefore,

$$\begin{aligned}
&\int_{\mathbb{R}^N} w(x)|\nabla u|^p b_k'(u)\psi^p \,\mathrm{d}x \\
&\quad\le p\int_{\mathbb{R}^N} w(x)|\nabla u|^{p-1} b_k(u)\psi^{p-1}|\nabla\psi|\,\mathrm{d}x + \int_{\mathbb{R}^N} f(x)e^u b_k(u)\psi^p \,\mathrm{d}x \\
&\quad\le \int_{\mathbb{R}^N} \varepsilon\left(w(x)^{\frac{p-1}{p}}|\nabla u|^{p-1} b_k'(u)^{\frac{p-1}{p}}\psi^{p-1}\right)^{\frac{p}{p-1}} \\
&\qquad + C_\varepsilon\left(w(x)^{\frac{1}{p}} b_k(u) b_k'(u)^{\frac{1-p}{p}}|\nabla\psi|\right)^p \,\mathrm{d}x \\
&\qquad + \int_{\mathbb{R}^N} f(x)e^u b_k(u)\psi^p \,\mathrm{d}x \\
&\quad= \varepsilon\int_{\mathbb{R}^N} w(x)|\nabla u|^p b_k'(u)\psi^p \,\mathrm{d}x \\
&\qquad + C_\varepsilon\int_{\mathbb{R}^N} w(x) b_k(u)^p b_k'(u)^{1-p}|\nabla\psi|^p \,\mathrm{d}x \\
&\qquad + \int_{\mathbb{R}^N} f(x)e^u b_k(u)\psi^p \,\mathrm{d}x,
\end{aligned}$$

which implies (2.3).

Step 2 For any $\varepsilon \in (0,1)$, any $k \in \mathbb{N}$ and any nonnegative function $\psi \in C_c^1(\mathbb{R}^N)$, there exists a constant $C_\varepsilon = C(p,\varepsilon) > 0$ such that

$$\begin{aligned}
\int_{\mathbb{R}^N} f(x)e^u a_k(u)^2\psi^p \,\mathrm{d}x \le (p-1+\varepsilon)&\int_{\mathbb{R}^N} w(x)|\nabla u|^p a_k'(u)^2\psi^p \,\mathrm{d}x \\
&+ C_\varepsilon\int_{\mathbb{R}^N} w(x) a_k(u)^p a_k'(u)^{2-p}|\nabla\psi|^p \,\mathrm{d}x. \quad (2.4)
\end{aligned}$$

To prove this, we use the stability assumption with $\varphi = a_k(u)\psi^{\frac{p}{2}}$. Since

$$\nabla\varphi = a_k'(u)\psi^{\frac{p}{2}}\nabla u + \frac{p}{2}a_k(u)\psi^{\frac{p-2}{2}}\nabla\psi,$$

using (1.4) we get

$$\begin{aligned}
\int_{\mathbb{R}^N} f(x)e^u a_k(u)^2\psi^p \,\mathrm{d}x \le (p-1)&\int_{\mathbb{R}^N} w(x)|\nabla u|^p a_k'(u)^2\psi^p \,\mathrm{d}x \\
&+ (p-1)p\int_{\mathbb{R}^N} w(x)|\nabla u|^{p-1} a_k(u) a_k'(u)\psi^{p-1}|\nabla\psi|\,\mathrm{d}x \\
&+ \frac{(p-1)p^2}{4}\int_{\mathbb{R}^N} w(x)|\nabla u|^{p-2} a_k(u)^2\psi^{p-2}|\nabla\psi|^2 \,\mathrm{d}x.
\end{aligned} \quad (2.5)$$

Now we use Young inequality to estimate the last two terms

$$
\begin{aligned}
&(p-1)p\int_{\mathbb{R}^N} w(x)|\nabla u|^{p-1}a_k(u)a_k'(u)\psi^{p-1}|\nabla\psi|\,\mathrm{d}x\\
&\le \int_{\mathbb{R}^N}\frac{\varepsilon}{2}\left(w(x)^{\frac{p-1}{p}}|\nabla u|^{p-1}a_k'(u)^{\frac{2(p-1)}{p}}\psi^{p-1}\right)^{\frac{p}{p-1}}\\
&\quad + C_\varepsilon\left(w(x)^{\frac{1}{p}}a_k(u)a_k'(u)^{\frac{2-p}{p}}|\nabla\psi|\right)^p\,\mathrm{d}x\\
&=\frac{\varepsilon}{2}\int_{\mathbb{R}^N} w(x)|\nabla u|^p a_k'(u)^2\psi^p\,\mathrm{d}x\\
&\quad + C_\varepsilon\int_{\mathbb{R}^N} w(x)a_k(u)^p a_k'(u)^{2-p}|\nabla\psi|^p\,\mathrm{d}x
\end{aligned}
$$

and if $p>2$ we have

$$
\begin{aligned}
&\frac{(p-1)p^2}{4}\int_{\mathbb{R}^N} w(x)|\nabla u|^{p-2}a_k(u)^2\psi^{p-2}|\nabla\psi|^2\,\mathrm{d}x\\
&\le \int_{\mathbb{R}^N}\frac{\varepsilon}{2}\left(w(x)^{\frac{p-2}{p}}|\nabla u|^{p-2}a_k'(u)^{\frac{2(p-2)}{p}}\psi^{p-2}\right)^{\frac{p}{p-2}}\\
&\quad + C_\varepsilon\left(w(x)^{\frac{2}{p}}a_k(u)^2a_k'(u)^{\frac{2(2-p)}{p}}|\nabla\psi|^2\right)^{\frac{p}{2}}\,\mathrm{d}x\\
&=\frac{\varepsilon}{2}\int_{\mathbb{R}^N} w(x)|\nabla u|^p a_k'(u)^2\psi^p\,\mathrm{d}x\\
&\quad + C_\varepsilon\int_{\mathbb{R}^N} w(x)a_k(u)^p a_k'(u)^{2-p}|\nabla\psi|^p\,\mathrm{d}x.
\end{aligned}
$$

Plugging these two estimates into (2.5), we obtain (2.4).

Step 3 We claim that there exists a constant $C=C(p,\alpha)>0$ such that for any nonnegative function $\psi\in C_c^1(\mathbb{R}^N)$ we have

$$
\int_{\mathbb{R}^N} f(x)e^{(\alpha+1)u}\psi^p\,\mathrm{d}x \le C\int_{\mathbb{R}^N} w(x)e^{\alpha u}|\nabla\psi|^p\,\mathrm{d}x. \tag{2.6}
$$

To prove this, we set $\beta_\varepsilon = 1-\dfrac{(p-1+\varepsilon)\alpha}{4(1-\varepsilon)}$. Since $\lim_{\varepsilon\to 0^+}\beta_\varepsilon = 1-\dfrac{\alpha(p-1)}{4}>0$, we can find and fix some $\varepsilon\in(0,1)$ depending on p and α such that $\beta_\varepsilon>0$.

Collecting (2.3), (2.4) and with the help of (2.2) we obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^N} f(x)e^u a_k(u)^2\psi^p\,\mathrm{d}x\\
&\le \frac{(p-1+\varepsilon)\alpha}{4}\int_{\mathbb{R}^N} w(x)|\nabla u|^p b_k'(u)\psi^p\,\mathrm{d}x\\
&\quad + C_\varepsilon\int_{\mathbb{R}^N} w(x)a_k(u)^p a_k'(u)^{2-p}|\nabla\psi|^p\,\mathrm{d}x\\
&\le \frac{(p-1+\varepsilon)\alpha}{4(1-\varepsilon)}\int_{\mathbb{R}^N} f(x)e^u b_k(u)\psi^p\,\mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&+ C_\varepsilon \int_{\mathbb{R}^N} w(x) \left[a_k(u)^p a_k'(u)^{2-p} + b_k(u)^p b_k'(u)^{1-p}\right] |\nabla\psi|^p \,\mathrm{d}x \\
&\leq \frac{(p-1+\varepsilon)\alpha}{4(1-\varepsilon)} \int_{\mathbb{R}^N} f(x) e^u a_k(u)^2 \psi^p \,\mathrm{d}x \\
&+ C_\varepsilon \int_{\mathbb{R}^N} w(x) e^{\alpha u} |\nabla\psi|^p \,\mathrm{d}x.
\end{aligned}
$$

Therefore,

$$
\beta_\varepsilon \int_{\mathbb{R}^N} f(x) e^u a_k(u)^2 \psi^p \,\mathrm{d}x \leq C_\varepsilon \int_{\mathbb{R}^N} w(x) e^{\alpha u} |\nabla\psi|^p.
$$

Letting $k \to \infty$, by Fatou's lemma we obtain

$$
\int_{\mathbb{R}^N} f(x) e^{(\alpha+1)u} \psi^p \,\mathrm{d}x \leq C \int_{\mathbb{R}^N} w(x) e^{\alpha u} |\nabla\psi|^p, \tag{2.7}
$$

where C depends only on p and α.

Step 4 We are now in the position to prove a priori estimate (2.1). Applying (2.6) for $\psi = \eta^{\alpha+1}$ to obtain

$$
\begin{aligned}
\int_{\mathbb{R}^N} f(x) e^{(\alpha+1)u} \eta^{p(\alpha+1)} \,\mathrm{d}x &\leq C \int_{\mathbb{R}^N} w(x) e^{\alpha u} \eta^{p\alpha} |\nabla\eta|^p \,\mathrm{d}x \\
&\leq \int_{\mathbb{R}^N} \frac{1}{2} \left(f(x)^{\frac{\alpha}{\alpha+1}} e^{\alpha u} \eta^{p\alpha}\right)^{\frac{\alpha+1}{\alpha}} \\
&\quad + C \left(w(x) f(x)^{-\frac{\alpha}{\alpha+1}} |\nabla\eta|^p\right)^{\alpha+1} \,\mathrm{d}x \\
&= \frac{1}{2} \int_{\mathbb{R}^N} f(x) e^{(\alpha+1)u} \eta^{p(\alpha+1)} \,\mathrm{d}x \\
&\quad + C \int_{\mathbb{R}^N} w(x)^{\alpha+1} f(x)^{-\alpha} |\nabla\eta|^{p(\alpha+1)} \,\mathrm{d}x.
\end{aligned}
$$

Hence, (2.1) follows at once. □

Proof of Theorem 1.4. By contradiction, we suppose that (1.1) admits a stable solution u in dimension $N < \dfrac{(p-a)(p+3)+4b}{p-1}$. Applying Proposition 2.1 for a test function $\eta_R \in C_c^1(\mathbb{R}^N)$:

$$
\eta_R(x) = \begin{cases} 1 & \text{if } |x| < R, \\ 0 & \text{if } |x| > 2R \end{cases}
$$

which satisfies $0 \leq \eta_R \leq 1$ in $\mathbb{R}^N$ and $|\nabla\eta_R| \leq \dfrac{C}{R}$ in $B_{2R} \backslash B_R$.

Consequently, for all $R > R_0$ there exists a constant C independent of R such that

$$
\int_{B_R} f(x) e^{(\alpha+1)u} \,\mathrm{d}x \leq C R^{N + a(\alpha+1) - b\alpha - p(\alpha+1)}. \tag{2.8}
$$

Since

$$
\lim_{\alpha \to \frac{4}{p-1}} N + a(\alpha+1) - b\alpha - p(\alpha+1) = N - \frac{(p-a)(p+3)+4b}{p-1} < 0,
$$

we may find some $\alpha \in \left(0, \dfrac{4}{p-1}\right)$ such that $N + a(\alpha+1) - b\alpha - p(\alpha+1) < 0$. Letting $R \to \infty$ in (2.8) we get $\int_{\mathbb{R}^N} f(x)e^{(\alpha+1)u}\, \mathrm{d}x = 0$, a contradiction. This concludes the proof. $\square$

3. A Counter Example and a Classification Result

In this section we will prove Proposition 1.6 and Theorem 1.8.

Proof of Proposition 1.6. Direct calculation yields that U is a weak solution of (1.5). In order to show that U is stable, we need the following inequality (see [3]).

Lemma 3.1 (Caffarelli–Kohn–Nirenberg inequality). *Let $r < \dfrac{N-2}{2}$, then for all $\varphi \in C_c^1(\mathbb{R}^N)$ we have*

$$\int_{\mathbb{R}^N} \frac{|\nabla\varphi|^2}{|x|^{2r}}\, \mathrm{d}x \geq \left(\frac{N-2-2r}{2}\right)^2 \int_{\mathbb{R}^N} \frac{\varphi^2}{|x|^{2r+2}}\, \mathrm{d}x. \tag{3.1}$$

Applying (3.1) with $r = \dfrac{p-a-2}{2}$ we get

$$\int_{\mathbb{R}^N} \frac{|\nabla\varphi|^2}{|x|^{p-a-2}}\, \mathrm{d}x \geq \left(\frac{N+a-p}{2}\right)^2 \int_{\mathbb{R}^N} \frac{\varphi^2}{|x|^{p-a}}\, \mathrm{d}x. \tag{3.2}$$

Since U is radially symmetric and decreasing in $|x|$, by arguing as in [1, Remark 1.7] it is necessary to check stability condition of U for all radially symmetric test function $\varphi \in C_c^1(\mathbb{R}^N)$. For such φ we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} \left[|x|^a|\nabla U|^{p-2}|\nabla\varphi|^2 + (p-2)|x|^a|\nabla U|^{p-4}(\nabla U, \nabla\varphi)^2 - |x|^b e^U \varphi^2\right] \mathrm{d}x\\
&= \int_{\mathbb{R}^N} \left[(p-1)|x|^a|\nabla U|^{p-2}|\nabla\varphi|^2 - |x|^b e^U \varphi^2\right] \mathrm{d}x\\
&= \int_{\mathbb{R}^N} \left[(p-1)(p-a+b)^{p-2}\frac{|\nabla\varphi|^2}{|x|^{p-a-2}} - (p-a+b)^{p-1}(N+a-p)\frac{\varphi^2}{|x|^{p-a}}\right] \mathrm{d}x\\
&\geq \frac{(N+a-p)(p-a+b)^{p-2}}{4}\left[(N+a-p)(p-1) - 4(p-a+b)\right] \int_{\mathbb{R}^N} \frac{\varphi^2}{|x|^p}\, \mathrm{d}x,
\end{aligned}$$

where we have used (3.2) in the last estimate. On the other hand, the assumption $N \geq \dfrac{(p-a)(p+3)+4b}{p-1}$ is equivalent to $(N+a-p)(p-1) - 4(p-a+b) \geq 0$. Thus, U is stable. $\square$

Proof of Theorem 1.8. Let u be a solution of (1.7) and suppose that u is stable outside a compact set. Then for sufficiently large enough $R_0 > 0$, we may use the following test function $\xi_R \in C_c^1(\mathbb{R}^N \backslash \overline{B_{R_0}})$ in Proposition 2.1:

$$\xi_R(x) = \begin{cases} 0 & \text{if } |x| < R_0 + 1,\\ 1 & \text{if } R_0 + 2 < |x| < R,\\ 0 & \text{if } |x| > 2R, \end{cases}$$

which satisfies

$$0 \le \xi_R \le 1 \text{ in } \mathbb{R}^N, \ |\nabla \xi_R| < \frac{C}{R} \text{ in } B_{2R}\backslash B_R, \ |\nabla \xi_R| < C_{R_0} \text{ in } B_{R_0+2}\backslash B_{R_0+1}.$$

Therefore, for $R > R_0 + 3$ and $\alpha \in \left(0, \frac{4}{p-1}\right)$ we get

$$\int_{B_R\backslash B_{R_0+2}} e^{(\alpha+1)u} \,\mathrm{d}x \le C_{R_0} + CR^{-N\alpha}.$$

Let $R \to \infty$ we obtain

$$\int_{\mathbb{R}^N\backslash B_{R_0+2}} e^{(\alpha+1)u} \,\mathrm{d}x \le C_{R_0}.$$

Now let $\alpha \to 0^+$ and applying Fatou's lemma to get

$$\int_{\mathbb{R}^N\backslash B_{R_0+2}} e^{u} \,\mathrm{d}x \le C_{R_0}.$$

Together with the local integrability of e^u, we have $\int_{\mathbb{R}^N} e^u \,\mathrm{d}x < +\infty$. On the other hand, since $p = N$, we obtain $u \in C^{1,\alpha}(\mathbb{R}^N)$ by standard elliptic estimates. Therefore, u must be of the form (1.7) by a recent classification result of Esposito [9].

Conversely, let u be of the form (1.7), which we may assume $x_0 = 0$. We will show that, if R_1 is sufficiently large, then

$$L_u(\varphi, \varphi) = \int_{\mathbb{R}^N} \left[|\nabla u|^{N-2}|\nabla \varphi|^2 + (N-2)|\nabla u|^{N-4}(\nabla u, \nabla \varphi)^2 - e^u\varphi^2\right] \,\mathrm{d}x \ge 0$$

for all $\varphi \in C_c^1(\mathbb{R}^N\backslash\overline{B_{R_1}})$. Indeed, as $|x| \to \infty$ we note that

$$e^u = \frac{\lambda^N N^{2N-1}}{(N-1)^{N-1}\left(1+\lambda^{\frac{N}{N-1}}|x|^{\frac{N}{N-1}}\right)^N} = O\left(\frac{1}{|x|^{\frac{N^2}{N-1}}}\right)$$

and

$$|\nabla u| = \frac{\lambda^{\frac{N}{N-1}} N^2 |x|^{\frac{1}{N-1}}}{(N-1)\left(1+\lambda^{\frac{N}{N-1}}|x|^{\frac{N}{N-1}}\right)} = O\left(\frac{1}{|x|}\right).$$

Therefore, for sufficiently large $R_1 > 1$ and some $C_0 > 0$ we have

$$|\nabla u|^{N-2} \ge \frac{C_0}{|x|^{N-2}} \quad \text{and} \quad e^u \le \frac{C_0}{4|x|^N \ln^2|x|} \quad \text{for all } |x| > R_1.$$

Using the fact that $N \ge 2$, we deduce for $\varphi \in C_c^1(\mathbb{R}^N\backslash\overline{B_{R_1}})$

$$\begin{aligned} L_u(\varphi,\varphi) &\ge \int_{\mathbb{R}^N} \left[|\nabla u|^{N-2}|\nabla\varphi|^2 - e^u\varphi^2\right] \,\mathrm{d}x \\ &\ge C_0 \int_{\mathbb{R}^N} \left(\frac{|\nabla\varphi|^2}{|x|^{N-2}} - \frac{\varphi^2}{4|x|^N \ln^2|x|}\right) \,\mathrm{d}x \\ &\ge 0. \end{aligned}$$

The last inequality follows immediately from the fact that $\ln^{\frac{1}{2}}|x|$ is a positive solution of

$$-\mathrm{div}\left(\frac{\nabla u}{|x|^{N-2}}\right) = \frac{u}{4|x|^N \ln^2|x|}$$

outside the closed unit ball of $\mathbb{R}^N$. Therefore, u is stable outside $\overline{B_{R_1}}$. □

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