Mediterr. J. Math. (2018) 15:93
https://doi.org/10.1007/s00009-018-1137-5
1660-5446/18/030001-21
*published online* April 24, 2018


Mediterranean Journal of Mathematics

# On the Arithmetic and Geometric Means of the First $n$ Prime Numbers

Christian Axler

**Abstract.** In this paper, we establish explicit upper and lower bounds for the ratio of the arithmetic and geometric means of the first $n$ prime numbers, which improve the current best estimates. Furthermore, we prove several conjectures related to this ratio stated by Hassani. To do this, we use explicit estimates for the prime counting function, Chebyshev's $\vartheta$-function, and the sum of the first $n$ primes.

**Mathematics Subject Classification.** Primary 11N05; Secondary 11A41.

**Keywords.** Arithmetic mean, Chebyshev's $\vartheta$-function, geometric mean.

## 1. Introduction

Let $a_n$ be the arithmetic mean and $g_n$ be the geometric mean of the first $n$ positive integers, respectively. Stirling's approximation for $n!$ implies that $a_n/g_n \to e/2$ as $n \to \infty$. In his paper [11], Hassani studied the arithmetic and geometric means of the first $n$ prime numbers, that is

$$A_n = \frac{1}{n}\sum_{k=1}^{n} p_k, \quad G_n = \left(\prod_{k=1}^{n} p_k\right)^{1/n}.$$

Here, as usual, $p_k$ denotes the $k$th prime number. Chebyshev's $\vartheta$-function is defined by $\vartheta(x) = \sum_{p \leq x} \log p$, where $p$ runs over primes not exceeding $x$. By setting $D(n) = \log p_n - \vartheta(p_n)/n$ and $R(n) = \sum_{k \leq n} p_k/n - p_n/2$, Hassani [11, p. 1595] derived the identity:

$$\log \frac{A_n}{G_n} = D(n) + \log\left(1 + \frac{2R(n)}{p_n}\right) - \log 2 \tag{1.1}$$

for the ratio of $A_n$ and $G_n$, which plays an important role in this paper. First, we establish asymptotic formulae for the quantities $D(n)$ and $G_n$ which help us to find the following asymptotic formula for the ratio of $A_n$ and $G_n$. Here, let $r_t = (t-1)!(1-1/2^t)$ and the positive integers $k_1, \ldots, k_s$, where $s$ is a positive integer, are defined by the recurrence formula $k_s + 1!k_{s-1} + \cdots + (s-1)!k_1 = s \cdot s!$.

Birkhäuser

**Theorem 1.1.** *For each positive integer $m$, we have*

$$\frac{A_n}{G_n} = e\left(\frac{1}{2} + \sum_{i=1}^{m} \frac{1}{\log^i p_n}\left(-r_{i+1} + r_i + \sum_{s=1}^{i-1} r_s k_{i-s}\right)\right) \cdot \exp\left(\sum_{j=1}^{m} \frac{k_j}{\log^j p_n}\right) + O\left(\frac{1}{\log^{m+1} p_n}\right).$$

One of Hassani's results [11, p. 1602] is that $A_n/G_n = e/2 + O(1/\log n)$ which implies that the ratio of $A_n$ and $G_n$ also tends to $e/2$ as $n \to \infty$. Setting $m = 2$ in Theorem 1.1, we get the following more accurate asymptotic formula:

$$\frac{A_n}{G_n} = \frac{e}{2} + \frac{e}{4\log p_n} + \frac{e}{\log^2 p_n} + O\left(\frac{1}{\log^3 p_n}\right). \tag{1.2}$$

Let $\pi(x)$ denote the prime counting function, which is defined by $\pi(x) = \sum_{p \leq x} 1$, where $p$ runs over primes not exceeding $x$. Using explicit estimates for the prime counting function $\pi(x)$ and the $n$th prime number $p_n$, Hassani [11, Theorem 1.1] found some explicit estimates for the ratio of $A_n$ and $G_n$. The proof of these estimates consists of three steps. First, he gave some explicit estimates for the quantities $D(n)$ and $\log(1 + 2R(n)/p_n)$, and then, he used (1.1). We follow this method to refine Hassani's estimates by showing the following both results in the direction of (1.2).

**Theorem 1.2.** *For every integer $n \geq 62$, we have*

$$\frac{A_n}{G_n} > \frac{e}{2} + \frac{e}{4\log p_n} + \frac{0.61e}{\log^2 p_n}.$$

**Theorem 1.3.** *For every integer $n \geq 294\,635$, we have*

$$\frac{A_n}{G_n} < \frac{e}{2} + \frac{e}{4\log p_n} + \frac{1.52e}{\log^2 p_n}.$$

Since the computation of $p_n$ is difficult for large $n$, the estimates for the ratio of $A_n$ and $G_n$ obtained in Theorems 1.2 and 1.3 are ineffective for large $n$. Hence, we are interested in explicit estimates for $A_n/G_n$ in terms of $n$. For this purpose, we find the following estimates.

**Theorem 1.4.** *For every integer $n \geq 139$, we have*

$$\frac{A_n}{G_n} > \frac{e}{2} + \frac{e}{4\log n} - \frac{e(\log\log n - 2.8)}{4\log^2 n}.$$

**Theorem 1.5.** *For every integer $n \geq 2$, we have*

$$\frac{A_n}{G_n} < \frac{e}{2} + \frac{e}{4\log n} - \frac{e(\log\log n - 6.44)}{4\log^2 n}.$$

In particular, we prove several conjectures concerning $D(n)$, $G_n$, and the ratio of $A_n$ and $G_n$ stated by Hassani [11]. For instance, we use Theorem 1.2 to show that the ratio of $A_n$ and $G_n$ is always greater than $e/2$ (see Corollary 7.1).

## 2. Several Asymptotic Formulae

Before we give a proof of Theorem 1.1, we derive asymptotic formulae for the quantities $D(n)$ and $G_n$.

### 2.1. Two Asymptotic Formulae for $D(n)$

To find the first asymptotic formula for

$$D(n) = \log p_n - \frac{\vartheta(p_n)}{n}$$

in terms of $p_n$, we introduce the following definition.

**Definition.** Let $m$ be a positive integer. We define the positive integers $k_1, \ldots, k_m$ by the recurrence formula:

$$k_m + 1!k_{m-1} + 2!k_{m-2} + \cdots + (m-1)!k_1 = m \cdot m!. \tag{2.1}$$

In particular, we have $k_1 = 1$, $k_2 = 3$, $k_3 = 13$, and $k_4 = 71$.

Then, we obtain the following result.

**Proposition 2.1.** *Let $r$ be a nonnegative integer. Then*

$$D(n) = 1 + \frac{k_1}{\log p_n} + \frac{k_2}{\log^2 p_n} + \cdots + \frac{k_r}{\log^r p_n} + O\left(\frac{1}{\log^{r+1} p_n}\right).$$

*Proof.* Using a result of Panaitopol [14], we get

$$\log x = \frac{x}{\pi(x)} + 1 + \frac{k_1}{\log x} + \frac{k_2}{\log^2 x} + \cdots + \frac{k_r}{\log^r x} + O\left(\frac{x}{\pi(x)\log^{r+2} x}\right). \tag{2.2}$$

The Prime Number Theorem states that $\pi(x) \sim x/\log x$ as $x \to \infty$. Therefore, we can simplify the error term in (2.2) as follows:

$$\log x = \frac{x}{\pi(x)} + 1 + \frac{k_1}{\log x} + \frac{k_2}{\log^2 x} + \cdots + \frac{k_r}{\log^r x} + O\left(\frac{1}{\log^{r+1} x}\right). \tag{2.3}$$

A well-known asymptotic formula for Chebyshev's $\vartheta$-function is given by $\vartheta(x) = x + O(x \exp(-c \log^{1/10} x))$, where $c$ is an absolute positive constant (see Brüdern [6, p. 41]). Now, the Prime Number Theorem and the fact that $\exp(-c\log^{1/10} x) = O(1/\log^s x)$ for every positive integer $s$ indicate that

$$\frac{\vartheta(p_n)}{n} = \frac{p_n}{n} + O\left(\frac{1}{\log^{r+1} p_n}\right). \tag{2.4}$$

Finally, we combine (2.4) with (2.3) to arrive at the end of the proof. □

Next, we establish another asymptotic formula for the quantity $D(n)$. To do this, we first note two useful results of Cipolla [8] concerning asymptotic formulae for the $n$th prime number $p_n$ and $\log p_n$, respectively. In this paper $lc(P)$ denotes the leading coefficient of a polynomial $P$.

**Lemma 2.2.** (Cipolla [8]) *Let $m$ be a positive integer. Then, there exist uniquely determined polynomials $Q_1, \ldots, Q_m \in \mathbb{Z}[x]$ with $\deg(Q_k) = k$ and $lc(Q_k) = (k-1)!$, so that*

$$\begin{aligned} p_n &= n\left(\log n + \log\log n - 1 + \sum_{k=1}^{m} \frac{(-1)^{k+1}Q_k(\log\log n)}{k!\log^k n}\right) \\ &\quad + O\left(\frac{n(\log\log n)^{m+1}}{\log^{m+1} n}\right). \end{aligned}$$

*The polynomials $Q_k$ can be computed explicitly. In particular, $Q_1(x) = x-2$, $Q_2(x) = x^2 - 6x + 11$ and $Q_3(x) = 2x^3 - 21x^2 + 84x - 131$.*

**Lemma 2.3.** (Cipolla [8]) *Let $m$ be a positive integer. Then, there exist uniquely determined polynomials $R_1, \ldots, R_m \in \mathbb{Z}[x]$ with $\deg(R_k) = k$ and $lc(R_k) = (k-1)!$, so that*

$$\log p_n = \log n + \log\log n + \sum_{k=1}^{m} \frac{(-1)^{k+1}R_k(\log\log n)}{k!\log^k n} + O\left(\frac{(\log\log n)^{m+1}}{\log^{m+1} n}\right).$$

*The polynomials $R_k$ can be computed explicitly. In particular, $R_1(x) = x-1$, $R_2(x) = x^2 - 4x + 5$ and $R_3(x) = 2x^3 - 15x^2 + 42x - 47$.*

Now, we give another asymptotic formula for the quantity $D(n)$.

**Proposition 2.4.** *Let $r$ be a positive integer and let $T_k(x) = R_k(x) - Q_k(x)$ for every $k \in \{1, \ldots, r\}$. Then, we have $\deg(T_k) = k-1$, $lc(T_k) = k!$, and*

$$D(n) = 1 + \sum_{k=1}^{r} \frac{(-1)^{k+1}T_k(\log\log n)}{k!\log^k n} + O\left(\frac{(\log\log n)^r}{\log^{r+1} n}\right).$$

*In particular, $T_1(x) = 1$, $T_2(x) = 2x - 6$, and $T_3(x) = 6x^2 - 42x + 84$.*

*Proof.* Let $k$ be an integer with $1 \leq k \leq r$. Since $\deg(Q_k) = \deg(R_k) = k$ and $lc(Q_k) = lc(R_k) = (k-1)!$, we have $\deg(T_k) \leq k-1$. Following Cipolla's notation (see [8, p. 144]), we write

$$Q_k(x) = (k-1)!x^k - a_{k,1}x^{k-1} + \sum_{j=2}^{k}(-1)^j a_{k,j}x^{k-j}$$

and

$$R_k(x) = (k-1)!x^k - b_{k,1}x^{k-1} + \sum_{j=2}^{k}(-1)^j b_{k,j}x^{k-j},$$

where $a_{i,j}, b_{i,j} \in \mathbb{Z}$. By Cipolla [8, p. 150], we have $-(b_{k,1} - a_{k,1}) = k! \neq 0$. Hence, $\deg(T_k) = k-1$ and $lc(T_k) = k!$. Using (2.4), we get

$$D(n) = \log p_n - \frac{p_n}{n} + O\left(\frac{1}{\log^{r+1} p_n}\right).$$

Now, we substitute the asymptotic formulae given in Lemmata 2.2 and 2.3 to obtain

$$D(n) = 1 + \sum_{k=1}^{r+1} \frac{(-1)^{k+1} T_k(\log\log n)}{k! \log^k n} + O\left(\frac{1}{\log^{r+1} p_n}\right).$$

To complete the proof, it suffices to note that $\deg(T_{r+1}) = r$ and $1/\log^{r+1} p_n = O(1/\log^{r+1} n)$. □

*Remark.* Proposition 2.4 gives a refinement of Hassani's [11] asymptotic formula $D(n) = 1 + O(1/\log n)$.

### 2.2. An Asymptotic Formula for $G_n$

Next, we derive an asymptotic formula for $G_n$, the geometric mean of the first $n$ prime numbers. By the defining formulas for $G_n$ and $D(n)$, we see that

$$G_n = \frac{p_n}{e^{D(n)}}. \tag{2.5}$$

Proposition 2.1 implies that $\lim_{n\to\infty} D(n) = 1$. Hence

$$G_n \sim \frac{p_n}{e} \qquad (n \to \infty), \tag{2.6}$$

which was conjectured by Vrba [15] and proved by Sándor and Verroken [18, Theorem 2.1]. In [17, Corollary 2.1], Sándor gave another proof of (2.6). Using (2.5) and Proposition 2.1, we get the following refinement of (2.6). Here, the positive integers $k_1, \ldots, k_r$ are defined by the recurrence formula (2.1).

**Proposition 2.5.** *Let $r$ be a positive integer. Then*

$$G_n = \frac{p_n}{\exp\left(1 + \frac{k_1}{\log p_n} + \frac{k_2}{\log^2 p_n} + \cdots + \frac{k_r}{\log^r p_n}\right)} + O\left(\frac{p_n}{\log^{r+1} p_n}\right). \tag{2.7}$$

*Proof.* The claim follows from (2.5), Proposition 2.1, and the formula $\exp(c/x) = 1 + O(1/x)$ that holds for every $c \in \mathbb{R}$. □

*Remark.* The asymptotic formula (2.7) was independently found by Kourbatov [12, Remark (ii)].

## 3. A Proof of Theorem 1.1

We use (2.5), Proposition 2.1, and an asymptotic formula for $A_n$ given in [1, Theorem 2] to give a proof of Theorem 1.1. Below, we use the notation:

$$r_i = (i-1)!\left(1 - \frac{1}{2^i}\right),$$

the positive integers $k_i$ are defined by (2.1).

**Theorem 3.1.** *For each positive integer $m$, we have*

$$\frac{A_n}{G_n} = e\left(\frac{1}{2} + \sum_{i=1}^{m} \frac{1}{\log^i p_n}\left(-r_{i+1} + r_i + \sum_{s=1}^{i-1} r_s k_{i-s}\right)\right) \cdot \exp\left(\sum_{j=1}^{m} \frac{k_j}{\log^j p_n}\right) + O\left(\frac{1}{\log^{m+1} p_n}\right).$$

*Proof.* By [1, Theorem 2], we have

$$A_n = p_n - \sum_{i=1}^{m-1} \frac{r_i p_n^2}{n \log^i p_n} + O\left(\frac{p_n^2}{n \log^m p_n}\right).$$

We combine this asymptotic formula with (2.5) and Proposition 2.1 to see that

$$\frac{A_n}{G_n} = \left(1 - \sum_{i=1}^{m+1} \frac{r_i p_n}{n \log^i p_n} + O\left(\frac{p_n}{n \log^{m+2} p_n}\right)\right) \cdot \left(\exp\left(1 + \sum_{j=1}^{m} \frac{k_j}{\log^j p_n}\right) + O\left(\frac{1}{\log^{m+1} p_n}\right)\right).$$

The Prime Number Theorem implies that $p_n \sim n \log p_n$ as $n \to \infty$. It follows:

$$\frac{A_n}{G_n} = e\left(1 - \sum_{i=1}^{m+1} \frac{r_i p_n}{n \log^i p_n} + O\left(\frac{1}{\log^{m+1} p_n}\right)\right) \cdot \exp\left(\sum_{j=1}^{m} \frac{k_j}{\log^j p_n}\right) + O\left(\frac{1}{\log^{m+1} p_n}\right). \tag{3.1}$$

Applying (2.3) with $x = p_n$ and $r = m - 1$ to (3.1), we get

$$\frac{A_n}{G_n} = e\left(1 - \sum_{i=1}^{m+1} \frac{r_i}{\log^i p_n}\left(\log p_n - 1 - \sum_{s=1}^{m-1} \frac{k_s}{\log^s p_n}\right)\right) \cdot \exp\left(\sum_{j=1}^{m} \frac{k_j}{\log^j p_n}\right) + O\left(\frac{1}{\log^{m+1} p_n}\right).$$

Hence, we have

$$\frac{A_n}{G_n} = e\left(1 - \sum_{i=1}^{m+1} \frac{r_i}{\log^{i-1} p_n} + \sum_{i=1}^{m+1} \frac{r_i}{\log^i p_n} + \sum_{i=1}^{m+1} \frac{k_1 r_i}{\log^{i+1} p_n} + \cdots + \sum_{i=1}^{m+1} \frac{k_{m-1} r_i}{\log^{m-1+i} p_n}\right) \times \exp\left(\sum_{j=1}^{m} \frac{k_j}{\log^j p_n}\right) + O\left(\frac{1}{\log^{m+1} p_n}\right).$$

To complete the proof, we separate the terms in the first parentheses which are $O(1/\log^{m+1} p_n)$. □

Now, we use Theorem 3.1 and the asymptotic formula

$$\exp\left(\sum_{j=1}^{m}\frac{k_j}{\log^j p_n}\right)=\sum_{i=1}^{m}\frac{1}{i!}\left(\sum_{j=1}^{m}\frac{k_j}{\log^j p_n}\right)^i+O\left(\frac{1}{\log^{m+1} p_n}\right)$$

to implement the following Maple code:

```
> restart:
```

*Computation of the values $k_i$:*

```
> for j from 1 to m do
    K[j] := j*j!-sum(s!*K[j-s], s=1..j-1):
```

*Computation of the values $r_i$:*

```
> for i from 1 to m+1 do
    R[i] := (i-1)!*(1-1/2^{i}):
  end do:
> AsymptoticExpansion := proc(n) local S1,S2;
    S1 := 1/2 + sum(b^{w}*(-R[w+1]+R[w]+sum(R[v]*K[w-v],
   v = 1.. (w-1))), w = 1..n);
    S2 := sum(1/t!*(sum(K[z]*b^{z}, z = 1..n))^{t}, t = 0..n));
    RETURN(subs(b = 1/log(p_n), convert(series(S1*S2, b,n+1),
   polynom)));
    end;
```

To give the explicit asymptotic expansion for the ratio of $A_n$ and $G_n$ up to some positive integer $m$, it suffices to write

```
> expand(exp(1)*AsymptoticExpansion(m));
```

For instance, we set $m = 5$ to obtain

$$\frac{A_n}{G_n}=\frac{e}{2}+\frac{e}{4\log p_n}+\frac{e}{\log^2 p_n}+\frac{61e}{12\log^3 p_n}+\frac{1463e}{48\log^4 p_n}+\frac{100367e}{480\log^5 p_n}+O\left(\frac{1}{\log^6 p_n}\right). \tag{3.2}$$

One of Hassani's results [11, p. 1602] is that $A_n/G_n = e/2+O(1/\log n)$. The asymptotic expansion given in (3.2) improves this result.

## 4. New Estimates for the Quantity $D(n)$

After giving two asymptotic formulae for the quantity $D(n)$ in Sect. 2.1, we are interested in finding some explicit estimates for $D(n)$.

### 4.1. Explicit Estimates for $D(n)$ in Terms of $p_n$

In this subsection, we give some explicit estimates for $D(n)$ in terms of $p_n$ which correspond to the first three terms of the asymptotic expansion given in Proposition 2.1. We start with the following lower bound.

**Proposition 4.1.** *For every integer $n \geq 218$, we have*

$$D(n) > 1 + \frac{1}{\log p_n} + \frac{2.7}{\log^2 p_n}. \tag{4.1}$$

*Proof.* Substituting $x = p_n$ in [3, Corollary 3.9], we get

$$\log p_n > \frac{p_n}{n} + 1 + \frac{1}{\log p_n} + \frac{2.85}{\log^2 p_n} \tag{4.2}$$

for every integer $n \geq 2\,324\,692$. In [3, Theorem 1.1], it is shown that $\vartheta(x) < x + 0.15x/\log^3 x$ for every $x > 1$. Applying the last inequality to (4.2), we see that

$$D(n) > 1 + \frac{1}{\log p_n} + \frac{2.85}{\log^2 p_n} - \frac{0.15 p_n}{n \log^3 p_n} \tag{4.3}$$

for every integer $n \geq 2\,324\,692$. Setting $x = p_k$ in [16, Corollary 1], we get

$$p_k \leq k \log p_k \tag{4.4}$$

for every integer $k \geq 7$. Now, it suffices to apply (4.4) to (4.3) to see that the inequality (4.1) holds for every integer $n \geq 2\,324\,692$. For smaller values of $n$, we use a computer. □

In the following proposition, we give two lower bounds for $D(n)$ which improve the inequality (4.1) for all sufficiently large values of $n$.

**Proposition 4.2.** *For every positive integer $n$, we have*

$$D(n) > 1 + \frac{1}{\log p_n} + \frac{3}{\log^2 p_n} - \frac{187}{\log^3 p_n} \tag{4.5}$$

*and*

$$D(n) > 1 + \frac{1}{\log p_n} + \frac{3}{\log^2 p_n} + \frac{13}{\log^3 p_n} - \frac{1160159}{\log^4 p_n}. \tag{4.6}$$

*Proof.* We start with the proof of (4.5). By [3, Proposition 2.5], we have $|\vartheta(x) - x| < 100x/\log^4 x$ for every $x \geq 70\,111 = p_{6946}$. Furthermore, in [3, Proposition 3.10] it is found that the inequality

$$\pi(x) > \frac{x}{\log x - 1 - \frac{1}{\log x} - \frac{3}{\log^2 x} + \frac{87}{\log^3 x}}$$

holds for every $x \geq 19\,423$. With an argument similar to the one used in the proof of Proposition 4.1, we get the inequality (4.5) for every integer $n \geq 6\,946$. We conclude by direct computation.

Next, we give the proof of (4.6). In [4, Corollary 2.2], it is shown that the inequality $|\vartheta(x) - x| < 580115x/\log^5 x$ holds for every $x \geq 2$. Furthermore, we see from the proof of [4, Theorem 1.1] that

$$\pi(x) > \frac{x}{\log x - 1 - \frac{1}{\log x} - \frac{3}{\log^2 x} - \frac{13}{\log^3 x} + \frac{580044}{\log^4 x}}$$

for every $x \geq 10^{13}$. Again, with an argument similar to the one used in the proof of Proposition 4.1, we get the inequality (4.6) for every integer $n \geq \pi(10^{13}) + 1$. For smaller values of $n$, we use (4.5). □

Since $k_1 = 1$ and $k_2 = 3$, Proposition 2.1 implies that there is a smallest positive integer $N_0$, so that

$$D(n) > 1 + \frac{1}{\log p_n} + \frac{3}{\log^2 p_n} \tag{4.7}$$

for every integer $n \geq N_0$. In the following corollary, we make the first step to find this $N_0$.

**Corollary 4.3.** *For every integer* $n$ *satisfying* $264 \leq n \leq \pi(10^{19}) = 234\,057\,667\,276\,344\,607$ *and* $n \geq \pi(e^{1160159/13}) + 1$*, the inequality* (4.7) *holds.*

*Proof.* The inequality (4.6) implies the correctness of (4.7) for every $n \geq \pi(e^{1160159/13}) + 1$. So it suffices to prove that the inequality (4.7) holds for every integer $n$ with $264 \leq n \leq \pi(10^{19})$. By [4, Theorem 1.1], we have

$$\pi(x) > \frac{x}{\log x - 1 - \frac{1}{\log x} - \frac{3}{\log^2 x}} \tag{4.8}$$

for every $x$ satisfying $65\,405\,887 \leq x \leq 5.5 \cdot 10^{25}$ and $x \geq e^{580044/13}$. Büthe [7, Theorem 2] found that $\vartheta(x) < x$ for every $x$, such that $1 \leq x \leq 10^{19}$. With an argument similar to the one used in the proof of Proposition 4.1, we use (4.8) and Büthe's result to see that the inequality (4.7) holds for every integer $n$ with $\pi(65\,405\,887) \leq n \leq \pi(10^{19})$. Finally, we check the remaining cases with a computer. □

The following conjecture is based on Corollary 4.3.

**Conjecture 4.4.** *The inequality* (4.7) *holds for every integer* $n \geq 264$.

Next, we establish some explicit upper bounds for $D(n)$ in terms of $p_n$. From Proposition 2.1, it follows that for each $\varepsilon > 0$, there is a positive integer $N_1 = N_1(\varepsilon)$, such that

$$D(n) < 1 + \frac{1}{\log p_n} + \frac{3 + \varepsilon}{\log^2 p_n}$$

for every integer $n \geq N_1$. We find the following upper bound for $D(n)$.

**Proposition 4.5.** *For every integer* $n \geq 74\,004\,585$*, we have*

$$D(n) < 1 + \frac{1}{\log p_n} + \frac{3.84}{\log^2 p_n}, \tag{4.9}$$

*and for every positive integer* $n$*, we have*

$$D(n) < 1 + \frac{1}{\log p_n} + \frac{3}{\log^2 p_n} + \frac{213}{\log^3 p_n}. \tag{4.10}$$

*Proof.* We start with the proof of (4.9) and first consider the case, where $n \geq 841\,508\,302$. From [3, Corollary 3.3], it follows that:

$$\log p_n < \frac{p_n}{n} + 1 + \frac{1}{\log p_n} + \frac{3.69}{\log^2 p_n}. \tag{4.11}$$

Furthermore, by [3, Theorem 1.1], we have $\vartheta(x) > x - 0.15x/\log^3 x$ for every $x \geq 19\,035\,709\,163 = p_{841\,508\,302}$. We combine the last inequality involving $\vartheta(x)$ and (4.11) to get

$$D(n) < 1 + \frac{1}{\log p_n} + \frac{3.69}{\log^2 p_n} + \frac{0.15 p_n}{n \log^3 p_n}.$$

Now, we use (4.4) to see that the inequality (4.9) holds for every integer $n \geq 841\,508\,302$. For smaller values of $n$, we check the required inequality with a computer.

Next, we establish the inequality (4.10). In [3, Proposition 3.5], it is shown that

$$\pi(x) < \frac{x}{\log x - 1 - \frac{1}{\log x} - \frac{3}{\log^2 x} - \frac{113}{\log^3 x}}$$

for every $x \geq 41$. By [3, Proposition 2.5], we have $|\vartheta(x) - x| < 100x/\log^4 x$ for every $x \geq 70\,111$. Now, we argue as in the proof of Proposition 4.2. For the remaining cases, we use a computer. □

### 4.2. Explicit Estimates for $D(n)$ in Terms of $n$

Since computation of $p_n$ is difficult for large $n$, the estimates for $D(n)$ obtained in Sect. 4.1 are ineffective for large $n$. Hence, we are interested in estimates for $D(n)$ in terms of $n$. First, we note that Proposition 2.4 implies the asymptotic formula:

$$D(n) = 1 + \frac{1}{\log n} - \frac{\log\log n - 3}{\log^2 n} + O\left(\frac{(\log\log n)^2}{\log^3 n}\right). \tag{4.12}$$

The goal of this subsection is to find upper and lower bounds for $D(n)$ in the direction of (4.12). We start with lower bounds. Hassani [11, Proposition 1.6] showed that the inequality $D(n) > 1 - 17/(5\log n)$ is valid for every integer $n \geq 2$. Here, we give the following refinement.

**Proposition 4.6.** *For every integer* $n \geq 591$, *we have*

$$D(n) > 1 + \frac{1}{\log n} - \frac{\log\log n - 2.5}{\log^2 n}. \tag{4.13}$$

*Proof.* We denote the right-hand side of (4.13) by $f(n)$. First, let $n$ be an integer satisfying $n \geq \pi(10^{19}) = 234\,057\,667\,276\,344\,607$. By [5, Corollary 3.3], we have

$$\frac{1}{\log p_n} \geq \frac{1}{\log n} - \frac{\log\log n}{\log^2 n} + \frac{(\log\log n)^2 - \log\log n + 1}{\log^2 n \log p_n}, \tag{4.14}$$

which implies that the weaker inequality

$$\frac{1}{\log p_n} \geq \frac{1}{\log n} - \frac{\log\log n}{\log^2 n} \tag{4.15}$$

also holds. We combine (4.14) and (4.15) to get

$$\frac{1}{\log p_n} \geq \frac{1}{\log n} - \frac{\log\log n}{\log^2 n} + \frac{(\log\log n)^2 - \log\log n + 1}{\log^2 n} \times \left(\frac{1}{\log n} - \frac{\log\log n}{\log^2 n}\right). \tag{4.16}$$

Applying the last inequality and (4.15) to (4.1), we see that

$$D(n) > f(n) + \frac{0.2}{\log^2 n} + \frac{(\log\log n)^2 - 5.6\log\log n + 1}{\log^3 n} - \frac{(\log\log n)^3 - 3.3(\log\log n)^2 + \log\log n}{\log^4 n},$$

which completes the proof for every integer $n \geq \pi(10^{19})$.

With an argument similar to the one used in the case $n \geq \pi(10^{19})$, we combine (4.15), (4.16), and Corollary 4.3 to get

$$D(n) > f(n) + \frac{0.5}{\log^2 n} + \frac{(\log\log n)^2 - 7\log\log n + 1}{\log^3 n} - \frac{(\log\log n)^3 - 4(\log\log n)^2 + \log\log n}{\log^4 n}$$

for every integer $n$ with $264 \leq n \leq \pi(10^{19})$, which implies that the required inequality holds for every integer $n$, such that $2\,426\,927\,728 \leq n \leq \pi(10^{19})$. We verify the remaining cases with a computer. □

We immediately get the following corollary.

**Corollary 4.7.** *For every $\alpha < 1$, there exists a positive integer $n_0 = n_0(\alpha)$, so that $D(n) > 1 + \alpha/\log n$ for every integer $n \geq n_0$.*

*Remark.* Hassani [11, Conjecture 1.7] conjectured that there exist a real number $\beta$ with $0 < \beta < 5.25$ and a positive integer $n_0$, so that $D(n) > 1+\beta/\log n$ for every integer $n \geq n_0$. This conjecture is proved in Corollary 4.7. The inequality (4.13) implies that $D(n) > 1$ for every integer $n \geq 591$. A computer check shows that the last inequality for $D(n)$ also holds for every integer $n$ with $10 \leq n \leq 591$. Thus, we have

$$D(n) > 1 \qquad (n \geq 10), \tag{4.17}$$

which was also conjectured by Hassani [11, Conjecture 1.7].

Finally, we establish some new upper bounds for $D(n)$ in terms of $n$. Using estimates for the $n$th prime number and Chebyshev's $\vartheta$-function, Hassani [11, Proposition 1.6] found that $D(n) < 1 + 21/(4\log n)$ for every integer $n \geq 2$. We give the following improvement of Hassani's upper bound.

**Proposition 4.8.** *For every integer $n \geq 2$, we have*

$$D(n) < 1 + \frac{1}{\log n} - \frac{\log\log n - 4.2}{\log^2 n}.$$

*In particular, for every $\beta \geq 1$, there exists a positive integer $n_1 = n_1(\beta)$, so that $D(n) < 1 + \beta/\log n$ for every integer $n \geq n_1$.*

*Proof.* By [5, Corollary 3.6], we have

$$\frac{1}{\log p_n} \leq \frac{1}{\log n} - \frac{\log\log n}{\log^2 n} + \frac{(\log\log n)^2 - \log\log n + 1}{\log^2 n \log p_n} + \frac{P_8(\log\log n)}{2\log^3 n \log p_n} - \frac{P_9(\log\log n)}{2\log^4 n \log p_n}$$

for every integer $n \geq 2$, where $P_8(x) = 3x^2 - 6x + 5.2$ and $P_9(x) = x^3 - 6x^2 + 11.4x - 4.2$. Since $P_9(x) > 0$ for every $x \geq 0.5$, we get

$$\frac{1}{\log p_n} \leq \frac{1}{\log n} - \frac{\log\log n}{\log^2 n} + \frac{(\log\log n)^2 - \log\log n + 1}{\log^3 n} + \frac{P_8(\log\log n)}{2\log^4 n} \tag{4.18}$$

for every $n \geq 6$. Now, we use Proposition 4.5 and the fact that $3.84/\log^2 p_n \leq 3.84/\log^2 n$ to obtain

$$D(n) < 1 + \frac{1}{\log n} - \frac{\log\log n - 3.84}{\log^2 n} + \frac{(\log\log n)^2 - \log\log n + 1}{\log^3 n} + \frac{P_8(\log\log n)}{2\log^4 n} \tag{4.19}$$

for every integer $n \geq 74\,004\,585$. Notice that the inequality

$$\frac{(\log\log x)^2 - \log\log x + 1}{\log^3 x} + \frac{P_8(\log\log x)}{2\log^4 x} < \frac{0.36}{\log^2 x} \tag{4.20}$$

holds for every $x \geq 1\,499\,820\,545$. Applying (4.20) to (4.19), we get the required inequality for every $n \geq 1\,499\,820\,545$. Finally, we use a computer to check the required inequality for smaller values of $n$. □

## 5. New Estimates for the Geometric Mean of the First $n$ Prime Numbers

In the following, we use the identity (2.5) and the explicit estimates for $D(n)$ obtained in the previous section to find new bounds for $G_n$, the geometric mean of the first $n$ prime numbers. First, we see that (2.5) and (4.17) imply $G_n < p_n/e$ for every integer $n \geq 10$, which was already proved by Panaitopol [13] and Sándor [17, Theorem 2.1]. In the direction of Proposition 2.5, Kourbatov [12, Theorem 2] used explicit estimates for $\pi(x)$ and Chebyshev's $\vartheta$-function to find that

$$G_n < \frac{p_n}{\exp\left(1 + \frac{1}{\log p_n} + \frac{1.62}{\log^2 p_n}\right)}$$

for every prime number $p_n \geq 32059$, i.e., for every integer $n \geq 3439$. Actually, this inequality holds for every integer $n$ with $92 \leq n \leq 3438$, as well. In the next proposition, we give sharper estimates for $G_n$.

**Proposition 5.1.** *For every integer $n \geq 218$, we have*

$$G_n < \frac{p_n}{\exp\left(1 + \frac{1}{\log p_n} + \frac{2.7}{\log^2 p_n}\right)}.$$

*For every positive integer $n$, we have*

$$G_n < \frac{p_n}{\exp\left(1 + \frac{1}{\log p_n} + \frac{3}{\log^2 p_n} - \frac{187}{\log^3 p_n}\right)}$$

*and*

$$G_n < \frac{p_n}{\exp\left(1 + \frac{1}{\log p_n} + \frac{3}{\log^2 p_n} + \frac{13}{\log^3 p_n} - \frac{1160159}{\log^4 p_n}\right)}.$$

*Proof.* The first inequality is a direct consequence of (2.5) and Proposition 4.1. Furthermore, we apply the inequalities obtained in Proposition 4.2 to the identity (2.5) and get the remaining inequalities. □

Next, we use Corollary 4.3 to get the following upper bound.

**Proposition 5.2.** *For every integer $n$ with $264 \leq n \leq \pi(10^{19}) = 234\,057\,667\,276\,344\,607$ and $n \geq \pi(e^{1160159/13}) + 1$, we have*

$$G_n < \frac{p_n}{\exp\left(1 + \frac{1}{\log p_n} + \frac{3}{\log^2 p_n}\right)}.$$

*Proof.* We combine (2.5) with Corollary 4.3. □

Proposition 2.5 and the Prime Number Theorem imply that

$$G_n = \frac{p_n}{e} + O(n), \tag{5.1}$$

which was already obtained by Hassani [11, p. 1602]. To find new upper bounds for $G_n$ in the direction of (5.1), we first state the following result.

**Proposition 5.3.** *For every integer $n \geq 47$, we have*

$$G_n < \frac{p_n}{e}\left(1 - \frac{1}{\log p_n}\right).$$

*Proof.* Using Proposition 5.1 and the inequality $e^x \geq 1 + x$ which holds for every real $x$, we get

$$G_n < \frac{p_n}{e}\left(1 - \frac{\log p_n + 2.7}{\log^2 p_n + \log p_n + 2.7}\right)$$

for every integer $n \geq 218$. Since $\log p_m > 2.7/1.7$ for every integer $m \geq 3$, we obtain the required inequality for every integer $n \geq 218$. For the remaining cases of $n$, we use a computer. □

In the direction of (5.1), we find the following upper bound for $G_n$.

**Corollary 5.4.** *For every integer $n \geq 31$, we have*

$$G_n < \frac{p_n}{e} - \frac{n}{e}\left(1 - \frac{1}{\log p_n} - \frac{1}{\log^2 p_n} - \frac{3.69}{\log^3 p_n}\right).$$

*In particular, for every real $\gamma$ with $0 < \gamma < 1/e$, there is a positive integer $n_2 = n_2(\gamma)$, so that $G_n < p_n/e - \gamma n$ for every integer $n \geq n_2$.*

*Proof.* We use (4.11) and Proposition 5.3 to get the required inequality for every integer $n \geq 456\,441\,574$. We conclude by direct computation. □

*Remark.* The second part of Corollary 5.4 proves a conjecture stated by Hassani [11, Conjecture 4.3].

Next, we find new lower bounds for $G_n$. In view of Proposition 2.5, Kourbatov [12, Theorem 2] used explicit estimates for the prime counting function $\pi(x)$ and Chebyshev's $\vartheta$-function to find that

$$G_n > \frac{p_n}{\exp\left(1 + \frac{1}{\log p_n} + \frac{4.83}{\log^2 p_n}\right)}$$

for every integer $n \geq 3\,439$. In the next proposition, we give two sharper lower bounds.

**Proposition 5.5.** *For every integer $n \geq 74\,004\,585$, we have*

$$G_n > \frac{p_n}{\exp\left(1 + \frac{1}{\log p_n} + \frac{3.84}{\log^2 p_n}\right)}, \tag{5.2}$$

*and for every positive integer $n$, we have*

$$G_n > \frac{p_n}{\exp\left(1 + \frac{1}{\log p_n} + \frac{3}{\log^2 p_n} + \frac{213}{\log^3 p_n}\right)}.$$

*Proof.* We use (2.5) and Proposition 4.5 to obtain the required inequalities. □

To derive a lower bound for $G_n$ in the direction of (5.1), we first establish the following result.

**Proposition 5.6.** *For every positive integer $n$, we have*

$$G_n > \frac{p_n}{e}\left(1 - \frac{1}{\log p_n} - \frac{4.74}{\log^2 p_n}\right).$$

*Proof.* First, we consider the case where $n \geq 883\,051\,281 = \pi(e^{23.72}) + 1$. It is easy to see that

$$e^t < 1 + t + \frac{2t^2}{3} \tag{5.3}$$

for every $t$ satisfying $0 < t < \log(4/3)$. Hence, we obtain

$$\exp\left(\frac{1}{x} + \frac{3.84}{x^2}\right) < 1 + \frac{1}{x} + \frac{13.52}{3x^2} + \frac{5.12}{x^3} + \frac{9.8304}{x^4}$$

for every $x \geq 6$. For $x \geq 23.72$, we have $5.12/x + 9.8304/x^2 < 0.23333$ and get

$$\exp\left(\frac{1}{x} + \frac{3.84}{x^2}\right) < 1 + \frac{1}{x} + \frac{4.74}{x^2} \tag{5.4}$$

for every $x \geq 23.72$. Since $\log p_n \geq 23.72$, it follows from (5.2) and the inequality (5.4) that

$$G_n > \frac{p_n}{e}\left(1 - \frac{\log p_n + 4.74}{\log^2 p_n + \log p_n + 4.74}\right).$$

Since the right-hand side of the last inequality is greater than the right-hand side of the required inequality, the corollary is proved for every integer

$n \geq 883\,051\,281$. A computer check shows that the desired inequality holds for every integer $n$ with $1 \leq n \leq 883\,051\,280$, as well. □

In view of (5.1), Hassani [11, Corollary 4.2] found that

$$G_n > \frac{p_n}{e} - 2.37n$$

for every positive integer $n$. The following corollary improves this inequality.

**Corollary 5.7.** *For every integer $n \geq 3$, we have*

$$G_n > \frac{p_n}{e} - \frac{n}{e}\left(1 + \frac{3.74}{\log p_n} - \frac{5.74}{\log^2 p_n} - \frac{7.59}{\log^3 p_n}\right).$$

*In particular, for every $\delta > 1/e$, there is a positive integer $n_3 = n_3(\delta)$, so that $G_n > p_n/e - \delta n$ for every integer $n \geq n_3$.*

*Proof.* First, we consider the case where $n \geq 2\,324\,692$. We use (4.2) and the inequality obtained in Proposition 5.6 to get

$$G_n > \frac{p_n}{e} - \frac{n}{e}\left(1 - \frac{1}{\log p_n} - \frac{1}{\log^2 p_n} - \frac{2.85}{\log^3 p_n}\right) - \frac{4.74 p_n}{e \log^2 p_n}. \tag{5.5}$$

We apply the inequality (4.2) to (5.5) and obtain the required inequality. For every integer $n$ satisfying $3 \leq n \leq 2\,324\,692$ we check the required inequality with a computer. □

*Remark.* Compared with (5.1), Corollaries 5.4 and 5.7 yield the more accurate asymptotic formula:

$$G_n = \frac{p_n}{e} - \frac{n}{e} + O\left(\frac{n}{\log p_n}\right).$$

## 6. New Estimates for the Quantity $\log(1 + 2R(n)/p_n)$

First, we derive an asymptotic formula for $\log(1 + 2R(n)/p_n)$ as $n \to \infty$, where

$$R(n) = \frac{1}{n}\sum_{k \leq n} p_k - \frac{p_n}{2}.$$

**Proposition 6.1.** *We have*

$$\log\left(1 + \frac{2R(n)}{p_n}\right) \sim -\frac{1}{2\log n} \qquad (n \to \infty).$$

*Proof.* We have $p_n \sim n \log n$ and, by [2], $R(n) \sim -n/4$ as $n \to \infty$. Hence, $\log(1 + 2R(n)/p_n) \sim \log(1 - 1/(2\log n))$ as $n \to \infty$. Since $\log(1 - 1/(2x)) \sim -1/(2x)$ as $x \to \infty$, the proposition is proved. □

Hassani [11, Corollary 1.5] found that

$$-\frac{15}{2\log n} < \log\left(1 + \frac{2R(n)}{p_n}\right) < -\frac{5}{36\log n}, \tag{6.1}$$

where the left-hand side inequality holds for every integer $n \geq 2$, and the right-hand side inequality holds for every integer $n \geq 10$. In Proposition 6.1, we gave a more suitable approximation for the quantity $\log(1 + 2R(n)/p_n)$ as $n \to \infty$. In this direction, we improve the inequalities found in (6.1). The following proposition is about a lower bound for $\log(1 + 2R(n)/p_n)$.

**Proposition 6.2.** *For every integer* $n \geq 26\,220$, *we have*

$$\log\left(1 + \frac{2R(n)}{p_n}\right) > -\frac{1}{2\log n} + \frac{\log\log n - 2.25}{2\log^2 n} - \frac{(\log\log n)^2 - 4.5\log\log n + 22.51/3}{2\log^3 n}. \tag{6.2}$$

*Proof.* By Dusart [10], we have

$$p_n \geq r(n) \tag{6.3}$$

for every integer $n \geq 2$, where $r(x) = x(\log x + \log\log x - 1)$. We set

$$s_1(x) = -\frac{x}{4} - \frac{x}{4\log x} + \frac{x(\log\log x - 4.42)}{4\log^2 x}.$$

Now, we use [2, Theorem 1.8] and [11, Corollary 1.5] to obtain

$$s_1(n) < R(n) < 0 \tag{6.4}$$

for every integer $n \geq 256\,376$. We define $h(x) = \log(1 + 2s_1(x)/r(x))$. By (6.3) and (6.4), it suffices to show that $h(x)$ is greater than the right-hand side of (6.2). For this purpose, we set

$$\begin{aligned} f(y) = {} & (2\log^3 y - 13\log^2 y + 33.09\log y - 29.1575)y^3 \\ & + (1.5\log^4 y - 11.5\log^3 y + 34.63\log^2 y - 41.1575\log y + 9.9075)y^2 \\ & + (-0.5\log^3 y + 1.52\log^2 y + 3.59\log y - 17.01605)y \\ & + 0.75\log^4 y - 7.94\log^3 y + 31.07\log^2 y - 53.72605\log y + 29.84605 \end{aligned}$$

and

$$g(y) = y^3 + y^2\log y - 1.5y^2 - 0.5y + 0.5\log y - 2.21.$$

It is easy to see that $f(y)$ and $g(y)$ are positive for every $y \geq e^{2.4}$. Hence

$$\begin{aligned} & \left(h(x) + \frac{1}{2\log x} - \frac{\log\log x - 2.25}{2\log^2 x} + \frac{(\log\log x)^2 - 4.5\log\log x + 22.51/3}{2\log^3 x}\right)' \\ & \quad = -\frac{f(\log x)}{g(\log x)r(x)\log^4 x} < 0 \end{aligned}$$

for every $x \geq \exp(\exp(2.4))$. In addition, we have

$$\lim_{x\to\infty}\left(h(x) + \frac{1}{2\log x} - \frac{\log\log x - 2.25}{2\log^2 x} + \frac{(\log\log x)^2 - 4.5\log\log x + 22.51/3}{2\log^3 x}\right) = 0.$$

Thus, we get

$$h(x) > -\frac{1}{2\log x} + \frac{\log\log x - 2.25}{2\log^2 x} - \frac{(\log\log x)^2 - 4.5\log\log x + 22.51/3}{2\log^3 x}$$

for every $x \geq \exp(\exp(2.4))$. Hence, the desired inequality holds for every integer $n \geq \exp(\exp(2.4))$. The remaining cases are checked with a computer. $\square$

Next, we give an upper bound for $\log(1 + 2R(n)/p_n)$.

**Proposition 6.3.** *For every integer $n \geq 6077$, we have*

$$\log\left(1 + \frac{2R(n)}{p_n}\right) < -\frac{1}{2\log p_n} - \frac{1}{\log^2 p_n} - \frac{2.9}{2\log^2 n \log p_n}.$$

*Proof.* First, we consider the case where $n \geq 78\,150\,372 \geq \exp(\exp(2.9))$. We define

$$s_2(x) = -\frac{x}{4} - \frac{x}{4\log x} + \frac{x(\log\log x - 2.9)}{4\log^2 x}.$$

By [2, Theorem 1.7] and the definition of $R(n)$, we obtain $R(n) < s_2(n) < 0$. Hence

$$\log\left(1 + \frac{2R(n)}{p_n}\right) < \log\left(1 + \frac{2s_2(n)}{p_n}\right).$$

Since $2s_2(n)/p_n > -1$, we apply the inequality $\log(1+x) \leq x$, which holds for every $x > -1$, to get

$$\log\left(1 + \frac{2R(n)}{p_n}\right) < -\frac{n}{2p_n} - \frac{n}{2p_n\log n} + \frac{n(\log\log n - 2.9)}{2p_n\log^2 n}.$$

Now, we use a lower bound for the prime counting function given by Dusart [9], namely that $\pi(x) \geq x/\log x + x/\log^2 x$ for every $x \geq 599$. Substituting $x = p_n$, we obtain

$$\log\left(1 + \frac{2R(n)}{p_n}\right) < -\frac{1}{2\log p_n} - \frac{1}{2\log^2 p_n} - \frac{1}{2\log n\log p_n} - \frac{1}{2\log n\log^2 p_n} + \frac{n(\log\log n - 2.9)}{2p_n\log^2 n}. \tag{6.5}$$

Again by Dusart [9], we have $\pi(x) \leq x/\log x + 2x/\log^2 x$ for every $x > 1$. Applying this and the fact that $\log\log n \geq 2.9$ to (6.5) we get

$$\log\left(1 + \frac{2R(n)}{p_n}\right) < -\frac{1}{2\log p_n} - \frac{1}{2\log^2 p_n} - \frac{1}{2\log n\log p_n} - \frac{1}{2\log n\log^2 p_n} + \frac{\log\log n - 2.9}{2\log^2 n\log p_n} + \frac{\log\log n - 2.9}{\log^2 n\log^2 p_n}.$$

Finally, we use (4.18) to get

$$\log\left(1 + \frac{2R(n)}{p_n}\right) < -\frac{1}{2\log p_n} - \frac{1}{\log^2 p_n} + \frac{(\log\log n)^2 + \log\log n - 4.8}{2\log^2 n\log^2 p_n} + \frac{P_8(\log\log n)}{4\log^3 n\log^2 p_n} - \frac{1}{2\log n\log^2 p_n} - \frac{2.9}{2\log^2 n\log p},$$

which implies the required inequality. For smaller values for $n$, we use a computer. □

In the direction of Proposition 6.1, we find the following upper bound for $\log(1 + 2R(n)/p_n)$ in terms of $n$, which leads to an improvement in the right-hand side inequality of (6.1).

**Corollary 6.4.** *For every integer $n \geq 92$, we have*

$$\log\left(1 + \frac{2R(n)}{p_n}\right) < -\frac{1}{2\log n} + \frac{\log\log n - 2}{2\log^2 n} + \frac{4\log\log n - 2.9}{\log^3 n} + \frac{2.9\log\log n}{2\log^4 n}.$$

*Proof.* First, we consider the case where $n \geq 6077$. From (4.16), it follows that

$$-\frac{1}{\log p_n} \leq -\frac{1}{\log n} + \frac{\log\log n}{\log^2 n}. \tag{6.6}$$

Applying this to Proposition 6.3, we get

$$\log\left(1 + \frac{2R(n)}{p_n}\right) < -\frac{1}{2\log n} + \frac{\log\log n}{2\log^2 n} - \frac{1}{\log n\log p_n} + \frac{\log\log n}{\log^2 n\log p_n} - \frac{2.9}{2\log^3 n} + \frac{2.9\log\log n}{2\log^4 n}.$$

Again, we use (6.6) to obtain the required inequality for every integer $n \geq 6077$. We conclude by direct computation. □

We now establish the following more precise result compared with Proposition 6.1.

**Corollary 6.5.** *We have*

$$\log\left(1 + \frac{2R(n)}{p_n}\right) = -\frac{1}{2\log n} + \frac{\log\log n}{2\log^2 n} + O\left(\frac{1}{\log^2 n}\right).$$

*Proof.* The claim follows directly from Proposition 6.2 and Corollary 6.4. □

## 7. The Proofs of Theorems 1.2–1.5

Now, we use the identity (1.1) and explicit estimates for $D(n)$ and $\log(1 + 2R(n)/p_n)$ obtained in Section 3 and Section 5 to prove Theorems 1.2–1.5.

*Proof of Theorem 1.4.* We first consider the case where $n \geq 465\,944\,315$. By (1.1), Proposition 4.6, and Proposition 6.2, we get

$$\frac{A_n}{G_n} > \frac{e}{2}\cdot\exp\left(\frac{1}{2\log n} - \frac{\log\log n - 2.75}{2\log^2 n} - \frac{(\log\log n)^2 - 4.5\log\log n + 22.51/3}{2\log^3 n}\right).$$

Notice that $0.15\log x > (\log\log x)^2 - 4.5\log\log x + 22.51/3$ for every $x \geq 465\,944\,315$. Hence

$$\frac{A_n}{G_n} > \frac{e}{2}\cdot\exp\left(\frac{1}{2\log n} - \frac{\log\log n - 2.6}{2\log^2 n}\right).$$

Now, we use the inequality $e^x \geq 1 + x + x^2/2$, which holds for every nonnegative $x$, to get

$$\frac{A_n}{G_n} > \frac{e}{2} \cdot \left(1 + \frac{1}{2\log n} - \frac{\log\log n - 2.85}{2\log^2 n} - \frac{\log\log n}{4\log^3 n} + \frac{0.65}{\log^3 n}\right).$$

Since the function $2 \cdot 0.05 - (\log\log t - 4 \cdot 0.65)/(4\log t)$ is positive for every $t \geq 2$, the required lower bound for the ratio of $A_n$ and $G_n$ holds. A direct computation for every integer $n$, such that $139 \leq n \leq 465\,944\,314$ completes the proof. □

*Remark.* Hassani [11] conjectured that there exist a real number $\alpha$ with $0 < \alpha < 9.514$ and a positive integer $n_0$, such that $A_n/G_n > e/2 + \alpha/\log n$ for every integer $n \geq n_0$. Theorem 1.4 proves this conjecture.

Now, we give a proof of Theorem 1.2. To do this, we use (4.18) and Theorem 1.4.

*Proof of Theorem* 1.2. Let $n$ be an integer with $n \geq 1\,499\,820\,545$. Using (4.18) and Theorem 1.4, we get

$$\frac{A_n}{G_n} > \frac{e}{2} + \frac{e}{4\log p_n} + \frac{e}{4}\left(\frac{2.8}{\log^2 n} - \frac{(\log\log n)^2 - \log\log n + 1}{\log^3 n} - \frac{P_8(\log\log n)}{2\log^4 n}\right),$$

where $P_8(x) = 3x^2 - 6x + 5.2$. Now, we apply (4.20) to see that the required inequality holds. We complete the proof by verifying the remaining cases with a computer. □

The following corollary confirms that the ratio of the arithmetic and geometric means of the prime numbers is always greater than $e/2$, as conjectured by Hassani [11].

**Corollary 7.1.** *For every positive integer $n$, we have*

$$\frac{A_n}{G_n} > \frac{e}{2}.$$

*Proof.* From Theorem 1.2, it follows that the required inequality holds for every integer $n \geq 62$. We verify the remaining cases with a computer. □

Next, we use Propositions 4.5 and 6.3 to give the following proof of Theorem 1.3.

*Proof of Theorem* 1.3. First, let $n$ be an integer with $n \geq 74\,004\,585$. By (1.1), (5.3), and Propositions 4.5 and 6.3, we obtain the inequality:

$$\frac{A_n}{G_n} < \frac{e}{2} \cdot \left(1 + \frac{1}{2\log p_n} + \frac{9.02}{3\log^2 p_n} + \frac{1.33}{3\log^3 p_n} + \frac{13.2312}{3\log^4 p_n}\right).$$

Since $\log p_n \geq 19.937$, we have $9.02/3 + 1.33/(3\log p_n) + 13.2312/(3\log^2 p_n) < 3.04$, which completes the proof for every integer $n \geq 74\,004\,585$. We conclude by direct computation. □

Finally, we use Theorem 1.3 and the inequality (4.18) to prove Theorem 1.5.

*Proof of Theorem* 1.5. Using Theorem 1.3 and the inequality (4.18), we get

$$\frac{A_n}{G_n} < \frac{e}{2} + \frac{e}{4\log n} - \frac{e(\log\log n - 6.08)}{4\log^2 n} + \frac{e((\log\log n)^2 - \log\log n + 1)}{4\log^3 n} + \frac{eP_8(\log\log n)}{8\log^4 n} \tag{7.1}$$

for every integer $n \geq 294\,635$. Applying (4.20) to (7.1), we see that the claim is true for every integer $n \geq 1\,499\,820\,545$. A computer check shows the correctness of the required inequality for every integer $n$ satisfying $2 \leq n \leq 1\,499\,820\,544$. □

*Remark.* One of the conjectures concerning the ratio of $A_n$ and $G_n$ stated by Hassani [11] is still open, namely that the sequence $(A_n/G_n)_{n\in\mathbb{N}}$ is strictly decreasing for every integer $n \geq 226$.

**Acknowledgements**

I would like to thank Mehdi Hassani for drawing my attention to the present subject. I would also like to express my great appreciation to Marc Deléglise for the computation of several special values of Chebyshev's $\vartheta$-function.

## References

[1] Axler, C.: On a sequence involving prime numbers. J. Integer Seq. **18**(7), Article 15.7.6 (2015)

[2] Axler, C.: New bounds for the sum of the first $n$ prime numbers. J. Théor. Nombres Bordeaux **(to appear)**

[3] Axler, C.: New estimates for some functions defined over primes. Integers **(to appear)**

[4] Axler, C.: Estimates for $\pi(x)$ for large values of $x$ and Ramanujan's prime counting inequality. Integers **(to appear)**

[5] Axler, C.: New estimates for the $n$th prime number (2017) **(preprint)**. arXiv:1706.03651

[6] Brüdern, J.: Einführung in die analytische Zahlentheorie, Springer Lehrbuch (1995)

[7] Büthe, J.: An analytic method for bounding $\psi(x)$. Math. Comput. **(to appear)**

[8] Cipolla, M.: La determinazione assintotica dell' $n^{imo}$ numero primo. Rend. Accad. Sci. Fis-Mat. Napoli (3) **8**, 132–166 (1902)

[9] Dusart, P.: Inégalités explicites pour $\psi(X)$, $\theta(X)$, $\pi(X)$ et les nombres premiers. C. R. Math. Acad. Sci. Soc. R. Can. **21**(2), 53–59 (1999)

[10] Dusart, P.: The $k$-th prime is greater than $k(\ln k + \ln\ln k - 1)$ for $k \geq 2$. Math. Comput. **68**(225), 411–415 (1999)

[11] Hassani, M.: On the ratio of the arithmetic and geometric means of the prime numbers and the number $e$. Int. J. Number Theory **9**(6), 1593–1603 (2013)

[12] Kourbatov, A.: On the geometric mean of the first $n$ primes (2016) **(preprint)**. arXiv:1603.00855

[13] Panaitopol, L.: An inequality concerning the prime numbers. Notes Number Theory Discret. Math. **5**(2), 52–54 (1999)

[14] Panaitopol, L.: A formula for $\pi(x)$ applied to a result of Koninck-Ivić. Nieuw Arch. Wiskd. (5) **1**(1), 55–56 (2000)

[15] Rivera, C. (ed.): Conjecture 67. Primes and $e$ (2010). http://primepuzzles.net/conjectures/conj_067.htm

[16] Rosser, J.B., Schoenfeld, L.: Approximate formulas for some functions of prime numbers. Ill. J. Math. **6**(1), 64–94 (1962)

[17] Sándor, J.: On certain bounds and limits for prime numbers. Notes Number Theory Discret. Math. **18**(1), 1–5 (2012)

[18] Sándor, J., Verroken, A.: On a limit involving the product of prime numbers. Notes Number Theory Discret. Math. **17**(2), 1–3 (2011)



Christian Axler
Institute of Mathematics
Heinrich-Heine University Düsseldorf
40225 Düsseldorf
Germany
e-mail: christian.axler@hhu.de



Received: August 25, 2017.
Revised: March 27, 2018.
Accepted: April 13, 2018.