



ϕ -Parabolicity and the Uniqueness of Spacelike Hypersurfaces Immersed in a Spatially Weighted GRW Spacetime

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Abstract. In this paper, we extend a technique due to Romero et al. (Class Quantum Gravity 30:1–13, 2013; Int J Geom Methods Mod Phys 10:1360014, 2013; J Math Anal Appl 419:355–372, 2014) establishing sufficient conditions to guarantee the parabolicity of complete spacelike hypersurfaces immersed in a weighted generalized Robertson–Walker spacetime whose fiber has ϕ -parabolic universal Riemannian covering. As some applications of this criteria, we obtain uniqueness results concerning spacelike hypersurfaces immersed in spatially weighted generalized Robertson–Walker spacetimes. Furthermore, Calabi–Bernstein type results are also given.

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1. Introduction

The study of immersed spacelike hypersurfaces with constant mean curvature in a Lorentzian manifold has attracted the interest of a considerable group of geometers, as evidenced by the amount of works that it has generated in the last decades. This is due not only to its mathematical interest, but also to its relevance in General Relativity. For example, constant mean curvature spacelike hypersurfaces are particularly suitable for studying the propagation of gravitational radiation (cf. [30]; see also [22] for a summary of several reasons justifying this interest).

From the mathematical point of view, the study of the geometry of constant mean curvature spacelike hypersurfaces is mostly due to the fact that they exhibit nice Bernstein type properties. In this context, several authors

have recently treated the problem of uniqueness for complete constant mean curvature spacelike hypersurfaces of generalized Robertson–Walker (GRW) spacetimes, that is, Lorentzian warped products with 1-dimensional negative definite base and Riemannian fiber. Along this branch, we point out the works of Romero, Rubio and Salamanca [26–28], where the authors studied non-compact complete spacelike hypersurfaces in GRW spacetimes whose fiber has a parabolic universal Riemannian covering. In this setting, they guaranteed the parabolicity of complete spacelike hypersurfaces, under boundedness assumptions on the warping function restricted to a spacelike hypersurface and on the hyperbolic angle function of the hypersurface. As an application of this new technique, they obtained several uniqueness results on complete maximal spacelike hypersurfaces.

Going a step further, our aim in this paper is to extend the previous mentioned parabolicity criteria to the context of complete constant weighted mean curvature spacelike hypersurfaces immersed in a spatially weighted GRW spacetime. As some applications of this criteria, we obtain uniqueness results concerning these spacelike hypersurfaces in the case where the GRW spacetime is spatially weighted.

We recall that, given a pseudo-Riemannian manifold (M, g) and a smooth function $\phi \in C^\infty(M)$, the weighted manifold M_ϕ associated to M and ϕ is the triple $(M, g, d\mu = e^{-\phi}dM)$, where dM is the standard volume element of M . In this context, Bakry and Émery introduced in [8] the Bakry–Émery–Ricci tensor as a suitable generalization of the classical Ricci tensor, Ric , defined by,

$$\text{Ric}_\phi = \text{Ric} + \text{Hess } \phi,$$

where Hess stands for the Hessian with respect to the metric g . So, it seems natural to try to extend results stated in terms of the Ricci curvature tensor to analogous results for the Bakry–Émery–Ricci tensor.

It is also interesting to remark that weighted manifolds are closely related to some classical mathematical concepts, as they can be used as a powerful mathematical tool in order to obtain new results related to them. Specifically, in the case where Ric_f is constant, we can induce on M a structure of a gradient Ricci soliton. Its mathematical relevance is due to the Perelman’s solution of the Poincaré conjecture since gradient Ricci solitons correspond to self-similar solutions to the Hamilton’s Ricci flow and often arise as limits of dilations of singularities developed along the Ricci flow. For an overview of results in this context one can consult [29]. On the other hand, weighted manifolds have also been considered when studying harmonic heat flows and heat kernels. For instance, Grigor’yan and Saloff-Coste established in [15] a result which relates the heat kernel on a complete, noncompact Riemannian manifold M with the Dirichlet heat kernel on the exterior of a compact set of M . They got a relation between both heat kernels either in the case when M is transient or when it is recurrent, and this was possible since they applied in the proof the theory of weighted manifolds.

This manuscript is organized as follows: In Sect. 2, we introduce some basic notions and facts related to spacelike hypersurfaces in GRW spacetimes

and weighted manifolds in general. In Sect. 3, we establish our ϕ -parabolicity criteria concerning complete spacelike hypersurfaces immersed in a spatially weighted GRW spacetime. Specifically, in Theorem 1 we prove that

Let S be a complete spacelike hypersurface in a spatially weighted GRW spacetime M_ϕ with weight function ϕ , whose fiber F is complete with ϕ -parabolic universal Riemannian covering. If the hyperbolic angle function Θ of S is bounded and the restriction $f(h)$ on S of the warping function f of M satisfies $\sup_S f(h) < \infty$ and $\inf_S f(h) > 0$, then S is ϕ -parabolic.

In Sect. 4, we apply the above criteria in order to get uniqueness results for spacelike hypersurfaces in spatially weighted GRW spacetimes. We also establish some nice consequences for the particular cases where M is static (i.e. with constant warping function) or a steady state type spacetime (i.e. with warping function $f(t) = e^t$). Finally, in Sect. 5 we present new Calabi–Bernstein type results establishing non-parametric versions of the results obtained in Sect. 4.

2. Set Up

Let (F, g_F) be a connected, n -dimensional, oriented Riemannian manifold, $I \subset \mathbb{R}$ an open interval endowed with the metric $-dt^2$ and $f: I \rightarrow \mathbb{R}$ a positive smooth function. Let us consider the product manifold $M = I \times F$. The class of Lorentzian manifolds which will be of our concern here is the one obtained by furnishing M with the Lorentzian metric,

$$\bar{g} = -\pi_I^*(dt^2) + f^2(\pi_I)\pi_F^*(g_F), \tag{2.1}$$

where π_I and π_F are the projections onto the factors I and F , respectively. In this case, we simply write,

$$M = -I \times_f F. \tag{2.2}$$

Let us observe that M is a Lorentzian warped product with warping function f and fiber F .

When F has constant sectional curvature, the warped product (2.2) has been known in the mathematical literature as a Robertson–Walker (RW) spacetime, an allusion to the fact that, for $n = 3$, it is an exact solution of the Einstein’s field equations (cf. Chapter 12 of [25]). In the general case, after [6], the warped product (2.2) has usually been referred to as a *generalized Robertson–Walker (GRW) spacetime*, and we will stick to this usage along this paper.

Let S be an n -dimensional connected manifold. A smooth immersion $x: S \rightarrow M$ is said to be a *spacelike hypersurface* if S , furnished with the metric induced from (2.1) via x , is a Riemannian manifold. If this is so, we will always assume that the metric on S is the induced one, which will be denoted by g_S . In this setting, it follows from the connectedness of S that one can uniquely choose a globally defined timelike normal unit vector field $N \in \mathfrak{X}(S)^\perp$, having the same time-orientation of ∂_t , i.e., such that $\bar{g}(N, \partial_t) \leq -1$, where ∂_t is the

coordinate vector field induced by the universal time on M . One then says that N is the *future-pointing Gauss map* of S .

We will consider two particular functions naturally attached to a space-like hypersurface S immersed into a GRW spacetime M , namely, the (*vertical*) *height function* $h = (\pi_I)|_S$ and the *hyperbolic angle function* $\Theta = \bar{g}(N, \partial_t)$. Given any vector field $V \in \mathfrak{X}(M)$, we denote the projection V^F of V onto F by,

$$V^F = (\pi_F)_*(V) = V + \bar{g}(V, \partial_t)\partial_t. \tag{2.3}$$

Since ∂_t is globally defined on M , from the last term of (2.3) we see that V^F is always well-defined. In particular, $N^F = N + \Theta\partial_t$, and therefore,

$$|N^F|^2 = \bar{g}(N^F, N^F) = \Theta^2 - 1. \tag{2.4}$$

Let us denote by $\bar{\nabla}$ and ∇ the gradients with respect to the metrics of M and S , respectively. Then, a simple computation shows that the gradient of π_I on M is given by,

$$\bar{\nabla}\pi_I = -\bar{g}(\bar{\nabla}\pi_I, \partial_t)\partial_t = -\partial_t,$$

so that the gradient of h on S is,

$$\nabla h = (\bar{\nabla}\pi_I)^\top = -\partial_t^\top = -\partial_t - \Theta N. \tag{2.5}$$

Thus, from (2.5) we get,

$$|\nabla h|^2 = \Theta^2 - 1, \tag{2.6}$$

where $|\nabla h|^2 = g_S(\nabla h, \nabla h)$ denotes the norm of the vector field ∇h on S .

At this point we recall that, given a semi-Riemannian manifold (P, g) and a smooth function ϕ on P , the *weighted manifold* P_ϕ associated to P and ϕ is the triple $(P, g, d\mu = e^{-\phi}dP)$, where dP is the canonical volume element of P . In this setting, we will consider the so-called *Bakry-Émery-Ricci tensor*, introduced by Bakry and Émery in [8] as a suitable extension of the standard Ricci tensor Ric , which is defined by,

$$\text{Ric}_\phi = \text{Ric} + \text{Hess } \phi. \tag{2.7}$$

For a spacelike hypersurface S immersed in a weighted GRW spacetime M , the ϕ -divergence operator on S is defined by,

$$\text{div}_\phi(X) = e^\phi \text{div}(e^{-\phi}X),$$

for any tangent vector field X on S and, given a smooth function $u: S \rightarrow \mathbb{R}$, its *drifted Laplacian* is defined by,

$$\Delta_\phi u = \text{div}_\phi(\nabla u) = \Delta u - g_S(\nabla u, \nabla \phi). \tag{2.8}$$

Furthermore, according to Gromov [16], the ϕ -mean curvature H_ϕ of S is defined by,

$$nH_\phi = nH - \bar{g}(\bar{\nabla}\phi, N), \tag{2.9}$$

where H denotes the standard mean curvature of S with respect to its future-pointing Gauss map N .

It follows from a splitting theorem due to Case [9, Theorem 1.2] that if M is a weighted GRW spacetime endowed with a weight function ϕ which is

bounded and such that $\overline{\text{Ric}}_\phi(V, V) \geq 0$ for any timelike vector field V , then ϕ must be constant along \mathbb{R} . Motivated by this result, in what follows we will deal with *spatially weighted GRW spacetimes* M , which means that the weight function ϕ does not depend on the parameter $t \in I$, that is, $\overline{g}(\nabla\phi, \partial_t) = 0$. We also observe that we can construct spatially weighted GRW spacetimes in a natural way taking the standard product of $(I, -dt^2)$ with a Riemannian weighted manifold F_ϕ and considering the weight function on $M_{\tilde{\phi}} = -I \times F_\phi$ as being $\tilde{\phi}(t, x) = \phi(x)$.

Remark 1. We note that the ϕ -mean curvature of a slice $\{t_0\} \times F$ in a spatially weighted GRW spacetime is given by,

$$H_\phi(t_0) = (\log f)'(t_0).$$

Indeed, since ∂_t is a timelike normal vector field to the slices $\{t\} \times F$, from (2.9) we have that $H_\phi(t_0) = H(t_0) = (\log f)'(t_0)$.

3. ϕ -Parabolicity of Spacelike Hypersurfaces

A smooth function u on a weighted manifold P_ϕ is said to be ϕ -superharmonic if $\Delta_\phi u \leq 0$. Taking this into account, the weighted manifold $(P, g, d\mu = e^{-\phi}dP)$ is called ϕ -parabolic if there is no nonconstant, nonnegative, ϕ -superharmonic function on P . On the other hand, for any compact subset $K \subset P$, we define the ϕ -capacity of K as,

$$\text{cap}_\phi(K) = \inf \left\{ \int_P |\nabla u|^2 d\mu : u \in \text{Lip}_0(P) \text{ and } u|_K \equiv 1 \right\},$$

where $\text{Lip}_0(P)$ is the set of all compactly supported Lipschitz functions on P . The following statement relates the notion of ϕ -capacity to the concept of ϕ -parabolicity (cf. [14, Proposition 2.1]).

Lemma 1. *The weighted manifold $(P, g, d\mu = e^{-\phi}dP)$ is ϕ -parabolic if, and only if, $\text{cap}_\phi(K) = 0$ for any compact set $K \subset P$.*

Let us recall that given two Riemannian manifolds (P, g) and (P', g') , a diffeomorphism ψ from P onto P' is called a *quasi-isometry* if there exists a constant $c \geq 1$ such that,

$$c^{-1}|v|_g \leq |d\psi(v)|_{g'} \leq c|v|_g$$

for all $v \in T_pP, p \in P$ (cf. [20] for more details). Suppose that we can endow both P and P' with the same weight function ϕ . We can reason as in Section 5 of [13] to verify that the ϕ -capacity changes under a quasi-isometry at most by a constant factor. So, from Lemma 1, we can state the following result.

Lemma 2. *Let (P, g) and (P', g') be two Riemannian manifolds endowed with the same weight function ϕ . If P and P' are quasi-isometric, then P and P' are ϕ -parabolic or not simultaneously.*

Given a spacelike hypersurface $x: S \rightarrow M$ in a GRW spacetime $M = -I \times_f F$, the following lemma provides sufficient conditions to guarantee that the hypersurface S and the fiber F are quasi-isometric (cf. [26, Lemma 4.1]).

Lemma 3. *Let $x: S \rightarrow M$ be a spacelike hypersurface in a GRW spacetime $M = -I \times_f F$, whose hyperbolic angle function Θ is bounded. If the warping function f on S satisfies,*

- (i) $\sup_S f(h) < \infty$ and
- (ii) $\inf_S f(h) > 0$,

then $\pi = \pi_F \circ x$ is a quasi-isometry from S onto F .

We can now present the main result of this section.

Theorem 1. *Let S be a complete spacelike hypersurface in a spatially weighted GRW spacetime M_ϕ with weight function ϕ , whose fiber F is complete with ϕ -parabolic universal Riemannian covering. If the hyperbolic angle function Θ of S is bounded and the restriction $f(h)$ on S of the warping function f of M satisfies,*

- (i) $\sup_S f(h) < \infty$ and
- (ii) $\inf_S f(h) > 0$,

then S is ϕ -parabolic.

The proof of Theorem 1 follows the same steps of [26, Theorem 4.4] (see also [27, Theorem 1]). For this, we will need a standard result onto covering spaces (cf. [17] for instance).

Lemma 4. *Let $\rho: (\tilde{E}, \tilde{x}_0) \rightarrow (E, x_0)$ be a covering space and let $h: (W, y_0) \rightarrow (E, x_0)$ be a continuous map, where W is a path connected and locally path connected topological space. Then, there exists a lift $\tilde{h}: (W, y_0) \rightarrow (\tilde{E}, \tilde{x}_0)$ of h if, and only if, $h_*(\pi_1(W, y_0)) \subset \rho_*(\pi_1(\tilde{E}, \tilde{x}_0))$.*

Proof of Theorem 1. Using [6, Lemma 3.1], we know that the projection on the fiber, $\pi: S \rightarrow F$, is a covering map. Moreover, by Lemma 3, we can find a constant $c \geq 1$ such that,

$$c^{-1}g_F(d\pi(v), d\pi(v)) \leq g_S(v, v) \leq c g_F(d\pi(v), d\pi(v)), \tag{3.1}$$

for all $v \in T_p S$ and $p \in S$.

Let $(\tilde{S}, g_{\tilde{S}})$ be the universal Riemannian covering of (S, g_S) and denote by $\tilde{\pi}_S: \tilde{S} \rightarrow S$ the corresponding Riemannian covering map. From Lemma 4 we conclude that there exists a lift $\tilde{h}: \tilde{S} \rightarrow \tilde{F}$ of the map $h = \pi \circ \tilde{\pi}_S: \tilde{S} \rightarrow F$. It is easy to check that \tilde{h} is, in fact, a diffeomorphism from \tilde{S} to \tilde{F} . Note that (3.1) implies

$$c^{-1}g_{\tilde{F}}(d\tilde{h}(\tilde{v}), d\tilde{h}(\tilde{v})) \leq g_{\tilde{S}}(\tilde{v}, \tilde{v}) \leq c g_{\tilde{F}}(d\tilde{h}(\tilde{v}), d\tilde{h}(\tilde{v})),$$

for any $\tilde{v} \in T_{\tilde{p}} \tilde{S}, \tilde{p} \in \tilde{S}$, which means that \tilde{h} is a quasi-isometry from $(\tilde{S}, g_{\tilde{S}})$ onto $(\tilde{F}, g_{\tilde{F}})$.

Finally, let u be a nonnegative ϕ -superharmonic function on S , and put $\tilde{u} = u \circ \tilde{\pi}_S$. The function \tilde{u} is a nonnegative ϕ -superharmonic function on the ϕ -parabolic Riemannian manifold \tilde{S} . Therefore, \tilde{u} must be constant and, consequently, u is also constant. □

As a direct consequence of Theorem 1 we get the following corollaries.

Corollary 1. *Let S be a complete spacelike hypersurface in a spatially weighted GRW spacetime M_ϕ with weight function ϕ and whose fiber F is complete, simply connected and ϕ -parabolic. If the hyperbolic angle function Θ of S is bounded and the warping function on S , $f(h)$, is bounded and it satisfies $\inf_S f(h) > 0$, then S is ϕ -parabolic.*

We recall that a GRW is said to be *static* when its warping function is constant, which, without loss of generality, can be supposed equal to 1.

Corollary 2. *Let S be a complete spacelike hypersurface in a static spatially weighted GRW spacetime M_ϕ with weight function ϕ and whose fiber F is complete with ϕ -parabolic universal Riemannian covering. If the hyperbolic angle function Θ of S is bounded, then S is ϕ -parabolic.*

4. Uniqueness Results

As an application of Theorem 1, we will prove in this section some uniqueness results concerning spacelike hypersurfaces immersed in a spatially weighted GRW spacetime. For this, we will also need the following lemma.

Lemma 5. *Let S be a spacelike hypersurface immersed in a spatially weighted GRW spacetime M_ϕ with weight function ϕ . Then,*

$$\Delta_\phi h = -(\log f)'(h)(n + |\nabla h|^2) - n\Theta H_\phi, \tag{4.1}$$

$$\Delta_\phi \mathcal{F}(h) = -n(f'(h) + f(h)\Theta H_\phi), \tag{4.2}$$

where $\mathcal{F}(t) = \int_{t_0}^t f(s)ds$, and

$$\begin{aligned} \Delta_\phi(f(h)\Theta) &= nf(h)\bar{g}(\nabla H_\phi, \partial_t) + nf'(h)H_\phi + f(h)\Theta|A|^2 \\ &\quad + f(h)\Theta\overline{\text{Hess}}\phi(N, N) \\ &\quad + f(h)\Theta(\text{Ric}^F(N^F, N^F) - (n - 1)(\log f)''(h)|\nabla h|^2), \end{aligned} \tag{4.3}$$

where Ric^F stands for the Ricci curvature tensor of the fiber F .

Proof. Equations (4.1) and (4.2) correspond to [10, Lemma 1], and (4.3) corresponds to [4, Lemma 1]. However, until the moment reference [4] is not yet published, so for the sake of completeness we present it here a proof of (4.3).

In [5, Corollary 8.2] it is proven that

$$\begin{aligned} \Delta(f(h)\Theta) &= nf(h)\bar{g}(\nabla H, \partial_t) + nf'(h)H + f(h)\Theta|A|^2 \\ &\quad + f(h)\Theta(\text{Ric}^F(N^F, N^F) - (n - 1)(\log f)''(h)|\nabla h|^2). \end{aligned} \tag{4.4}$$

Taking into account (2.9), it follows that,

$$nf(h)\bar{g}(\nabla H, \partial_t) = nf(h)\bar{g}(\nabla H_\phi, \partial_t) + f(h)\partial_t^\top \bar{g}(\bar{\nabla}\phi, N).$$

Moreover, from a straightforward computation we get,

$$\partial_t^\top (\bar{g}(\bar{\nabla}\phi, N)) = -\frac{f'}{f}(h)\bar{g}(\bar{\nabla}\phi, N) + \Theta\overline{\text{Hess}}\phi(N, N) - \bar{g}(\bar{\nabla}\phi, A\partial_t^\top),$$

and $\nabla(f(h)\Theta) = -f(h)A\partial_t^\top$. So (4.4) can be written as,

$$\begin{aligned} \Delta(f(h)\Theta) &= nf(h)\bar{g}(\nabla H_\phi, \partial_t) - f'(h)\bar{g}(\bar{\nabla}\phi, N) + f(h)\Theta\overline{\text{Hess}}\phi(N, N) \\ &\quad + \bar{g}(\bar{\nabla}\phi, \nabla(f(h)\Theta)) + nf'(h)H + f(h)\Theta|A|^2 \\ &\quad + f(h)\Theta(\text{Ric}^F(N^F, N^F) - (n-1)(\log f)''(h)|\nabla h|^2). \end{aligned} \tag{4.5}$$

Finally, (4.3) follows from (4.5) and (2.8). □

We can now prove our first uniqueness result. A spacelike hypersurface is called ϕ -maximal if its ϕ -mean curvature H_ϕ is identically zero. A slab $[t_1, t_2] \times F = \{(t, q) \in M : t_1 \leq t \leq t_2\}$ of a GRW spacetime M is called a *timelike bounded region*.

Theorem 2. *Let M_ϕ be a spatially weighted GRW spacetime whose fiber F is complete with ϕ -parabolic universal Riemannian covering, and such that the warping function f is monotone. The only ϕ -maximal complete spacelike hypersurfaces contained in a timelike bounded region of M and with bounded hyperbolic angle function Θ are the slices $\{t_0\} \times F$, where $t_0 \in I$ is such that $f'(t_0) = 0$.*

Proof. Let S be such a spacelike hypersurface. From Lemma 5, we obtain,

$$\Delta_\phi \mathcal{F}(h) = -nf'(h).$$

Consequently, the monotonicity of f implies that $\Delta_\phi \mathcal{F}(h)$ is globally either nonpositive or nonnegative signed. Since S is contained in a timelike bounded region of M and the warping function f is monotone, the function $\mathcal{F}(h)$ is clearly bounded on S . From Theorem 1 we know that S is ϕ -parabolic, so $\mathcal{F}(h)$ is constant in S and, hence, h must also be constant in S . □

In what follow, we will assume that the ambient space obeys the so-called *null convergence condition* (NCC). We recall that a GRW spacetime M satisfies NCC if,

$$\text{Ric}^F \geq (n-1)f^2(\log f)''g_F, \tag{4.6}$$

which is equivalent to the Ricci curvature of M being nonnegative on null or lightlike directions (cf. [23]). In our next result we will also suppose that the weight function ϕ is convex (that is, $\bar{\text{Hess}}\phi \geq 0$). For a throughout discussion concerning convex functions on Riemannian manifolds, we refer the reader to [31, Chapter 3].

Theorem 3. *Let M_ϕ be a spatially weighted GRW spacetime satisfying (4.6), with convex weight function ϕ and whose fiber F is complete with ϕ -parabolic universal Riemannian covering. Let S be a complete ϕ -maximal spacelike hypersurface immersed in M , with bounded hyperbolic angle function Θ and such that the restriction $f(h)$ on S of the warping function f of M satisfies,*

- (i) $\sup f(h) < \infty$ and
- (ii) $\inf f(h) > 0$.

Then S is totally geodesic. In addition, if the inequality (4.6) is strict for all non-zero vector fields on F or ϕ is strictly convex on F , then S is a slice $\{t_0\} \times F$, where $t_0 \in I$ is such that $f'(t_0) = 0$.

Proof. From Lemma 5, we have that the drifted Laplacian of the bounded function $f\Theta$ is given by,

$$\begin{aligned} \Delta_\phi(f(h)\Theta) &= f(h)\Theta|A|^2 + f(h)\Theta\overline{\text{Hess}}\phi(N, N) \\ &\quad + f(h)\Theta(\text{Ric}^F(N^F, N^F) - (n - 1)(\log f)''(h)|\nabla h|^2). \end{aligned} \tag{4.7}$$

Since ϕ is convex and we are assuming that the null convergence condition (4.6) holds, it follows that $\Delta_\phi(f(h)\Theta) \leq 0$. Theorem 1 assures that S is ϕ -parabolic, so $f(h)\Theta$ must be constant. Therefore, returning to (4.7), we infer that $|A| \equiv 0$, that is, S is totally geodesic,

$$\overline{\text{Hess}}\phi(N, N) = \text{Hess}^F\phi(N^F, N^F) = 0, \tag{4.8}$$

and

$$\text{Ric}^F(N^F, N^F) - (n - 1)(\log f)''(h)|\nabla h|^2 = 0.$$

Consequently, if the inequality (4.6) is strict, or if ϕ is strictly convex on F , then (4.7) also gives that $|N^F| = |\nabla h| = 0$ on S , that is, S is a slice. \square

We recall that the Gaussian space \mathbb{G}^n corresponds to the Euclidean space \mathbb{R}^n endowed with the Gaussian probability density $e^{-\phi(x)} = (2\pi)^{-\frac{n}{2}} e^{-\frac{|x|^2}{2}}$. From Corollary 3 of [18] we have that \mathbb{G}^n has finite ϕ -volume. Consequently, taking into account Remark 3.8 of [19], we conclude that \mathbb{G}^n is ϕ -parabolic. On the other hand, it is not difficult to verify that the weight function ϕ of \mathbb{G}^n is strictly convex. Consequently, from Theorem 3 we get,

Corollary 3. *The only complete ϕ -maximal spacelike hypersurfaces of $-\mathbb{R} \times \mathbb{G}^n$ having bounded hyperbolic angle function, are the spacelike hyperplanes $\{t\} \times \mathbb{G}^n$.*

Now, considering as ambient space a static GRW spacetime, we obtain the following result,

Theorem 4. *Let M_ϕ be a static spatially weighted GRW spacetime, whose fiber F is complete with ϕ -parabolic universal Riemannian covering and such that its Bakry–Émery–Ricci tensor Ric_ϕ^F is nonnegative. Let S be a complete spacelike hypersurface immersed in M with constant ϕ -mean curvature H_ϕ . If the hyperbolic angle function Θ of S is bounded, then S is totally geodesic. In addition, if Ric_ϕ^F is definite positive at some point of S , then S is a slice $\{t\} \times F$.*

Proof. From Lemma 5, (2.7), and (4.8) we get that,

$$\Delta_\phi\Theta = (\text{Ric}_\phi^F(N^F, N^F) + |A|^2)\Theta. \tag{4.9}$$

Consequently, since we are supposing that Ric_ϕ^F is nonnegative and that Θ is negative and bounded on S , we can apply Corollary 2 to conclude that Θ is constant on S . Thus, returning to (4.9) we get that $|A| \equiv 0$, that is, S is totally geodesic. Moreover, if Ric_ϕ^F is definite positive at some $p \in F$, considering once more Eq. (4.9) and taking into account relation (2.4), we conclude that $\Theta \equiv -1$ on S , which means that S is a slice of M . \square

We recall that Xin [33] and Aiyama [1] proved simultaneously and independently that the only spacelike hypersurfaces of the Lorentz–Minkowski space $\mathbb{L}^{n+1} = -\mathbb{R} \times \mathbb{R}^n$, with constant mean curvature and having bounded hyperbolic angle function, are the spacelike hyperplanes. Hence, from Theorem 4 we get the following extension of this Xin–Aiyama result.

Corollary 4. *The only complete spacelike hypersurfaces of $-\mathbb{R} \times \mathbb{G}^n$, with constant ϕ -mean curvature and having bounded hyperbolic angle function, are the spacelike hyperplanes $\{t\} \times \mathbb{G}^n$.*

Proceeding, we will use a Bochner’s formula due to Wei and Wylie [32] to obtain the following theorem.

Theorem 5. *Let M_ϕ be a static spatially weighted GRW spacetime endowed with a convex weight function ϕ and whose fiber F is complete, with nonnegative sectional curvature, and such that its universal Riemannian covering is ϕ -parabolic. Let S be a complete spacelike hypersurface lying in a semi-space of M and with constant ϕ -mean curvature H_ϕ . If the hyperbolic angle function Θ is bounded, then S is a slice $\{t\} \times F$.*

Proof. Since we are supposing that F has nonnegative sectional curvature, it follows from inequalities (3.3) and (3.4) of [12] that,

$$\text{Ric}(X, X) \geq nH_\phi g_S(AX, X) + |AX|^2. \tag{4.10}$$

On the other hand, taking into account that,

$$\text{Hess } \phi(X, X) = \overline{\text{Hess } \phi}(X, X) - \bar{g}(\bar{\nabla} \phi, N)g_S(AX, X),$$

from the convexity of the weight function ϕ we get,

$$\text{Hess } \phi(X, X) \geq -\bar{g}(\bar{\nabla} \phi, N)g_S(AX, X). \tag{4.11}$$

From (4.10) and (4.11) we get the following lower bound for Ric_ϕ

$$\text{Ric}_\phi(X, X) \geq nH_\phi g_S(AX, X) + |AX|^2. \tag{4.12}$$

Inequality (4.12) provides us,

$$\text{Ric}_\phi(\nabla h, \nabla h) \geq nH_\phi g_S(A(\nabla h), \nabla h) + |A(\nabla h)|^2. \tag{4.13}$$

Since H_ϕ is constant, we have,

$$\nabla \Delta_\phi h = -nH_\phi A(\nabla h). \tag{4.14}$$

On the other hand, from Bochner’s formula (cf. [32]),

$$\frac{1}{2} \Delta_\phi |\nabla h|^2 = |\text{Hess } h|^2 + g_S(\nabla h, \nabla \Delta_\phi h) + \text{Ric}_\phi(\nabla h, \nabla h). \tag{4.15}$$

Consequently, from (2.6), (4.13)–(4.15) we get,

$$\frac{1}{2} \Delta_\phi \Theta^2 = \frac{1}{2} \Delta_\phi |\nabla h|^2 \geq |\text{Hess } h|^2 \geq 0. \tag{4.16}$$

Thus, from Corollary 2 we have that Θ is constant and, returning to (4.16), we get $|\text{Hess } h|^2 = 0$ in S . Then, since $n|\text{Hess } h|^2 \geq (\Delta h)^2$, we have that h is harmonic. So, since $\Delta h = -nH\Theta$, we also get that $H = 0$ in S and from (4.10) we have that S has nonnegative Ricci curvature. Therefore, since we are supposing that S lies in a semi-space of M , we can apply the strong

Liouville property due to Yau in [34] (see also [21, Theorem 4.8]) to conclude that h must be constant and, hence, S is a slice of M . \square

Extending the ideas of [2], we now consider *spatially weighted steady state type spacetimes*, that is, GRW spacetimes of the type $-\mathbb{R} \times_{e^t} F$ whose fiber F is endowed with a weight function ϕ . In this setting, our next result is an extension of Theorem 8 of [2].

Theorem 6. *Let M_ϕ be a spatially weighted steady state type spacetime, whose fiber F is complete, with nonnegative sectional curvature, and let S be a complete spacelike hypersurface which lies in a timelike bounded region of M . Suppose that $|\bar{\nabla}\phi|$ is bounded on S . If the ϕ -mean curvature H_ϕ of S is constant and the hyperbolic angle function Θ is bounded, then $H_\phi = 1$. In addition, if the universal Riemannian covering of F is ϕ -parabolic, then S is a slice $\{t\} \times F$.*

Proof. We claim that the mean curvature H of S is bounded. Indeed, since $\bar{g}(\bar{\nabla}\phi, \partial_t) = 0$, from (2.9) we have that,

$$\begin{aligned} n|H| &\leq n|H_\phi| + |\bar{g}(\bar{\nabla}\phi, N)| \\ &= n|H_\phi| + |\bar{g}(\bar{\nabla}\phi, N^F)|. \end{aligned} \tag{4.17}$$

Thus, from (2.4) and (4.17) we get,

$$n|H| \leq n|H_\phi| + |\bar{\nabla}\phi|(\Theta^2 - 1). \tag{4.18}$$

Consequently, since H_ϕ is constant and $\bar{\nabla}\phi$ and Θ are supposed to be bounded on Σ^n , it follows from (4.18) that H is also bounded on S .

On the other hand, from inequality (16) of [2] we have that the Ricci curvature of S satisfies,

$$\text{Ric}(X, X) \geq n - 1 - \frac{n^2 H^2}{4}.$$

So, we conclude that the Ricci curvature of S is bounded from below.

Thus, we can apply the generalized maximum principle of Omori [24] and Yau [34] to guarantee that there exists a sequence $\{p_k\}$ in S such that,

$$\lim_k h(p_k) = \sup_S h, \quad \lim_k |\nabla h(p_k)| = 0 \quad \text{and} \quad \lim_k \sup \Delta h(p_k) \leq 0.$$

But, from (2.6) and (2.8), we also get that,

$$\lim_k \Theta(p_k) = -1 \quad \text{and} \quad \lim_k \sup \Delta_\phi h(p_k) \leq 0.$$

Hence, taking into account formula (4.1), we can reason in a similar way to the proof of Theorem 8 of [2] to obtain that $H_\phi = 1$. Furthermore, from formula (4.2) we also have that,

$$\Delta_\phi e^h = -ne^h(1 + \Theta) \geq 0.$$

Therefore, assuming that the universal Riemannian covering of F is ϕ -parabolic, we can apply Theorem 1 to conclude that h is constant in S . \square

Our next result extends Theorem 5.3 of [11].

Theorem 7. *Let M_ϕ be a spatially weighted steady state type spacetime whose fiber F is complete, with ϕ -parabolic universal Riemannian covering, and let S be a complete spacelike hypersurface which lies in a timelike bounded region of M , with $1 \leq H_\phi \leq \alpha$ for a certain $\alpha \geq 1$. If the hyperbolic angle function Θ of S satisfies $-\Theta \leq H_\phi$, then S is a slice $\{t\} \times F$.*

Proof. From formula (4.1) we get,

$$\begin{aligned} \Delta_\phi e^{-h} &= e^{-h} (|\nabla h|^2 - \Delta_\phi h) \\ &\leq ne^{-h} (|\nabla h|^2 + 1 + H_\phi \Theta) \\ &= ne^{-h} \Theta (\Theta + H_\phi). \end{aligned} \tag{4.19}$$

Hence, our hypothesis on Θ guarantees that the function e^{-h} is a ϕ -superharmonic positive function on S . Therefore, we can apply once more Theorem 1 to get that h is constant on S . □

5. Calabi–Bernstein Type Results

Let $\Omega \subseteq F$ be a connected domain and let $u \in C^\infty(\Omega)$ be a smooth function. Then, $S(u)$ will denote the vertical graph over Ω determined by u , that is,

$$S(u) = \{(u(x), x) : x \in \Omega\} \subset M = -I \times_f F.$$

The graph is said to be entire if $\Omega = F$. The metric induced on Ω from the Lorentzian metric of the ambient space via $S(u)$ is,

$$g_{S(u)} = -du^2 + f^2(u)g_F. \tag{5.1}$$

It can be easily seen that a graph $S(u)$ is a spacelike hypersurface if, and only if, $|Du|_F^2 < f^2(u)$, Du being the gradient of u in F and $|Du|_F$ its norm, both with respect to the metric g_F . It is well known (cf. [6, Lemma 3.1]) that in the case where F is a simply connected manifold, every complete spacelike hypersurface S immersed in M such that the warping function f is bounded on S is an entire spacelike graph over F . In particular, this happens for complete spacelike hypersurfaces contained in a timelike bounded region of M . It is interesting to observe that, in contrast to the case of graphs into a Riemannian space, an entire spacelike graph $S(u)$ in a Lorentzian spacetime is not necessarily complete, in the sense that the induced Riemannian metric is not necessarily complete on F . However, it can be proved that if F is complete and $|Du|_F^2 \leq f^2(u) - c$ for certain positive constant $c > 0$, then $S(u)$ is complete. Although a particular case of this claim is proven in [3, Theorem 4.1], and the general proof is given in ([4, Proposition 9]), we will expose it here for the sake of completeness.

Proposition 1. *Let F be a complete Riemannian manifold and $S(u)$ an entire spacelike vertical graph in $M = -I \times_f F$. If,*

$$|Du|_F^2 \leq f^2(u) - c$$

for certain positive constant $c > 0$, then $\Sigma^n(u)$ is complete.

Proof. From (5.1), the Cauchy–Schwarz inequality and the assumptions of the proposition we get,

$$g_{S(u)}(X, X) = -g_F(Du, X)^2 + f^2(u)g_F(X, X) \geq (f^2(u) - |Du|_F^2)g_F(X, X) \geq cg_F(X, X),$$

for every $X \in \mathfrak{X}(S(u))$. This implies that $L \geq \sqrt{c}L_F$, where L and L_F denote the length of a curve on $S(u)$ with respect to the Riemannian metrics $g_{S(u)}$ and g_F , respectively. As a consequence, as F is complete by assumption, the induced metric on $S(u)$ from the metric of M is also complete. \square

The future-pointing Gauss map of a spacelike vertical graph $S(u)$ over Ω is given by the vector field,

$$N(x) = \frac{f(u(x))}{\sqrt{f^2(u(x)) - |Du(x)|_F^2}} \left(\partial_t|_{(u(x),x)} + \frac{1}{f^2(u(x))} Du(x) \right), \quad x \in \Omega. \tag{5.2}$$

Moreover, the shape operator A of $S(u)$ with respect to its orientation (5.2) is given by,

$$AX = -\frac{1}{f(u)\sqrt{f^2(u) - |Du|_F^2}} D_X Du - \frac{f'(u)}{\sqrt{f^2(u) - |Du|_F^2}} X + \left(\frac{-g_F(D_X Du, Du)}{f(u)(f^2(u) - |Du|_F^2)^{3/2}} + \frac{f'(u)g_F(Du, X)}{(f^2(u) - |Du|_F^2)^{3/2}} \right) Du, \tag{5.3}$$

for any tangent vector field X tangent to Ω . Consequently, if $S(u)$ is a space-like vertical graph over a domain Ω of the fiber F of a spatially weighted GRW spacetime M endowed with a weight function ϕ , it is not difficult to verify from (2.9) and (5.3) that the ϕ -mean curvature function $H_\phi(u)$ of $S(u)$ is given by,

$$nH_\phi(u) = -\operatorname{div}_\phi \left(\frac{Du}{f(u)\sqrt{f(u)^2 - |Du|_F^2}} \right) - \frac{f'(u)}{\sqrt{f(u)^2 - |Du|_F^2}} \left(n + \frac{|Du|_F^2}{f(u)^2} \right).$$

The differential equation $H_\phi(u) = 0$ with the constraint $|Du|_F < f(u)$ is called the ϕ -maximal spacelike hypersurface equation in M , and its solutions provide ϕ -maximal spacelike graphs in M .

In [7] An et al. obtained a Calabi–Bernstein type result for $-\mathbb{R} \times \mathbb{G}^n$ showing that the graph $S(u)$ of a function $u(x) = t$ over \mathbb{G}^n , with $|Du|_{\mathbb{G}}$ bounded away from 1, is ϕ -maximal if and only if u is constant. Now, we present an extension of this result.

Theorem 8. *Let M_ϕ be a spatially weighted GRW spacetime, whose fiber F is complete with ϕ -parabolic universal Riemannian covering, and such that the warping function f is monotone. The only entire bounded solutions of the*

following modified ϕ -maximal spacelike hypersurface equation,

$$(\mathbf{E}) \begin{cases} \operatorname{div}_\phi \left(\frac{Du}{f(u)\sqrt{f(u)^2 - |Du|_F^2}} \right) = -\frac{f'(u)}{\sqrt{f(u)^2 - |Du|_F^2}} \left(n + \frac{|Du|_F^2}{f(u)^2} \right) \\ |Du|_F \leq \alpha f(u), \end{cases}$$

where $0 < \alpha < 1$ is constant, are the constant functions $u = t_0$, with $f'(t_0) = 0$.

Proof. We observe that the constraint on $|Du|_F$ assures the boundedness of the hyperbolic angle function $\Theta(u)$ of $S(u)$. Indeed, from (2.4), (2.6) and (5.2) we obtain that,

$$|\nabla h|^2 = \frac{|Du|_F^2}{f^2(u) - |Du|_F^2}. \tag{5.4}$$

Hence, using (2.6) and (5.4) we see that $|Du|_F \leq \alpha f(u)$ implies $\Theta(u) \geq \frac{-1}{\sqrt{1 - \alpha^2}}$.

Moreover, since we are looking for bounded solutions for (E), taking $c = (1 - \alpha^2) \inf_{S(u)} f^2(u)$ we can apply Proposition 1 to see that such a solution must be complete. Therefore, the result follows from Theorem 2. \square

It is not difficult to see that we can reason as in the proof of the previous result to obtain non-parametric versions of all others theorems of Sect. 4. For instance, we have the following non-parametric version of Theorem 7.

Theorem 9. *Let M_ϕ be a spatially weighted steady state type spacetime, whose fiber F is complete with ϕ -parabolic universal Riemannian covering. The only entire bounded solutions of the equation,*

$$(\mathbf{E}') \begin{cases} \operatorname{div}_\phi \left(\frac{Du}{e^u \sqrt{e^{2u} - |Du|_F^2}} \right) = -nH_\phi(u) - \frac{e^u}{\sqrt{e^{2u} - |Du|_F^2}} \left(n + \frac{|Du|_F^2}{e^{2u}} \right) \\ 1 \leq H_\phi(u) \leq \alpha \\ |Du|_F \leq \sqrt{1 - \frac{1}{H_\phi^2(u)}} e^u \end{cases}$$

where $\alpha \geq 1$ is a constant, are the constant functions $u = t_0$.

Proof. Let us observe first that, under the assumptions of the theorem, the solutions of (E') determine complete entire graphs $S(u)$. In fact, from the last inequality of (E') we easily obtain that

$$e^{2u} - |Du|_F^2 \geq c = e^{2 \inf_{S(u)} u} \sup_{S(u)} \frac{1}{H_\phi^2(u)} > 0,$$

and the completeness of $S(u)$ follows again from Proposition 1.

Therefore, using (5.4) and the last inequality of (E') we can easily verify that the hyperbolic angle function $\Theta(u)$ of $S(u)$ satisfies $-\Theta(u) \leq H_\phi(u)$ and, hence, the result follows from Theorem 7. \square

Remark 2. As it was observed by the referee of this paper, an important property of Robertson–Walker spacetimes is that they are locally conformally flat, a condition that does not hold in the generalized Robertson–Walker spacetimes. Hence, a nice generalization of Robertson–Walker spacetimes could be, instead of weakening the assumption on the geometry of the fiber, a more general assumption on the warped product structure, and to consider locally conformally flat multiply warped products. Therefore, it remains as an interesting open problem to obtain analogous versions of our previous results for this other context.

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