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Existence and Uniqueness of Solutions for a Boundary Value Problem of Fractional Type with Nonlocal Integral Boundary Conditions in Hölder Spaces

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Abstract. In this paper, we prove the existence and uniqueness of solutions for the following fractional boundary value problem

$$\begin{cases} {}^{c}D^{\alpha}_{0+}u(t) = \lambda f(t,u(t)), \quad t \in [0,1], \\ u(0) = \gamma I^{\rho}_{0+}u(\eta) = \gamma \displaystyle\int_0^{\eta} \frac{(\eta - s)^{\rho-1}}{\Gamma(\rho)} u(s)\mathrm{d}s, \end{cases}$$

where $0 < \alpha \leq 1, 0 < \eta < 1$ and $\lambda, \gamma, \rho \in \mathbb{R}$. Our solutions are placed in the space of functions satisfying the Hölder condition. Our analysis relies on a fixed point theorem in complete metric spaces. Moreover, we present some examples illustrating our results.

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1. Introduction

Differential equations of fractional order occur more frequently on different research areas such as physics, chemistry, engineering, and economics. Indeed, we can find numerous applications in viscoelasticity, electrochemistry control, porous media, electromagnetism, etc... [5,6,12–17,19,21,22].

For an extensive collection of results about this type of equations, we refer the reader to the monographs [13,19,21].

On the other hand, some basic theory for the initial value problems of fractional differential operators has been discussed in [14,15] and some other boundary value problems of fractional type have been studied in [1,2,23] (among others).

The main purpose of this paper is to study the existence and uniqueness of solutions for the following boundary value problem of fractional type with nonlocal integral boundary conditions

$$\begin{cases} {}^cD^{\alpha}_{0+}u(t) = \lambda f(t, u(t)), \quad t \in [0, 1], \\ u(0) = \gamma I^{\rho}_{0+}u(\eta) = \gamma \int_0^{\eta} \frac{(\eta - s)^{\rho-1}}{\Gamma(\rho)} u(s)\mathrm{d}s, \end{cases} \tag{1}$$

where ${}^cD^{\alpha}_{0+}$ denotes the Caputo fractional derivative and $0 < \alpha \leq 1$, $0 < \eta < 1$ and $\lambda, \gamma, \rho \in \mathbb{R}$.

For our knowledge, generally, the papers appearing in the literature study the existence of solutions for the fractional boundary value problems in the space of the continuous functions defined on the interval $[0, 1]$.

In the paper [1], the author studied the problem

$$D^s_{0+}u(t) = \lambda\, a(t)f(u(t)), \quad 0 < t < 1,$$

where $0 < s < 1$, D^s denotes the the Riemann–Lioville fractional derivative, and $f\colon [0, \infty) \to [0, \infty)$ is a continuous function with $f(0) > 0$, $a\colon [0, 1] \to \mathbb{R}$ and $\lambda > 0$.

In [2], the author studied the problem

$$\begin{cases} D^{\alpha}_{0+}u(t) + f(t, u(t)) = 0, \quad 0 < t < 1, \\ u(0) = 0, \quad \beta\, u(\eta) = u(1), \end{cases}$$

where $1 < \alpha \leq 2$, $0 < \beta\, \eta^{\alpha-1} < 1$, $0 < \eta < 1$ and f is a continuous function defined on $[0, 1] \times [0, \infty)$.

In [23], the authors researched the problem

$$\begin{cases} D^{\alpha}_{0+}u(t) + \lambda f(u(t)) = 0, \quad 0 < t < 1, \\ u(0) = u(1) = u'(0) = 0, \end{cases}$$

where $2 < \alpha \leq 3$, $\lambda > 0$ and $f\colon (0, \infty) \to (0, \infty)$ is continuous.

In these tree papers, the solutions of these problems are in the space of the continuous functions on $[0, 1]$ and the techniques used are the non-linear alternative of Leray–Schauder type, the fixed point index theory and Guo–Krasnosel'skii fixed point theorem on cones, respectively. However, in our study, the solutions of Problem (1) are placed in the space of functions satisfying the Hölder condition, and the main tool used in the proofs is a fixed point theorem in partially ordered sets which appears in [7].

We also refer the interested reader to [8–11] in which using variational methods and critical point theory, the existence of multiple solutions for some fractional boundary value problems has been discussed.

2. Preliminaries

For the convenience of the reader, we present some definitions, lemmas and basic facts about fractional calculus [13–15, 19].

Definition 1. For at least n times continuously differentiable function $g\colon [0, \infty) \to \mathbb{R}$, the Caputo derivative of fractional order q is defined as

$${}^cD^q_{0+}g(t) = \frac{1}{\Gamma(n - q)} \int_0^t (t - s)^{n-q-1} g^{(n)}(s)\mathrm{d}s,$$

where $n = [q] + 1$ and $[q]$ denotes the integer part of q. Here, $\Gamma(\alpha)$ denotes the classical gamma function.

Definition 2. The Riemann–Liouville fractional integral of order q of a function $g\colon (0,\infty) \to \mathbb{R}$, defined by

$$I_{0+}^{q} g(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{g(s)}{(t-s)^{1-q}} \mathrm{d}s,$$

provides that the right side is pointwise defined on $(0,\infty)$.

The following two lemmas give us some relations between the Riemann–Liouville fractional integrals and the Caputo fractional derivatives and they can be found in [13].

Lemma 1. *Suppose that $p, q \geq 0$ and $f \in L^1[0,1]$. Then,*

$$I_{0+}^{p} I_{0+}^{q} f(t) = I_{0+}^{p+q} f(t) \quad \textit{and} \quad {}^{c}D_{0+}^{q} I_{0+}^{q} f(t) = f(t),$$

for any $t \in [0,1]$.

Lemma 2. *Suppose that $\beta > \alpha > 0$ and $f \in L^1[0,1]$. Then,*

$${}^{c}D_{0+}^{\alpha} I_{0+}^{\beta} f(t) = I_{0+}^{\beta-\alpha} f(t), \quad \textit{for any } t \in [0,1].$$

The next lemma appears in [18] and it is a very interesting result for our study.

Lemma 3. *Suppose that $\gamma \neq \dfrac{\Gamma(\rho+1)}{\eta^{\rho}}$ and $\rho > 0$. Then, for a function $f \in \mathcal{C}([0,1])$, the solution of the fractional differential equation*

$${}^{c}D_{0+}^{\alpha} x(t) = f(t), \quad t \in [0,1]$$

with $0 < \alpha \leq 1$ and under the boundary condition

$$x(0) = \gamma I_{0+}^{\rho} x(\eta) = \gamma \int_0^{\eta} \frac{(\eta-s)^{\rho-1}}{\Gamma(\rho)} x(s) \mathrm{d}s,$$

where $0 < \eta < 1$, is given by

$$\begin{aligned} x(t) =& \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) \mathrm{d}s \\ &+ \frac{\gamma \Gamma(\rho+1)}{\Gamma(\rho+1) - \gamma \eta^{\rho}} \int_0^{\eta} \frac{(\eta-s)^{\rho+\alpha-1}}{\Gamma(\rho+\alpha)} f(s) \mathrm{d}s, \quad t \in [0,1]. \end{aligned}$$

In the sequel, we will introduce the space of functions satisfying the Hölder condition and some basic facts about these spaces. These results can be found in [3].

Let $[a,b]$ be a closed interval in $\mathbb{R}$, by $\mathcal{C}[a,b]$ we denote the space of the continuous functions on $[a,b]$ with real values equipped with the norm of the supremum, i.e.,

$$\|x\|_{\infty} = \sup\{|x(t)| \colon t \in [a,b]\} \quad \text{for } x \in \mathcal{C}[a,b].$$

For $0 < \alpha \leq 1$ fixed, by $\mathcal{H}_\alpha[a,b]$ we will denote the space of the real functions x defined on $[a,b]$ and satisfying the Hölder condition, that is, those functions x for which there exists a constant $\mathcal{H}_x^\alpha$ such that

$$|x(t) - x(p)| \leq \mathcal{H}_x^\alpha |t-p|^\alpha, \tag{2}$$

for any $t, p \in [a,b]$.

It is easily seen that $\mathcal{H}_\alpha[a,b]$ is a linear subspace of $\mathcal{C}[a,b]$. In the sequel, for $x \in \mathcal{H}_\alpha[a,b]$, by $\mathcal{H}_x^\alpha$ we will denote the least possible constant for which inequality (2) is satisfied. More precisely, we put

$$\mathcal{H}_x^\alpha = \sup\left\{ \frac{|x(t) - x(p)|}{|t-p|^\alpha} : t, p \in [a,b], t \neq p \right\}. \tag{3}$$

The spaces $\mathcal{H}_\alpha[a,b]$ with $0 < \alpha \leq 1$ can be normed by

$$\begin{aligned} \|x\|_\alpha &= |x(a)| + \sup\left\{ \frac{|x(t) - x(p)|}{|t-p|^\alpha} : t, p \in [a,b],\ t \neq p \right\} \\ &= |x(a)| + \mathcal{H}_x^\alpha, \end{aligned}$$

for any $x \in \mathcal{H}_\alpha[a,b]$. In [3], the authors proved that $(\mathcal{H}_\alpha[a,b], \|\cdot\|)$ with $0 < \alpha \leq 1$ is a Banach space.

The following two lemmas appear in [3].

Lemma 4. *For $x \in \mathcal{H}_\alpha[a,b]$ with $0 < \alpha \leq 1$, the following inequality,*

$$\|x\|_\infty \leq \max\left(1, (b-a)^\alpha\right) \|x\|_\alpha, \tag{4}$$

holds.

Lemma 5. *For $0 < \alpha < \gamma \leq 1$,*

$$\mathcal{H}_\gamma[a,b] \subset \mathcal{H}_\alpha[a,b] \subset \mathcal{C}[a,b]. \tag{5}$$

Moreover, for $x \in \mathcal{H}_\gamma[a,b]$ the following inequality,

$$\|x\|_\alpha \leq \max\left(1, (b-a)^{\gamma-\alpha}\right) \|x\|_\gamma, \tag{6}$$

holds.

Next, we present the results about fixed points which we will use later. The following result appears in [20].

Theorem 1. *Let (X,d) be a complete metric space and $T\colon X \to X$ a mapping satisfying*

$$d(Tx, Ty) \leq d(x,y) - \varphi(d(x,y)), \quad \textit{for any } x, y \in X, \tag{7}$$

where $\varphi\colon [0,\infty) \to [0,\infty)$ is nondecreasing and $\varphi(t) = 0$ if and only if $t = 0$. Then, T has a unique fixed point.

Remark 1. In [20], the author assumes the continuity of φ, but it is easily seen that such condition is superfluous [4].

In the sequel, we will denote by $\mathcal{A}$ the class of functions $\varphi\colon [0,\infty) \to [0,\infty)$ which are nondecreasing and such that $\varphi(t) = 0$ if and only if $t = 0$.

The following result is the version of Theorem 1 in the context of ordered metric spaces and it appears in [7].

Theorem 2. *Let $(X, \leq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Let $T\colon X \to X$ be a nondecreasing mapping such that*

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y)), \quad \text{for } x \geq y, \tag{8}$$

where $\varphi \in \mathcal{A}$.
Assume that T is continuous or that X satisfies the following condition:

$$\begin{aligned} &\text{if } (x_n) \text{ is a nondecreasing sequence in } X \text{ such that } x_n \to x \\ &\text{then } x_n \leq x \text{ for any } n \in \mathbb{N}. \end{aligned} \tag{9}$$

If there exists $x_0 \in X$ such that $x_0 \leq Tx_0$, then T has a fixed point. Moreover, if $(X, \leq)$ satisfies the condition:

$$\text{for each } x, y \in X \text{ there exists } z \in X \text{ which is comparable to } x \text{ and } y, \tag{10}$$

then the fixed point is unique.

3. Main Results

To study Problem (1), we start this section with the following proposition.

Proposition 1. *Assume that $0 < \alpha \leq 1$, $0 < \eta < 1$, $\lambda, \rho > 0$, $\gamma \in \mathbb{R}$ and $\gamma \neq \frac{\Gamma(\rho+1)}{\eta^\rho}$. Suppose that $f\colon [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function and $x \in \mathcal{H}_\alpha[0,1]$.*

Let Tx be the function defined by

$$\begin{aligned} (Tx)(t) = &\frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s)) \mathrm{d}s \\ &+ \frac{\lambda\gamma\Gamma(\rho+1)}{\Gamma(\rho+1) - \gamma\eta^\rho} \int_0^\eta \frac{(\eta-s)^{\rho+\alpha-1}}{\Gamma(\rho+\alpha)} f(s, x(s)) \mathrm{d}s, \end{aligned}$$

for any $t \in [0,1]$. Then, $Tx \in \mathcal{H}_\alpha[0,1]$.

Proof. Notice that, for any $t \in [0,1]$,

$$\begin{aligned} (Tx)(t) = \lambda \Bigg[&\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s)) \mathrm{d}s \\ &+ \frac{\gamma\Gamma(\rho+1)}{\Gamma(\rho+1) - \gamma\eta^\rho} \int_0^\eta \frac{(\eta-s)^{\rho+\alpha-1}}{\Gamma(\rho+\alpha)} f(s, x(s)) \mathrm{d}s \Bigg]. \end{aligned}$$

To prove that $Tx \in \mathcal{H}_\alpha[0,1]$, we take $t, t' \in [0,1]$ with $t \neq t'$. Without loss of generality, we can suppose that $t > t'$.

Then, we have the following estimate:

$$\begin{aligned} &\frac{|(Tx)(t) - (Tx)(t')|}{|t-t'|^\alpha} \\ &= \frac{\frac{\lambda}{\Gamma(\alpha)} \left| \int_0^t (t-s)^{\alpha-1} f(s, x(s)) \mathrm{d}s - \int_0^{t'} (t'-s)^{\alpha-1} f(s, x(s)) \mathrm{d}s \right|}{|t-t'|^\alpha} \end{aligned}$$

$$= \frac{\frac{\lambda}{\Gamma(\alpha)} \left| \int_0^{t'} \left[(t-s)^{\alpha-1} - (t'-s)^{\alpha-1} \right] f(s,x(s)) \mathrm{d}s + \int_t^{t'} (t-s)^{\alpha-1} f(s,x(s)) \mathrm{d}s \right|}{|t-t'|^\alpha}$$

$$\leq \frac{\frac{\lambda}{\Gamma(\alpha)} \left[\int_0^{t'} \left| (t-s)^{\alpha-1} - (t'-s)^{\alpha-1} \right| |f(s,x(s))| \mathrm{d}s + \int_{t'}^{t} (t-s)^{\alpha-1} |f(s,x(s))| \mathrm{d}s \right]}{|t-t'|^\alpha}.$$

As f is a continuous function and $x \in \mathcal{H}_\alpha[0,1] \subset \mathcal{C}[0,1]$, there exists $M = \sup\{|f(s,x)| \colon s \in [0,1], x \in [-\|x\|_\infty, \|x\|_\infty]\}$.

From the last inequality and, since $0 < \alpha \leq 1$ and $t > t'$, it follows that

$$\begin{aligned}
\frac{|(Tx)(t) - (Tx)(t')}{|t-t'|^\alpha} &\leq \frac{\frac{\lambda M}{\Gamma(\alpha)} \left[\int_0^{t'} \left[(t'-s)^{\alpha-1} - (t-s)^{\alpha-1} \right] \mathrm{d}s + \int_{t'}^{t} (t-s)^{\alpha-1} \mathrm{d}s \right]}{|t-t'|^\alpha} \\
&\leq \frac{\frac{\lambda M}{\Gamma(\alpha)} \left[\frac{(t-t')^\alpha}{\alpha} + \frac{t'^\alpha}{\alpha} - \frac{t^\alpha}{\alpha} + \frac{(t-t')^\alpha}{\alpha} \right]}{|t-t'|^\alpha} \\
&\leq \frac{2\lambda M}{\Gamma(\alpha)} \frac{(t-t')^\alpha}{\alpha(t-t')^\alpha} \\
&= \frac{2\lambda M}{\alpha\Gamma(\alpha)},
\end{aligned}$$

where we have used the fact that $\frac{t'^\alpha}{\alpha} - \frac{t^\alpha}{\alpha} \leq 0$ (because $t > t'$ and $0 < \alpha \leq 1$).

This proves that $Tx \in \mathcal{H}_\alpha[0,1]$. □

Proposition 2. *Under assumptions of Proposition 1, suppose that* $f \colon [0,1] \times \mathbb{R} \to \mathbb{R}$ *satisfies that*

$$|f(t,x) - f(t,y)| \leq \phi(|x-y|),$$

for any $t \in [0,1]$ *and* $x, y \in \mathbb{R}$, *where* $\phi \colon \mathbb{R}_+ \to \mathbb{R}_+$ *is a nondecreasing function.*

Then, we have the following estimate:

$$\begin{aligned}
\|Tx - Ty\|_\alpha \leq& \lambda \left[\left| \frac{\gamma\Gamma(\rho+1)}{\Gamma(\rho+1) - \gamma\eta^\rho} \right| \frac{1}{(\rho+\alpha)\Gamma(\rho+\alpha)} \eta^{\rho+\alpha} \right. \\
&\left. + \frac{2}{\alpha\Gamma(\alpha)} \right] \phi(\|x-y\|_\alpha).
\end{aligned}$$

Proof. Firstly, we estimate:

$$\sup\left\{ \frac{|[(Tx)(t) - (Ty)(t)] - [(Tx)(t') - (Ty)(t')]|}{|t-t'|^\alpha} \colon t, t' \in [0,1], t \neq t' \right\}.$$

To do this, we take $t, t' \in [0,1]$ with $t \neq t'$.

Without loss of generality, suppose that $t > t'$.

Then, we have:

$$\frac{|[(Tx)(t) - (Ty)(t)] - [(Tx)(t') - (Ty)(t')]|}{|t-t'|^\alpha}$$

$$= \frac{\frac{\lambda}{\Gamma(\alpha)} \left| \int_0^t (t-s)^{\alpha-1} [f(s,x(s)) - f(s,y(s))] \mathrm{d}s - \int_0^{t'} (t'-s)^{\alpha-1} [f(s,x(s)) - f(s,y(s))] \mathrm{d}s \right|}{|t-t'|^\alpha}$$

$$= \frac{\frac{\lambda}{\Gamma(\alpha)}\left|\int_0^{t'}\left[(t-s)^{\alpha-1}-(t'-s)^{\alpha-1}\right][f(s,x(s))-f(s,y(s))]\mathrm{d}s\right.}{|t-t'|^{\alpha}}$$
$$+\frac{\left.\int_{t'}^{t}(t-s)^{\alpha-1}[f(s,x(s))-f(s,y(s))]\mathrm{d}s\right|}{|t-t'|^{\alpha}}$$
$$\leq \frac{\frac{\lambda}{\Gamma(\alpha)}\left[\int_0^{t'}\left|(t-s)^{\alpha-1}-(t'-s)^{\alpha-1}\right||f(s,x(s))-f(s,y(s))|\mathrm{d}s\right.}{|t-t'|^{\alpha}}$$
$$+\frac{\left.\int_{t'}^{t}(t-s)^{\alpha-1}|f(s,x(s))-f(s,y(s))|\mathrm{d}s\right]}{|t-t'|^{\alpha}}$$
$$\leq \frac{\frac{\lambda}{\Gamma(\alpha)}\left[\int_0^{t'}\left((t'-s)^{\alpha-1}-(t-s)^{\alpha-1}\right)\phi(|x(s)-y(s)|)\mathrm{d}s+\int_{t'}^{t}(t-s)^{\alpha-1}\phi(|x(s)-y(s)|)\mathrm{d}s\right]}{|t-t'|^{\alpha}}$$
$$\leq \frac{\frac{\lambda}{\Gamma(\alpha)}\cdot\phi(\|x-y\|_{\infty})\left[\int_0^{t'}\left((t'-s)^{\alpha-1}-(t-s)^{\alpha-1}\right)\mathrm{d}s+\int_{t'}^{t}(t-s)^{\alpha-1}\mathrm{d}s\right]}{|t-t'|^{\alpha}}$$
$$\leq \frac{\frac{\lambda}{\Gamma(\alpha)}\cdot\phi(\|x-y\|_{\alpha})\left[\frac{(t-t')^{\alpha}}{\alpha}+\frac{t'^{\alpha}}{\alpha}-\frac{t^{\alpha}}{\alpha}+\frac{(t-t')^{\alpha}}{\alpha}\right]}{|t-t'|^{\alpha}}$$
$$\leq \frac{\frac{\lambda}{\Gamma(\alpha)}\cdot\phi(\|x-y\|_{\alpha})\left[2\frac{(t-t')^{\alpha}}{\alpha}\right]}{|t-t'|^{\alpha}}$$
$$\leq \frac{2\lambda}{\alpha\Gamma(\alpha)}\cdot\phi(\|x-y\|_{\alpha}), \tag{11}$$

where we have used $\|x-y\|_{\infty}\leq\|x-y\|_{\alpha}$ (Lemma 4), the nondecreasing character of ϕ and the fact that $\frac{t'^{\alpha}}{\alpha}-\frac{t^{\alpha}}{\alpha}\leq 0$.

Therefore, we have the following estimate:

$$\begin{aligned}\|Tx-Ty\|_{\alpha} &= |(Tx)(0)-(Ty)(0)| \\ &\quad+\sup\left\{\frac{|[(Tx)(t)-(Ty)(t)]-[(Tx)(t')-(Ty)(t')]|}{|t-t'|^{\alpha}}: t,t'\in[0,1], t\neq t'\right\} \\ &\leq \lambda\left|\frac{\gamma\Gamma(\rho+1)}{\Gamma(\rho+1)-\gamma\eta^{\rho}}\int_0^{\eta}\frac{(\eta-s)^{\rho+\alpha-1}}{\Gamma(\rho+\alpha)}\left(f(s,x(s))-f(s,y(s))\right)\mathrm{d}s\right| \\ &\quad+\frac{2\lambda}{\alpha\Gamma(\alpha)}\phi(\|x-y\|_{\alpha}) \\ &\leq \lambda\left|\frac{\gamma\Gamma(\rho+1)}{\Gamma(\rho+1)-\gamma\eta^{\rho}}\right|\int_0^{\eta}\frac{(\eta-s)^{\rho+\alpha-1}}{\Gamma(\rho+\alpha)}\left|f(s,x(s))-f(s,y(s))\right|\mathrm{d}s \\ &\quad+\frac{2\lambda}{\alpha\Gamma(\alpha)}\phi(\|x-y\|_{\alpha}) \\ &\leq \lambda\left|\frac{\gamma\Gamma(\rho+1)}{\Gamma(\rho+1)-\gamma\eta^{\rho}}\right|\phi(\|x-y\|_{\alpha})\frac{1}{(\rho+\alpha)\Gamma(\rho+\alpha)}\eta^{\rho+\alpha} \\ &\quad+\frac{2\lambda}{\alpha\Gamma(\alpha)}\phi(\|x-y\|_{\alpha}) \\ &= \lambda\left[\left|\frac{\gamma\Gamma(\rho+1)}{\Gamma(\rho+1)-\gamma\eta^{\rho}}\right|\frac{1}{(\rho+\alpha)\Gamma(\rho+\alpha)}\eta^{\rho+\alpha}+\frac{2\lambda}{\alpha\Gamma(\alpha)}\right]\phi(\|x-y\|_{\alpha}).\end{aligned}$$

This completes the proof. □

Next, we introduce the following class of functions $\mathcal{B}$ which we will use later. By $\mathcal{B}$ we will denote the class of those functions $\phi\colon [0,\infty) \to [0,\infty)$ which are nondecreasing and such that if $\varphi(x) = x - \phi(x)$ then $\varphi \in \mathcal{A}$, where $\mathcal{A}$ is the class of functions appearing in Theorem 1.

Examples of functions belonging to the class $\mathcal{B}$ are $\phi(t) = \arctan t$, $\phi(t) = \frac{t}{1+t}$ and $\phi(t) = \ln(1+t)$.

Now, we are ready to present one of the main results of the paper. Previously, we put

$$L = \left|\frac{\gamma\Gamma(\rho+1)}{\Gamma(\rho+1) - \gamma\eta^\rho}\right| \frac{1}{(\rho+\alpha)\Gamma(\rho+\alpha)}\eta^{\rho+\alpha} + \frac{2}{\alpha\Gamma(\alpha)}.$$

Theorem 3. *Suppose that* $0 < \alpha \leq 1$, $0 < \eta < 1$, $\lambda, \rho > 0$, $\gamma \in \mathbb{R}$ *and* $\gamma \neq \frac{\Gamma(\rho+1)}{\eta^\rho}$. *Let* $f\colon [0,1] \times \mathbb{R} \to \mathbb{R}$ *be a continuous function such that*

$$|f(t,x) - f(t,y)| \leq \phi(|x-y|),$$

for any $t \in [0,1]$ *and* $x, y \in \mathbb{R}$, *where* $\phi \in \mathcal{B}$.

Under assumption that $\lambda \leq \frac{1}{L}$, *Problem (1) has a unique solution in the space* $\mathcal{H}_\alpha[0,1]$.

Proof. Consider the operator T defined on $\mathcal{H}_\alpha[0,1]$ as

$$\begin{aligned}(Tx)(t) = \lambda \Bigg[&\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,x(s))\mathrm{d}s \\ &+ \frac{\gamma\Gamma(\rho+1)}{\Gamma(\rho+1)-\gamma\eta^\rho}\int_0^\eta \frac{(\eta-s)^{\rho+\alpha-1}}{\Gamma(\rho+\alpha)} f(s,x(s))\mathrm{d}s\Bigg],\end{aligned}$$

for any $x \in \mathcal{H}_\alpha[0,1]$ and $t \in [0,1]$.

In virtue of Lemma 3, a solution for Problem (1) is a fixed point of T.

By Proposition 1, T applies $\mathcal{H}_\alpha[0,1]$ into itself.

By Proposition 2, we have, for any $x, y \in \mathcal{H}_\alpha[0,1]$,

$$\begin{aligned}d(Tx,Ty) &= \|Tx - Ty\|_\alpha \\ &\leq \lambda\left[\left|\frac{\gamma\Gamma(\rho+1)}{\Gamma(\rho+1)-\gamma\eta^2}\right| \frac{1}{(\rho+\alpha)\Gamma(\rho+\alpha)}\ \eta^{\rho+\alpha} + \frac{2}{\alpha\Gamma(\alpha)}\right]\phi(\|x-y\|_\alpha) \\ &= \lambda\left[\left|\frac{\gamma\Gamma(\rho+1)}{\Gamma(\rho+1)-\gamma\eta^2}\right| \frac{1}{(\rho+\alpha)\Gamma(\rho+\alpha)}\ \eta^{\rho+\alpha} + \frac{2}{\alpha\Gamma(\alpha)}\right]\phi(d(x,y)) \\ &\leq \phi(d(x,y)),\end{aligned}$$

where we have used the assumption $\lambda \leq \frac{1}{L}$.

From the last inequality and, since $\phi \in \mathcal{B}$, we get

$$\begin{aligned}d(Tx,Ty) &\leq \phi(d(x,y)) \\ &= d(x,y) - [d(x,y) - \phi(d(x,y))] \\ &= d(x,y) - \varphi(d(x,y)),\end{aligned}$$

where $\varphi\colon [0,\infty) \to [0,\infty)$ is nondecreasing and $\varphi(t) = 0$ if and only if $t = 0$.

By Theorem 1, the operator T has a unique fixed point and this fixed point is the unique solution of Problem (1).

This finishes the proof. □

To illustrate our result, we present the following numerical example.

Example 1. Consider the following fractional boundary value problem:

$$\begin{cases} {}^cD_{0+}^{\frac{1}{2}}u(t) = \lambda\left(-t + \dfrac{|u(t)|}{1+|u(t)|}\right), \quad 0<t<1 \\ u(0) = 2I_{0+}^{\frac{1}{2}}u\left(\dfrac{1}{3}\right) = \dfrac{2}{\Gamma\left(\frac{1}{2}\right)}\displaystyle\int_0^{\frac{1}{3}} \dfrac{u(s)}{\left(\frac{1}{3}-s\right)^{\frac{1}{2}}}\mathrm{d}s. \end{cases} \tag{12}$$

Notice that Problem (12) is a particular case of Problem (1) with $\alpha = \frac{1}{2}$, $\eta = \frac{1}{3}$, $\rho = \frac{1}{2}$ and $\gamma = 2$. Moreover, in Problem (12), $f(t,x) = -t + \frac{|x|}{1+|x|}$. It is clear that $f\colon [0,1]\times\mathbb{R}\to\mathbb{R}$ is a continuous function and, for any $t\in[0,1]$ and $x,y\in\mathbb{R}$, we have

$$\begin{aligned} |f(t,x)-f(t,y)| &= \left|\frac{|x|}{1+|x|} + \frac{|y|}{1+|y|}\right| \\ &= \left|\frac{|x|-|y|}{(1+|x|)(1+|y|)}\right| \\ &= \frac{||x|-|y||}{(1+|x|)(1+|y|)} \\ &\le \frac{|x-y|}{1+|x-y|} \\ &= \phi(|x-y|), \end{aligned}$$

where $\phi\colon [0,\infty)\to[0,\infty)$ is defined by $\phi(t) = \frac{t}{1+t}$, and where we have used the inequalities $||x|-|y|| \le |x-y|$ and $(1+|x|)(1+|y|) \ge 1+|x-y|$.

It is easily seen that $\phi\in\mathcal{B}$.

Moreover, in our case,

$$\begin{aligned} L &= \left|\frac{\gamma\Gamma(\rho+1)}{\Gamma(\rho+1)-\gamma\eta^\rho}\right| \frac{1}{(\rho+\alpha)\Gamma(\rho+\alpha)}\,\eta^{\rho+\alpha} + \frac{2}{\alpha\Gamma(\alpha)} \\ &= \left|\frac{2\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2}\right)-2\sqrt{\frac{1}{3}}}\right| \frac{1}{\Gamma(1)}\,\frac{1}{3} + \frac{2}{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)} \\ &= \left|\frac{\sqrt{\pi}}{\frac{\sqrt{\pi}}{2}-\frac{2}{\sqrt{3}}}\right| \frac{1}{3} + \frac{4}{\sqrt{\pi}} \\ &\simeq 4'458. \end{aligned}$$

Therefore, for $\lambda \le \frac{1}{4'458}$, Problem (12) has a unique solution in the space $\mathcal{H}_{\frac{1}{2}}[0,1]$, in virtue of Theorem 3.

From a practical point of view, an interesting question is when the solution $u(t)$ of Problem (1) is positive, i.e., $u(t)>0$ for $t\in(0,1)$.

In the sequel, we will treat this question.

Theorem 4. *Suppose that* $0<\alpha\le 1$, $0<\eta<1$, $\gamma,\lambda,\rho>0$ *and* $\gamma < \frac{\Gamma(\rho+1)}{\eta^\rho}$. *Let* $f\colon [0,1]\times[0,\infty)\to[0,\infty)$ *be a function satisfying the following assumptions:*

(i) *f is continuous.*
(ii) *$f(t,x)$ is nondecreasing with respect to the second argument for any $t\in[0,1]$.*
(iii) *There exists $\phi\in\mathcal{B}$ such that*

$$f(t,y)-f(t,x)\leq\phi(y-x),$$

for any $x,y\in[0,\infty)$ with $y\geq x$ and $t\in[0,1]$.

If $\lambda\leq\frac{1}{L}$, then Problem (1) has a unique nonnegative solution in the space $\mathcal{H}_\alpha[0,1]$.

Proof. Consider the cone

$$P=\{u\in\mathcal{H}_\alpha[0,1]\colon u\geq 0\}.$$

It is easily seen that P is a closed subset of $\mathcal{H}_\alpha[0,1]$, since the convergence in $\mathcal{H}_\alpha[0,1]$ implies the convergence in the supremum norm (see, Lemma 4).

Moreover, P can be equipped with a partial order given by

$$x,y\in P,\quad x\leq y\Leftrightarrow x(t)\leq y(t)\quad\text{ for any } t\in[0,1].$$

By Lemma 4, it is easily seen that P with this partial order $\leq$ satisfies assumption (9) of Theorem 2.

Now, we consider the operator T defined on P as

$$\begin{aligned}(Tx)(t)=\lambda\Bigg[&\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s,x(s))\mathrm{d}s\\&+\frac{\gamma\Gamma(\rho+1)}{\Gamma(\rho+1)-\gamma\eta^\rho}\int_0^\eta\frac{(\eta-s)^{\rho+\alpha-1}}{\Gamma(\rho+\alpha)}f(s,x(s))\mathrm{d}s\Bigg],\end{aligned}$$

for $x\in P$ and $t\in[0,1]$.

In virtue of our assumptions and Proposition 1, it is clear that T applies P into itself.

Next, we will prove that T is a nondecreasing operator.

In fact, for $x\geq y$ and taking into account (ii), we have

$$\begin{aligned}(Tx)(t)&=\lambda\Bigg[\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s,x(s))\mathrm{d}s\\&\quad+\frac{\gamma\Gamma(\rho+1)}{\Gamma(\rho+1)-\gamma\eta^\rho}\int_0^\eta\frac{(\eta-s)^{\rho+\alpha-1}}{\Gamma(\rho+\alpha)}f(s,x(s))\mathrm{d}s\Bigg]\\&\geq\lambda\Bigg[\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s,y(s))\mathrm{d}s\\&\quad+\frac{\gamma\Gamma(\rho+1)}{\Gamma(\rho+1)-\gamma\eta^\rho}\int_0^\eta\frac{(\eta-s)^{\rho+\alpha-1}}{\Gamma(\rho+\alpha)}f(s,y(s))\mathrm{d}s\Bigg]\\&=(Ty)(t).\end{aligned}$$

Besides, for $x\geq y$, and by using a similar argument that in the proof of Theorem 3, we can obtain

$$\begin{aligned}d(Tx,Ty)&=\|Tx-Ty\|_\alpha\leq\phi(\|x-y\|_\alpha)=\phi(d(x,y))\\&=d(x,y)-(d(x,y)-\phi(d(x,y)).\end{aligned}$$

Since $\phi \in \mathcal{B}$, this proves that assumption (8) of Theorem 2 is satisfied.

Notice that the zero function satisfies $0 \leq T0$.

Moreover, notice that for $x, y \in P$, the function $x+y \in P$ and $x \leq x+y$ and $y \leq x+y$, this proves that $(P, \leq)$ satisfies condition (10) of Theorem 2.

Finally, by Theorem 2, Problem (1) has a unique nonnegative solution in $\mathcal{H}_\alpha[0,1]$. □

The following result gives us a sufficient condition for the existence and uniqueness of a positive solution for Problem (1) in the space $\mathcal{H}_\alpha[0,1]$.

Theorem 5. *Under assumptions of Theorem 4 and adding the condition*

$$f(t_0, 0) \neq 0 \text{ for certain } t_0 \in [0,1] \text{ with } t_0 \leq \eta,$$

we obtain the existence and uniqueness of a positive solution for Problem (1) in the space $\mathcal{H}_\alpha[0,1]$.

Proof. Consider the nonnegative solution $x(t)$ for Problem (1), guaranteed by Theorem 4.

Notice that, as $x(t)$ is a fixed point of the operator T, we have

$$x(t) = \lambda \left[\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s)) \mathrm{d}s \right. \\ \left. + \frac{\gamma\Gamma(\rho+1)}{\Gamma(\rho+1) - \gamma\eta^\rho} \int_0^\eta \frac{(\eta-s)^{\rho+\alpha-1}}{\Gamma(\rho+\alpha)} f(s, x(s)) \mathrm{d}s \right].$$

Next, we will prove that $x(t) > 0$ for $t \in (0,1)$.

In fact, in the contrary case we can find $0 < t^* < 1$ such that $x(t^*) = 0$ and, therefore,

$$0 = x(t^*) = \lambda \left[\frac{1}{\Gamma(\alpha)} \int_0^{t^*} (t^*-s)^{\alpha-1} f(s, x(s)) \mathrm{d}s \right. \\ \left. + \frac{\gamma\Gamma(\rho+1)}{\Gamma(\rho+1) - \gamma\eta^\rho} \int_0^\eta \frac{(\eta-s)^{\rho+\alpha-1}}{\Gamma(\rho+\alpha)} f(s, x(s)) \mathrm{d}s \right].$$

Since the summands in the right part are nonnegative, we have

$$\frac{\lambda\gamma\Gamma(\rho+1)}{\Gamma(\rho+1) - \gamma\eta^\rho} \int_0^\eta \frac{(\eta-s)^{\rho+\alpha-1}}{\Gamma(\rho+\alpha)} f(s, x(s)) \mathrm{d}s = 0$$

or, equivalently, since $\dfrac{\lambda\gamma\Gamma(\rho+1)}{\Gamma(\rho+1) - \gamma\eta^\rho} > 0$,

$$\int_0^\eta (\eta-s)^{\rho+\alpha-1} f(s, x(s)) \mathrm{d}s = 0.$$

By (ii) of Theorem 4 and the fact that $f\colon [0,1] \times [0,\infty) \to [0,\infty)$, it follows that

$$0 = \int_0^\eta (\eta-s)^{\rho+\alpha-1} f(s, x(s)) \mathrm{d}s \geq \int_0^\eta (\eta-s)^{\rho+\alpha-1} f(s, 0) \mathrm{d}s \geq 0.$$

Thus,

$$\int_0^\eta (\eta - s)^{\rho+\alpha-1} f(s,0)\mathrm{d}s = 0.$$

Since the integrand is nonnegative, we have

$$(\eta - s)^{\rho+\alpha-1} f(s,0) = 0 \quad \text{a.e.}(s) \text{ in } [0,\eta].$$

Since $(\eta - s)^{\rho+\alpha-1} \neq 0$ a.e.(s) in $[0,\eta]$, we have that

$$f(s,0) = 0 \quad \text{a.e.}(s) \text{ in } [0,\eta]. \tag{13}$$

Taking into account that $f(t_0,0) > 0$ with $t_0 \leq \eta$ and the fact that f is continuous, we can find a set $\Omega \subset [0,\eta]$ with $t_0 \in \Omega$ and $\mu(\Omega) > 0$, where μ is the Lebesgue measure, such that $f(t,0) > 0$ for any $t \in \Omega$. This contradicts (13).

Therefore, $x(t) > 0$ for $t \in (0,1)$. □

Remark 2. Under assumptions of Theorem 4 if the nonnegative solution $x(t)$ for Problem (1) (whose existence is guaranteed by Theorem 4) satisfies $x(0) = 0$, then since

$$x(0) = \gamma I_{0+}^{\rho} x(\eta) = \gamma \int_0^\eta \frac{(\eta - s)^{\rho-1}}{\Gamma(\rho)} x(s)\mathrm{d}s,$$

by a similar argument to the one used in the proof of Theorem 5, we get

$$x(s) = 0 \quad \text{a.e.}(s) \text{ in } [0,\eta],$$

and, therefore, the solution $x(t)$ is identically equal to zero.

In the sequel, we present an example illustrating the result obtained in Theorem 5.

Example 2. Consider the following fractional boundary value problem

$$\begin{cases} {}^{c}D_{0+}^{\frac{1}{2}} x(t) = \lambda\left(t + \arctan x(t)\right), \quad 0 < t < 1 \\ x(0) = I_{0+}^{\frac{1}{2}} x\left(\frac{1}{3}\right) = \dfrac{1}{\Gamma\left(\frac{1}{3}\right)} \displaystyle\int_0^{\frac{1}{3}} \frac{x(s)}{\left(\frac{1}{3} - s\right)^{\frac{1}{2}}}\mathrm{d}s. \end{cases} \tag{14}$$

Notice that in this case, $\alpha = \frac{1}{2}$, $\eta = \frac{1}{3}$, $\rho = \frac{1}{2}$ and $\gamma = 1$, and, moreover, it is satisfied $\gamma \neq \frac{\Gamma(\rho+\frac{1}{3})}{\eta^\rho}$ since $1 \neq \frac{\Gamma\left(\frac{3}{2}\right)}{\sqrt{\frac{1}{3}}}$.

The function $f(t,x)$ is given by $f(t,x) = t + \arctan x$ and it is clear that $f\colon [0,1] \times [0,\infty) \to [0,\infty)$ and satisfies (i) and (ii) of Theorem 4.

Moreover, for any $x, y \in [0,\infty)$ with $y \geq x$ and $t \in [0,1]$, we have

$$f(t,y) - f(t,x) = \arctan y - \arctan x \leq \arctan(y - x) = \phi(y - x),$$

where we have used the fact that the inverse tangent is subadditive (because it is concave and its value in zero is zero) and where $\phi(t) = \arctan t$. It is easily seen that $\phi \in \mathcal{B}$.

Moreover, in this case, the inequality $\gamma < \frac{\Gamma(\rho+1)}{\eta^\rho}$ is satisfied, since

$$1 < \frac{\Gamma\left(\frac{3}{2}\right)}{\sqrt{\frac{1}{3}}}.$$

The constant L is given by

$$\begin{aligned} L &= \left|\frac{\gamma\Gamma(\rho+1)}{\Gamma(\rho+1)-\gamma\eta^\rho}\right| \frac{1}{(\rho+\alpha)\Gamma(\rho+\alpha)}\ \eta^{\rho+\alpha} + \frac{2}{\alpha\Gamma(\alpha)} \\ &= \left|\frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2}\right)-\sqrt{\frac{1}{3}}}\right| \frac{1}{\Gamma(1)}\ \frac{1}{3} + \frac{2}{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)} \\ &= \left|\frac{\frac{1}{2}\sqrt{\pi}}{\frac{\sqrt{\pi}}{2}-\frac{2}{\sqrt{3}}}\right|\ \frac{1}{3} + \frac{4}{\sqrt{\pi}} \\ &\simeq 1'15643. \end{aligned}$$

Since $f(t,0) = t$ and $f(\frac{1}{4},0) = \frac{1}{4} > 0$ and $\frac{1}{4} = t_0 < n = \frac{1}{3}$, by Theorem 5, Problem (14) has a unique positive solution in the space $\mathcal{H}_{\frac{1}{2}}[0,1]$.

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