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Enumerating Some Stable Partitions Involving Stirling and r-Stirling Numbers of the Second Kind

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Abstract. The coefficient of the chromatic polynomial counts the number of partitions of the vertex set of a simple and finite graph G into k independent vertex sets, equivalently, it gives the number of proper colorings of G with exactly k colors subject to some constraints. In this work, we study this invariant, we establish new formulas in this context for some families of graphs and we treat some specific cases as Thorn graphs. Consequently, we derive identities for the classical Stirling numbers of the second kind, besides that, this gives rise to new explicit formulae for the r-Stirling numbers of the second kind.

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1. Introduction

Several problems in graph theory have combinatorial interpretations involving classical sequences in enumerative combinatorics, namely the Stirling numbers, the Fibonacci numbers, the Bell numbers, and their generalizations, see for instance [2,10,13,14,17]. Also, important identities and explicit formulas related to some classical sequences are established using elements of graph theory, [3,16]. In this work, we present a particular focus on the Stirling and the r-Stirling numbers of the second kind.

For n, k non negative integers such that $k \leq n$, the Stirling number of the second kind is the number of set partitions of an n-element set into k non-empty subsets, denoted by $S(n, k)$, with $S(n, 1) = 1$ for $n \geq 1$, $S(n, n) = 1$, $S(n, n-1) = \binom{n}{2}$ and $S(n, 2) = 2^{n-1} - 1$.

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The Stirling numbers of the second kind satisfy the triangular recurrence relation, for $n, k \geq 1$

$$S(n,k) = S(n-1,k-1) + kS(n-1,k).$$

Many research papers were devoted to the study of these numbers, their restrictions and their properties, such as, generating functions, recurrence relations and explicit formulas, see for instance [1,8,19] and references there in.

The ordinary generating function of the Stirling numbers of the second kind is given by

$$\sum_{n\geq k} S(n,k)x^n = \frac{x^k}{(1-x)(1-2x)\cdots(1-kx)}.$$

The generalized Stirling numbers of the second kind have been first introduced by Carlitz in 1978 and studied by Broder in 1984, who gave in [6] combinatorial interpretations and several algebraic properties.

The r-Stirling numbers of the second kind count the restricted set partitions of an n-element set into k non-empty blocks such that the r first elements belong to distinct blocks. We use $S_r(n,k)$ to denote these numbers.

Some particular values are

$$S_r(n,k) = \begin{cases} 0 & \text{for } n < r, \\ \delta_{k,r} & \text{for } n = r, \\ r^{n-r} & \text{for } n \geq r, \end{cases}$$

where $\delta_{k,r}$ is the Kronecker delta.

For $n \geq k > r \geq 0$, we have the triangular recurrence relation

$$S_r(n,k) = S_r(n-1,k-1) + kS_r(n-1,k).$$

For $r = 1$ and $r = 0$, these numbers coincide with the classical Stirling numbers of the second kind.

The ordinary generating function of the r-Stirling numbers of the second kind is given by

$$\sum_{n} S_r(n,k)z^n = \frac{z^k}{(1-rz)(1-(r+1)z)\cdots(1-kz)}.$$

Now, we give definitions of concepts in graph theory that we will need later.

Let $G = (V,E)$ be a simple and finite graph of vertex set $V = V(G)$ and edge set $E = E(G)$ with $|V(G)| = n$ and $|E(G)| = m$. A stable partition (or independant partition) of $V(G)$ is a partition of $V(G)$ into independent vertex subsets (or stables), where a subset of $V(G)$ is called a stable if no two vertices in the subset are adjacent. A proper coloring of G is an assignment of colors to $V(G)$ so that each two adjacent vertices receive different colors. A proper λ-coloring of G is a proper coloring using at most λ colors. The chromatic number of the graph G denoted by $\chi(G)$ is the smallest number λ such that G admits a λ-coloring.

Counting the number of ways to color a simple and finite graph G is of interest, besides the fact that it constitutes an important area in graph theory, it gives also attractive questions in enumerative combinatorics, in particular, when the permutations of the colors are disregarded. Furthermore, when the colorings are restricted in number of colors, we are in front of counting the number of stable partitions of the vertex set of G into a given number of stables. The sequence that enumerates these stable partitions appeared in the literature with different notations and under several denominations, it is considered by Goldman et al. [12] and referred as chromatic vector of G, we find it also under the name chromatic spectrum by Voloshin [20]. Also, it should be noted that in 1982, Prodinger and Tichy defined in [18] the Fibonacci number of a graph G to be the total number of these stable partitions including the empty set.

Throughout this paper, we shall use the notation $S(G,k)$ to denote the sequence that counts the number of stable partitions of $V(G)$ into k blocks. It has to be noted that it generalizes the classical Stirling numbers of the second kind, since for an empty graph E_n (graph without edges) $S(E_n,k) = S(n,k)$, that's why it was referred as the graphical Stirling number or Stirling numbers for graphs by [10,14], where they evaluated the Stirling numbers for some well-known graphs.

Indeed, $S(G,k) = 0$ for $0 \leq k \leq \chi(G)-1$, $S(G,n) = 1$ and $S(G,n-1) = \binom{n}{2} - m$.

The object that encodes the number of λ-colorings is called the Chromatic polynomial, it has been introduced in 1912 by Birkhoff [4] in an attempt to solve the four-color problem. Let $P(G,\lambda)$ be this polynomial, $S(G,k)$ for $\chi(G) \leq k \leq n$ constitutes the coefficient of the falling factorial $(\lambda)_k = \lambda(\lambda-1)(\lambda-2)\cdots(\lambda-k+1)$ in the chromatic polynomial's formula, described as

$$P(G,\lambda) = \sum_{k=\chi(G)}^{n} S(G,k)(\lambda)_k. \tag{1}$$

Maamra and Mihoubi in [15,16] have used the coefficients of the chromatic polynomial to derive some applications on Stirling numbers of the second kind.

In the present paper, a special focus on the coefficient $S(G,k)$ is given. In the next section, we start by some preliminary results on the Stirling numbers for some special graphs. In Sect. 3, we extend the study to a complementary class of graphs and we establish identities and explicit formulas for the Stirling numbers and the r-Stirling numbers of the second kind.

2. Preliminary Results

In combinatorics, references were made to treat the sequence that enumerates the number of proper colorings subject to some constraints for some well known graphs [10,11,13,14,21]. For this, the deletion–contraction principle was a valuable tool and had useful applications in that context. It was first used to compute the chromatic polynomial, its application remains also valid in the case of counting the number of partitions, this follows from the fact that coloring the vertex set of a graph amounts to partition their vertices into stables.

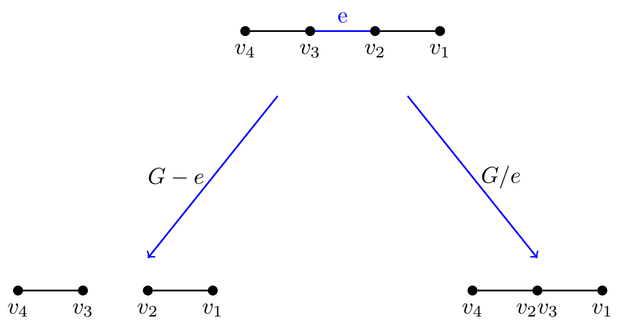

Figure 1. Deletion–contraction procedure

Theorem 2.1. *[14] Let G be a simple graph of order n, $e \in E(G)$ and $0 \leq k \leq n-1$, then*

$$S(G,k) = S(G-e,k) - S(G/e,k),$$

where $G-e$ and G/e are the transformation graphs obtained by deleting and contracting edge e from G, respectively.

See the illustration in Figure 1.

We must note that the graph G/e may not be a simple graph, but because of the fact that the contraction of distinct vertices will not create any loops, we can ignore multiple edges between vertices as this does not affect the calculation of the number of partitions (as two adjacent vertices remain adjacent regardless of the number of edges between them).

In the sequel, let us show some former formulas to calculate $S(G,k)$ for some well known graphs. For basic terminology in graph theory and some descriptions of special graph classes, see Brandstadt et al. [5].

For the complete graph (the graph in which all the vertices are pairwise adjacent), denoted by K_n, we have one coloring, subject to the above constraints when $k = n$ and no possible colorings, otherwise. Thus, the chromatic polynomial $P(K_n,k) = (k)_n$.

Let P_n, C_n, T_n and S_n be a path, a cycle, a tree and a star of order n, where a path of order n, P_n is an alternating sequence of vertices and edges beginning and ending with vertices such that the vertices are all distinct, a cycle is a closed path, a tree is a graph with no cycle and a star of order n is a tree with one central vertex and $n-1$ leaves ($n-1$ pendant vertices adjacent to the central one) and it is represented by one isolated vertex when $n = 1$.

Counting the number of stable partitions for paths, cycles and trees has also been considered by several authors with different interpretations [10,11,13,14,18].

For P_n, S_n and T_n of order n, the number of partitions into k stable sets is expressed in terms of the Stirling number of the second kind, for $1 \leq k \leq n$,

$$S(P_n,k) = S(S_n,k) = S(T_n,k) = S(n-1,k-1). \tag{2}$$

This has been generalized to the m-trees denoted by $T_n^{(m)}$ (where the m-tree is a graph defined by construction, starting with an $(m+1)$-vertex complete graph and then repeatedly adding vertices in such a way that each added vertex has exactly m neighbors), as

$$S(T_n^m, k) = S(n-m, k-m) \text{ for } m \le k \le n \text{ and } 0 \text{ otherwise.} \tag{3}$$

For a cycle of order n, C_n, it is defined by the following alternating sum

$$S(C_n, k) = \sum_{j=k-1}^{n-1} (-1)^{n-1-j} S(j, k-1), \text{ for } n \ge 3 \text{ and } 2 \le k \le n. \tag{4}$$

3. Counting the Number of Stable Partitions for Other Families of Graphs

Inspired by the previously cited works and exploiting the works carried out in this context, we propose in this section to calculate the number of stable partitions for other families of graphs. In what follows, inductive proofs using the deletion–contraction principle, bijective proofs and generating functions are used to demonstrate the results. As consequences, we give new identities concerning the Stirling number of the second kind. Besides that, important explicit formulas in terms of the generalized r-Stirling numbers are established.

We give some notations and definitions.

Given a simple and finite graph of order n, G_n. For any fixed $p \ge 1$, consider the super-graph obtained by joining a path of order p, P_p to G_n by a bridge as illustrated in Fig. 2 and let us denote the resultant graph by $G_{n,p}$. If G_n is a cycle of order n, C_n, then the resultant graph is called a tadpole.

The number of stable partitions of $G_{n,p}$ can be expressed in terms of the number of stable partitions of the disjoint unions of paths and the initial graph G_n, where the disjoint union of two simple and finite graphs H_1 and H_2 denoted by $H_1 \cup H_2$, is the graph H whose vertex set is $V(H) = V(H_1) \cup V(H_2)$ and whose edge set is $E(H) = E(H_1) \cup E(H_2)$.

Theorem 3.1. *For $1 \le k \le n$ and $p \ge 1$ we have,*

$$S(G_{n,p}, k) = \sum_{i=0}^{p} (-1)^i S(G_n \cup P_{p-i}, k),$$

where $P_0 = \emptyset$, thus $G_n \cup P_0 = G_n$.

Proof. The proof proceeds by induction on p using the deletion–contraction principle. The recurrence is valid for the trivial case ($p = 1$) with the convention that $G_n \cup P_0 = G_n$ and $P_1 = E_1$ (one isolated vertex) and can be verified using Theorem 2.1. Now, suppose the identity true for $G_{n,p}$ and let us establish it for $G_{n,p+1}$. From Theorem 2.1, we have,

$$S(G_{n,p+1}, k) = S(G_n \cup P_{p+1}, k) - S(G_{n,p}, k),$$

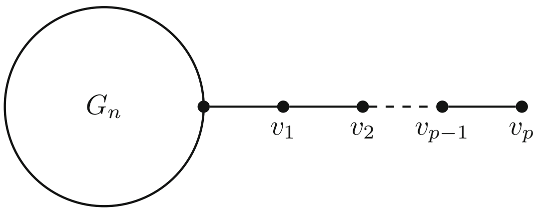

Figure 2. Representation of $G_{n,p}$

then, using the induction hypothesis, we get,

$$S(G_{n,p+1},k) = S(G_n \cup P_{p+1},k) - \sum_{i=0}^{p}(-1)^i S(G_n \cup P_{p-i},k),$$

we set $j = i+1$ and we obtain,

$$S(G_{n,p+1},k) = \sum_{j=1}^{p+1}(-1)^j S(G_n \cup P_{p-j+1},k) + S(G_n \cup P_{p+1},k), \quad (5)$$

which gives the result. □

Lemma 3.2. *[13] Let $H_1 \cup H_2$ be the disjoint union of H_1 and H_2. Then we have,*

$$S(H_1 \cup H_2,k) = \sum_{i=1}^{k}\sum_{j=0}^{i} S(H_1,i)S(H_2,k-j)\binom{i}{i-j}\binom{k-j}{i-j}(i-j)!. \quad (6)$$

As a corollary, an identity on the classical Stirling numbers of the second kind evaluated with three summations can be derived.

Corollary 3.3. *For $n \geq 4$, $0 \leq k \leq n$ and $1 \leq l < n$, we have*

$$S(n,k) = \sum_{i,s,t}(-1)^i S(l,s-1)S(n-l-i-1,k-t)\binom{s}{t}\binom{k+1-t}{s-t}(s-t)!,$$

where i,s,t satisfy $0 \leq i < n-l$, $1 \leq s \leq min(k,l)+1$ and $0 \leq t \leq s$.

Proof. Let P_n be a path of order n and P_l a subpath of P_n of order l, $1 \leq l < n$. Therefore, P_n can be written as $G_{l,n-l}$ for $1 \leq l < n$.
Applying Theorem 3.1 we obtain for $1 \leq k \leq n$, $n \geq 2$ and $1 \leq l < n$,

$$S(P_n,k) = S(G_{l,n-l},k) = \sum_{i=0}^{n-l}(-1)^i S(P_l \cup P_{n-l-i},k),$$

we use Lemma 3.2 (the indices i and j in relation (6) are changed to s and t, respectively), we get, for $0 \leq i \leq n-l$, $1 \leq s \leq k$ and $0 \leq t \leq s$,

$$S(P_n,k) = \sum_{i,s,t}(-1)^i S(P_l,s)S(P_{n-l},k-t)\binom{s}{s-t}\binom{k-t}{s-t}(s-t)!,$$

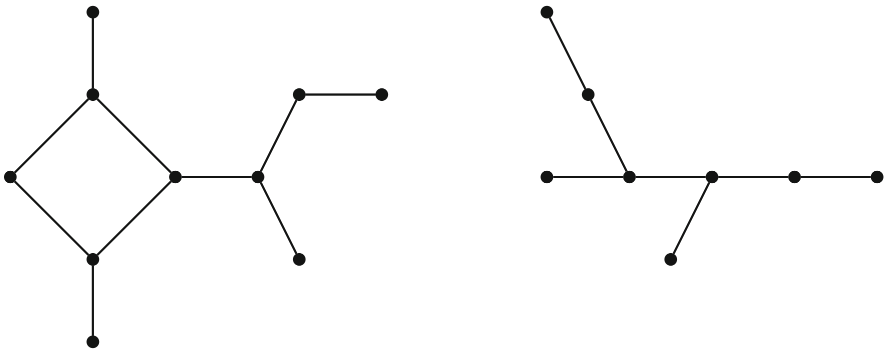

Figure 3. Some examples of generalized thorn graphs

we replace $S(P_n, k)$ by its value given in (2), we obtain, for $0 \le i < n - l$, $1 \le s \le k$ and $0 \le t \le s$

$$S(n-1, k-1) = \sum_{i,s,t} (-1)^i S(l-1, s-1) S(n-l-i-1, k-t-1) \times \binom{s}{s-t} \binom{k-t}{s-t} (s-t)!,$$

by changing of variables k and n, we get, for $n \ge 2$, $1 \le k \le n$ and $1 \le l < n$,

$$S(n, k) = \sum_{i,s,t} (-1)^i S(l-1, s-1) S(n-l-i, k-t) \binom{s}{s-t} \binom{k-t+1}{s-t} (s-t)!,$$

where i, s, t satisfy $0 \le i < n - l$, $1 \le s \le min(k, l) + 1$ and $0 \le t \le s$. $\square$

More generally, let us consider a graph $G^* = G(t_1, t_2, \ldots, t_n)$ of G_n which is obtained by attaching t_i (≥ 0) new vertices of degree 1 to a vertex v_i of G_n, $i = 1, \ldots, n$. This definition refers to a class of graphs known in the literature as Thorn graphs. For G_n being a tree $G^* = T^*$ is called a thorn tree.

Moreover, we define a generalized thorn graph $G^{(t)}$ to be a graph obtained from G by attaching new trees T_i of order $t_i \ge 0$ to a given vertex v_i of G_n, $i = 1, \ldots, n$ such that $t = \sum_{i=1}^{n} t_i$. Then, if all T_i's are single vertices then $G^{(t)}$ is a thorn graph. See for instance some generalized thorn graphs in Fig. 3.

Theorem 3.4. *For $t \ge 1$ and $1 \le k \le n + t$, we have,*

$$S(G^{(t)}, k) = \sum_{i=0}^{k-1} S(G_n, k-i) \sum_{j_1+\cdots+j_{i+1}=t-i} (k-1)^{j_1} \cdots (k-i-1)^{j_{i+1}}.$$

Proof. Recall that $G^{(t)}$ is a generalized thorn graph defined by attaching some trees to some vertices of a simple and finite graph of order n, G_n. Thus $|G^{(t)}| = n + t$. To simplify, we adopt the convention: $\beta_{t,k} = S(G^{(t)}, k)$ and

$\beta_{0,k} = S(G,k)$. First, we prove using a bijective combinatorial argument the following recurrence relation

$$\beta_{t+1,k} = (k-1)\beta_{t,k} + \beta_{t,k-1}. \tag{7}$$

As was agreed at the beginning of the proof, we have $\beta_{t+1,k} = S(G^{(t+1)}, k)$. Motivated by the connection to colorings, if we take v the vertex of the end of any attached tree T_i in $G^{(t+1)}$ then we have two possible situations. Either the end vertex has one color already used by other vertices or it has its own color, in this case there is $\beta_{t,k-1}$ colorings and it turns out that there is $(k-1)\beta_{t,k}$ possible colorings in the former case, since v has all possible colors except one used by its neighbor. Hence,

$$\beta_{t+1,k} = (k-1)\beta_{t,k} + \beta_{t,k-1}.$$

Now, we use the induction over t to prove the following recurrence

$$\beta_{t,k} = (k-1)^t \beta_{0,k} + \sum_{j_1=0}^{t-1} (k-1)^{j_1} \beta_{t-1-j_1,k-1}. \tag{8}$$

It is easy to verify using relation (7) for the trivial case ($t = 1$) that the identity (8) is true. By relation (7) and the induction hypothesis we have

$$\beta_{t+1,k} = (k-1)^{t+1} \beta_{0,k} + \sum_{j_1=0}^{t-1} (k-1)^{j_1+1} \beta_{t-1-j_1,k-1} + \beta_{t,k-1}, \tag{9}$$

we set $j_1' = j_1 + 1$ and we get,

$$\beta_{t+1,k} = (k-1)^{t+1} \beta_{0,k} + \sum_{j_1'=1}^{t} (k-1)^{j_1'} \beta_{t-j_1',k-1}, \tag{10}$$

thus the Relation (8) is true for $t \geq 1$. Also, we have,

$$\beta_{t-1-j_1,k-1} = (k-2)^{t-1-j_1} \beta_{0,k-1} + \sum_{j_1=0}^{t-1-j_1-1} (k-1)^{j_2} \beta_{t-2-j_1-j_2,k-2}, \tag{11}$$

hence, using the same approach as in (8), we establish by induction over t,

$$\begin{aligned}\beta_{t,k} = (k-1)^t \beta_{0,k} + \beta_{0,k-1} \sum_{j_1+j_2=t-1} (k-1)^{j_1} (k-2)^{j_2} \\ + \sum_{j_1,j_2/j_1+j_2 \leq t-2} (k-1)^{j_1+j_2} \beta_{t-2-j_1-j_2,k-2}.\end{aligned}$$

By developing the sum in the right hand side with the same way and applying the same inductive procedure, we get the result, for $i \geq 1$,

$$\beta_{t,k} = \sum_{i=0}^{k-1} \beta_{0,k-i} \sum_{j_1+j_2+\cdots+j_{i+1}=t-i} (k-1)^{j_1} \cdots (k-i-1)^{j_{i+1}}.$$

□

Paw, tadpole, subdivided star and caterpillar graphs are particular cases of the generalized thorn graphs $G^{(t)}$. See definitions of these graph classes in Brandstadt et al. [5].

For $G_n = S_n$ (a star) and all T_i's are paths, we get subdivisions of star graphs.

For $G_n = C_n$ (a cycle), $i = 1$ and T_1 is a path, we obtain a tadpole graph, where a tadpole graph is the graph obtained by joining a path to a cycle with a bridge.

For $G^{(t)} = G^*$ a thorn graph and G_n a path of length n, we get a caterpillar, where a caterpillar is a tree with the property that the removal of its endpoints leaves a path.

For instance, we propose to count the number of stable partitions into k stable sets for some graphs cited above.

From Theorem 3.4 and Relation (4), the number of stable partitions for a tadpole graph $C_{n,p}$ constructed with a cycle of order $n \geq 3$ joined to a path of order t, is given by

$$\sum_{i=0}^{k-1} \sum_{j=k-i-1}^{n-1} (-1)^{n-1-j} S(j, k-i-1) \sum_{j_1+\cdots+j_{i+1}=t-i} (k-1)^{j_1} \cdots (k-i-1)^{j_{i+1}}.$$

For a caterpillar constructed with a path of order $n \geq 2$, attached to t pendent vertices, using Theorem 3.4 and Relation (2), the number of stable partitions is equal to

$$\sum_{i=0}^{k-1} S(n-1, k-i-1) \sum_{j_1+\cdots+j_{i+1}=t-i} (k-1)^{j_1} \cdots (k-i-1)^{j_{i+1}}.$$

Observe that the paths are particular cases of the considered class $G^{(t)}$. Moreover, the number of stable partitions of paths have already been done by several authors with several interpretations, cited previously. Considering this fact, the following identity holds.

Corollary 3.5. *For $n \geq 1, l \leq n$ and $1 \leq k \leq n$, we have,*

$$S(n+t-1, k-1) = \sum_{i=0}^{k-1} S(n-1, k-i-1) \sum_{j_1+\cdots+j_{i+1}=t-i} (k-1)^{j_1} \cdots (k-i-1)^{j_{i+1}}.$$

Proof. With the same way, as in Corollary 3.3, we consider a path of length $n+t$ and a path of length n as initial graph. Then, the identity results in by replacing $G^{(t)}$ and evaluating the sequence in Theorem 3.4. $\square$

Notice that the previous theorem can be extended to the following one, in which $\sum_{j_1+\cdots+j_{i+1}=t-i}(k-1)^{j_1} \cdots (k-i-1)^{j_{i+1}}$ is replaced by the r-Stirling numbers of the second kind.

Theorem 3.6. *For $t \geq 1$ and $1 \leq k \leq n+t$, we have,*

$$S(G^{(t)}, k) = \sum_{i=0}^{k-1} S(G_n, k-i) S_{k-i-1}(t+k-i-1, k-1).$$

Proof. The r-Stirling numbers of the second kind have the generating function denoted by $\phi_k(u)$ and described as follows

$$\phi_k(u) = \sum_{n \geq k} S_r(n+r, k+r) u^n = \frac{u^k}{(1-(r+1)u) \cdots (1-(r+k)u)}. \tag{12}$$

On other hand, it is well known that

$$\frac{1}{1-ju} = \sum_{n \geq 0} (ju)^n. \tag{13}$$

From Relations (12) and (13) we obtain,

$$\phi_k(u) = u^k \sum_{n_1 \geq 0} ((r+1)u)^{n_1} \sum_{n_2 \geq 0} ((r+2)u)^{n_2} \cdots \sum_{n_k \geq 0} ((r+k)u)^{n_k}, \tag{14}$$

summing by parts, relation (14) can be written as follows

$$\phi_k(u) = u^k \sum_{n_1, n_2, \cdots, n_k \geq 0} (r+1)^{n_1} (r+2)^{n_2} \cdots (r+k)^{n_k} u^{n_1+n_2+\cdots+n_k}, \tag{15}$$

also, relation (15) gives

$$\phi_k(u) = u^k \sum_{n \geq 0} (\sum_{n_1+n_2+\cdots+n_k=n} (r+1)^{n_1} (r+2)^{n_2} \cdots (r+k)^{n_k}) u^n, \tag{16}$$

thus,

$$\phi_k(u) = \sum_{m \geq k} (\sum_{n_1+n_2+\cdots+n_k=m-k} (r+1)^{n_1} (r+2)^{n_2} \cdots (r+k)^{n_k}) u^m, \tag{17}$$

by identification with the generating function we get,

$$S(n+r, m+r)_r = \sum_{n_1+n_2+\cdots+n_m=n-m} (r+1)^{n_1} (r+2)^{n_2} \cdots (r+m)^{n_m}, \tag{18}$$

by changing of variables, we obtain

$$\sum_{j_1+j_2+\cdots+j_k=t-i} (k-1)^{j_1} (k-2)^{j_2} \cdots (k-i-1)^{j_{i+1}} = S_{k-i-1}(t+k-i-1, k-1). \tag{19}$$

□

Consequently, we derive an identity of the Stirling numbers of the second kind in terms of Stirling and the r-Stirling numbers of the second kind.

Corollary 3.7. *For $t \geq 1$ and $0 \leq k \leq n+t$ we have,*

$$S(n+t, k) = \sum_{i=0}^{k} S(n, k-i) S_{k-i}(t+k-i, k).$$

Proof. From Corollary 3.5 and Theorem 3.6, we get the formula. □

This gives rise to an explicit formula related to binomial coefficient which expresses the Stirling numbers of the second kind in terms of the generalized r-Stirling numbers of the second kind, evaluated with two summations.

Corollary 3.8. *For $0 \leq k \leq n+t$, $n \geq 0$, $t \geq 1$ and $l \leq k$, we have*

$$S(n+t,k) = \sum_{i=0}^{k-l} S(n,k-i) \sum_{j} \binom{t}{j} S_l(t+l-j,i+l)(k-l-i)^j.$$

Proof. This is obtained using Theorem 3.6 combined with relation (33 See P. 249 in [6]) in Broder's explicit formulas for the r-Stirling numbers of the second kind [6]. □

Note that for $l = 1$, an identity of the Stirling numbers of the second kind can be deduced.

Corollary 3.9. *For $0 \leq k \leq n+t$, $n \geq 0$, $t \geq 1$, we have*

$$S(n+t,k) = \sum_{i=0}^{k-1} S(n,k-i) \sum_{j} \binom{t}{j} S(t+1-j,i+1)(k-i-1)^j.$$

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