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Study of the Regularity of Solutions for a Elasticity System with Integrable Data with Respect to the Distance Function to the Boundary

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Abstract. In this article, we are interested in the existence, uniqueness and regularity of the solution of the linear elasticity system. More precisely, the quasi-static elasticity system. In the first part, we study the existence of a weak solution and the regularity in the space $W_0^{1,p}(\Omega)$, $\forall p \in]1,+\infty[$ for a p-integrable source function. In the second part, the very weak solution is introduced which can be considered when the second member is a function with a very weak solution, for example, a locally integrable function. Such source functions lead to a lack of regularity for the solution in the fact that existence in classical spaces is no longer assured. So, to overcome this difficulty, the strategy consists in approaching it by another more regular problem "converging" towards the initial problem "in a direction to be specified".

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1. Introduction

Scientists have for centuries attempted to write some models describing the behavior of the material. More or less generally, accurate or robust, these models are based on the representation of the deformation phenomena using the vector fields and tensors. This describes, in particular, the deformation of the object as well as the internal constraints (the internal forces involved between portions) that it undergoes. The behavior laws then join the constraints with the resultant deformation.

The theory of linear elasticity lies within the framework of the description of slow deformable solids, and on the other hand it is imposed that the elastic constitutive law connecting the stress tensor to that of the deformations is linear. When the elastic solid has an isotropic behavior (it does not favor any direction of space),we obtain the law of behavior of Hooke,

$$\sigma_{ij} = \lambda \sum_{K=1}^{n} \varepsilon_{KK}(u)\delta_{ij} + 2\mu\varepsilon_{ij}(u).$$

with σ and ε, respectively, representing the stress tensor and the deformed tensor. In addition, λ and μ are the positive coefficients of Gabriel Lamé.

Still, in the case of a homogeneous material, the various coefficients introduced above are constants, taking into account that $\vec{u}$ is a field of displacement and the conservation of mass and the quality of the material is written as:

$$\rho \frac{\partial^2 \vec{u}}{\partial t^2} - \mu\Delta\vec{u} - (\lambda + \mu)\vec{\nabla}(\vec{\nabla} \cdot \vec{u}) = \vec{f},$$

where $\vec{f}$ represents the external force acting on the material and ρ is the constant representing the density of the medium homogeneous.

When the displacement is independent of time, we then speak of the quasi-static elasticity . We then come back to

$$(\lambda + \mu)\nabla(\operatorname{div} \vec{u}) + \mu\Delta\vec{u} + \vec{f} = 0.$$

This system differs from that of Laplace in the diffusion term $(\lambda + \mu)\nabla(\operatorname{div} \vec{u})$ which gives it all its richness, but also all its complexity.

As λ and μ are nonzero positive constants, we can simplify the constants and reduce the system to the following form:

$$(P) : \begin{cases} E\vec{u} = -\Delta\vec{u} - \lambda\nabla \operatorname{div} \vec{u} = \vec{f} & \text{in } \Omega. \\ \vec{u} = \vec{0} & \text{on } \partial\Omega, \end{cases}$$

where E denotes the elasticity operator which is a second-order linear elliptic operator.

In the literature, we find many works dedicated to the mathematical study of linear elasticity system. In fact, Ciarlet, in [4,5] explained the physical phenomenon and presented the well-posed character of the system. In short, by calculating the index of the operator associated with the elasticity, he showed the regularity $W^{2,p}(\Omega)$ for $p \geqslant 2$ in a bounded domain of class $\mathcal{C}^2$ subset of $\mathbb{R}^2$ (see Theorem 2.2.4 page 80 of [4]).

In addition, Grisvard was also interested in the problem, but in a polyhedric domain included in $\mathbb{R}^3$. He shows the regularity of solution in $W^{r,2}(\Omega)$ for $r \in [\frac{3}{2}, 2[$. It denotes by r the index of regularity which depends on the solid angles formed by the vertices of the polyhedron (see [12]). Also, Shi and Wright (see [20, page 295]) worked on the regularity in $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ with $1 < p < \infty$, but in a domain included in $\mathbb{R}^3$ having a regularity between $\mathcal{C}^1$ and $\mathcal{C}^{1,1}$. For applications on the elasticity problems, we refer the reader to [7–9].

We note that the elasticity problem is treated in $\mathbb{R}^3$ mostly in all references in the literature, simply because the natural phenomenon is done in a space of dimension 3. This does not mean that we can not consider the equation in $\mathbb{R}^n$, but in this case, the interest at the physical level is limited. Here, we are interested in this mathematical generalization by focusing on external forces depending on the distance to the boundary.

1.1. Notation and Definition

For a measurable set in $\mathbb{R}^n$, we denote by $|E|$ its Lebesgue measure and for a measurable function u from the open bounded set Ω into $\mathbb{R}^n$, we define the following auxiliary functions (see [15]):

1. The distributional function m of u defined as, $m : \mathbb{R} \to]0, \Omega[$ such that
$$m(t) = \text{ mes}\{x \in \Omega : u(x) > t\} = |u > t|,$$
2. The monotone rearrangement of u (denoted by u_*), is the generalized inverse of m, i.e.,
$$u_*(s) = \inf\{t \in \mathbb{R} : |u > t| \leq s, \}, s \in]0, |\Omega|[,$$
$$u_*(0) = ess \sup_{\Omega} u.$$

The Lorentz spaces $L^{p,q}(\Omega)$ are defined (see [6,7]), for $1 \leq p < +\infty, 1 \leq q < +\infty$, as,

$$L^{p,q}(\Omega) = \left\{ u : \Omega \to \mathbb{R} \text{ measurable } : \int_0^{|\Omega|} \left[t^{1/p}|u|_{**}(t)\right]^q \frac{\mathrm{d}t}{t} < +\infty \right\},$$
$$L^{p,\infty}(\Omega) = \left\{ u : \Omega \to \mathbb{R} \text{ measurable } : \sup_{t\leq|\Omega|} t^{1/p}|u|_{**}(t) < +\infty \right\},$$

with

$$|u|_{**} = \frac{1}{t}\int_0^t |u|_*(s)\,\mathrm{d}s, \quad \text{for } t > 0.$$

We note that we can associate with a banach space V a sobolev space,

$$W^1V := \{v \in L^1_{loc}(\Omega) \quad \text{such that} \quad |\nabla v| \in V\}.$$

In particular,

$$W_0^1L^{p,q}(\Omega) = \{u \in W_0^{1,1}(\Omega) : |\nabla u| \in L^{p,q}(\Omega)\}.$$

Definition 1.1 (*Lebesgue weighted spaces*) (*see [7]*). Let Ω be a bounded subset of $\mathbb{R}^n$. If $\omega : \ \Omega \to]0, +\infty[$ is an integrable function, we define the Lebesgue weighted spaces as

$$L^p(\Omega, \omega) := \left\{ f \text{ is measurable } : |f|^p_{L^p(\Omega,\omega)} = \int_{\Omega} |f(x)|^p \; \omega(x)\,\mathrm{d}x < \infty \right\}.$$

In particular, we define $L^p(\Omega, \delta^\alpha)$ with $0 \leqslant \alpha \leqslant 1$, $1 \leqslant p < +\infty$ and $\delta(x) = \text{dist }(x, \partial\Omega)$ in the following way:

- $f \in L^p(\Omega, \delta^p)$ if $\displaystyle\int_{\Omega} |f(x)|^p \delta(x)^\alpha \,\mathrm{d}x < +\infty.$
- $f \in L^1(\Omega, \delta(1 + |\ln \delta|))$ if $\displaystyle\int_{\Omega} |f(x)|\delta(1 + |\ln \delta|)\,\mathrm{d}x < +\infty.$

These spaces are complete associated with norm $|\cdot|_{L^p(\Omega,\omega)}$.

For Ω, an open bounded domain subset of $\mathbb{R}^n$, we will consider formulations equivalent to the initial elasticity system, for this reason, we define

$$\mathcal{M}_{n\times m}(\Omega;\mathbb{R}) = \Big\{A : A(x) \in \mathcal{M}_{n\times m}(\mathbb{R}), x \in \Omega\Big\},$$

$$\vec{u} = \begin{pmatrix} u^1 \\ \vdots \\ u^N \end{pmatrix}, \ \vec{f} = \begin{pmatrix} f^1 \\ \vdots \\ f^N \end{pmatrix}, \quad \mathcal{G} = \begin{pmatrix} \vec{G}^1 \\ \vdots \\ \vec{G}^n \end{pmatrix} = \begin{pmatrix} G_1^1 & \dots & G_N^1 \\ \vdots & \vdots & \vdots \\ G_1^n & \dots & G_N^n \end{pmatrix} = \left(G_\alpha^i\right)_{\substack{1\leqslant i\leqslant n \\ 1\leqslant\alpha\leqslant N}}$$

with

$$\|\mathcal{G}\|_1 = \max_\alpha \sum_{i=1}^n |G_\alpha^i|$$

and

$$\operatorname{div}\mathcal{G} = \begin{pmatrix} \operatorname{div}\vec{G}^1 \\ \vdots \\ \operatorname{div}\vec{G}^n \end{pmatrix} \in \mathcal{M}_{n\times 1}(\Omega;\mathbb{R}).$$

We will show (see Proposition 2.1) that (P) can be written in divergence form, for a suitable tensor A as

$$(Q) : \begin{cases} -D_\alpha(A_{ij}^{\alpha\beta} D_\beta u^j) = -D_\alpha G_\alpha^i + f^i & \text{in } \Omega \\ u^i = 0 & \text{on } \partial\Omega, \end{cases}$$

which is written with Einstein's notation of repeated indexes, but can be fully written as

$$\begin{cases} -\sum_{j=1}^N \sum_{\alpha,\ \beta=1}^n D_\alpha(A_{ij}^{\alpha\beta} D_\beta u^j) = -\sum_{\alpha=1}^n D_\alpha G_\alpha^i + f^i \\ u^i = 0, \end{cases} \tag{1.1}$$

where $D_\alpha = \dfrac{\partial}{\partial x_\alpha}$ with $1 \leqslant \alpha, \beta \leqslant n$, $1 \leqslant i, j \leqslant N$ and the coefficients $A_{ij}^{\alpha\beta}$ verify the Legendre–Hadamard coercivity conditions (see Proposition 2.1).

Definition 1.2. of weak solution (Q) or of (P). We call weak solution of (Q) or of (P) a function $\vec{u} \in W_0^{1,p}(\Omega)^n$ for a real $1 < p < +\infty$ which verifies the system in the sense of distributions.

In most cases, we will appeal to the following theorems of existence and regularity.

Theorem 1.1 (Morrey's Theorem) (see [14, Theorem 6.48 and 6.55], [10,19] Lemma 2 page 265]). *Let Ω be an open bounded domain subset of $\mathbb{R}^n$ of class $C^{1,1}$, for the coefficients $A_{ij}^{\alpha\beta}$ continuous on $\bar{\Omega}$ and satisfies the coercivity conditions in the sense of Legendre–Hadamard* (see below 2.1)*, we have:*

1. *If* $G = 0$ *and* $\vec{f} \in L^p(\Omega; \mathbb{R}^N)$ *for* $1 < p < n$, *then, the problem* (Q) *admits a unique solution* $\vec{u}$ *in the sense of distributions belonging to* $W_0^{1,p^*}(\Omega; \mathbb{R}^N)$ *with* $p^* = \dfrac{np}{n-p}$ *and in addition,*
$$\|\nabla \vec{u}\|_{L^{p^*}(\Omega)^{n\times N}} \leqslant C_p \|\vec{f}\|_{L^p(\Omega)^N}.$$
2. *If* $\vec{f} = 0$ *and* $G \in L^p(\Omega; \mathbb{R}^{n\times N})$ *with* $1 < p < +\infty$, *then, it admits one and only one solution* $\vec{u}$, *in the sense of distributions,* $\vec{u} \in W_0^{1,p}(\Omega, \mathbb{R}^N)$ *and*
$$\|\nabla \vec{u}\|_{L^p(\Omega;\mathbb{R}^{n\times N})} \leqslant C_p \|G\|_{L^p(\Omega)}.$$

2. Case of the Elasticity System: Classical Results of Existence and Regularity

2.1. Formulation Equivalent to the Elasticity System

The purpose of this section is to show that the system can be written in three equivalent manners. First, in the initial vectorial one (P), then by component and finally in the scalar form. First of all, we start initially by the vectorial and tensorial form already illustrated in the introduction:

$$(P)_{\text{div}} : \begin{cases} -\Delta \vec{u} - \lambda \nabla \operatorname{div} \vec{u} = \operatorname{div} \mathcal{G} & \text{in } \Omega, \\ \vec{u} = \vec{0} & \text{on } \partial\Omega. \end{cases}$$

By writing this equation component by component for $\vec{u}$ and $\mathcal{G}$,

$$(P)_c : \begin{cases} -\Delta u^i - \lambda \displaystyle\sum_{\beta=1}^{n} \frac{\partial^2 u^\beta}{\partial x_i \partial x_\beta} = \sum_{\beta=1}^{n} \frac{\partial G^i}{\partial x_\beta} & \text{in } \Omega, \\ u^i = 0 & \text{on } \partial\Omega, \end{cases}$$

with u^i being the i^{th} component of $\vec{u}$ and $\sum_{\beta=1}^{n} \dfrac{\partial G^i_\beta}{\partial x_\beta} = \operatorname{div} G^i$.

Finally, we will prove that the system can also be written in the condensed form:

$$(P)_s : \begin{cases} -D_\alpha \left(A_{ij}^{\alpha\beta} (D_\beta u^j) \right) = D_\alpha G^i_\alpha & \text{in } \Omega. \\ u^j = 0 & \text{on } \partial\Omega, \end{cases}$$

where $D_\alpha = \dfrac{\partial}{\partial x_\alpha}$, with $1 \leqslant \alpha, \beta \leqslant n$ and $1 \leqslant i, j \leqslant n$. To determine the coefficients $A_{ij}^{\alpha\beta}$, we have the following proposition.

Proposition 2.1. *We designate by* E *the elasticity operator such that,*
$$E\vec{u} = -\Delta \vec{u} - \lambda \nabla \operatorname{div} \vec{u}.$$
Then,
$$E\vec{u} = -D_\alpha (A_{ij}^{\alpha\beta} (D_\beta u^j)),$$
where, for all $1 \leqslant i, j, \alpha, \beta \leqslant n$, *we have*

1. *$A_{ij}^{\alpha\beta}(x) = \delta_{\alpha\beta}\delta_{ij} + \lambda\delta_{i\alpha}\delta_{j\beta}$, with δ_{ij} which is the Kronecker symbol.*
2. $0 \leqslant A_{ij}^{\alpha\beta} \leqslant 1 + \lambda$.
3. *For all $\eta \in \mathbb{R}^n$ and $\zeta \in \mathbb{R}^n$, $\sum_{i,\ j=1}^{n} \sum_{\alpha,\ \beta=1}^{n} A_{ij}^{\alpha,\ \beta}\zeta_\alpha\zeta_\beta\eta^i\eta^j \leqslant |\eta|^2|\zeta|^2$.*
4. *For all $P \in \mathbb{R}^{n\times n}$, $\sum_{i,j=1}^{n} \sum_{\alpha,\ \beta=1}^{n} A_{ij}^{\alpha\beta}P_\alpha^i P_\beta^j \geqslant \|P\|^2$ with $\|P\|_2 = \sum_{\alpha,i=1}^{n}(P_\alpha^i)^2$.*

Proof. 1. We have:

$$-\Delta u^i = -\sum_{\beta=1}^{n} D_\beta^2 u^i$$
$$= -\sum_{\alpha,\beta,j=1}^{n} D_\alpha(\delta_{\alpha\beta}\delta_{ij}D_\beta u^j),$$

and

$$-\lambda\frac{\partial}{\partial x_i}(\operatorname{div}\vec{u}) = -\lambda\Bigg(\sum_{\alpha,i=1}^{n}\delta_{i\alpha}D_\alpha\Bigg)\Bigg(\sum_{j,\ \beta=1}^{n}\delta_{\beta j}D_\beta u^j\Bigg)$$
$$= -\lambda\sum_{\alpha,\beta,j=1}^{n}\delta_{\alpha i}\delta_{j\beta}D_\alpha D_\beta u^j.$$

Consequently,

$$-\Delta u^i - \lambda\frac{\partial}{\partial x_i}(\operatorname{div}\vec{u}) = -\sum_{\alpha,\beta,j=1}^{n} D_\alpha\Big((\delta_{\alpha\beta}\delta_{ij} + \lambda\delta_{i\alpha}\delta_{j\beta})D_\beta u^j\Big).$$

Therefore,

$$A_{ij}^{\alpha\beta}(x) = \delta_{\alpha\beta}\delta_{ij} + \lambda\delta_{i\alpha}\delta_{j\beta}. \tag{2.1}$$

2. Due to (2.1), we observe that the coefficients $A_{ij}^{\alpha\beta}$ are bounded. In fact, if $\alpha = \beta = i = j$, then $A_{ij}^{\alpha\beta} \leqslant 1 + \lambda$. Otherwise, $A_{ij}^{\alpha\beta} = 0$.
3. Let us check the Legendre–Hadamard condition, $\forall\zeta \in \mathbb{R}^n$ and $\eta \in \mathbb{R}^n$ we have:

$$\sum_{i,j=1}^{n}\sum_{\alpha,\beta=1}^{n} A_{ij}^{\alpha\beta}\zeta_\alpha\zeta_\beta\eta^i\eta^j = \sum_{\alpha,\beta=1}^{n}\delta_{\alpha\beta}\zeta_\alpha\zeta_\beta\sum_{i,j=1}^{n}\delta_{ij}\eta^i\eta^j$$
$$+\lambda\sum_{\alpha,i=1}^{n}\delta_{i\alpha}\eta^i\zeta_\alpha\sum_{\beta,j=1}^{n}\delta_{j\beta}\eta^j\zeta_\beta$$
$$\geqslant |\zeta|^2|\eta|^2 + \lambda(\zeta.\eta)^2 \geqslant |\zeta|^2|\eta|^2.$$

This condition is weak, but it does not imply the uniform ellipticity of the coefficients. For that, $\forall P \in R^{n\times n}$, we have:

$$\sum_{\alpha,\beta=1}^{n}\sum_{i,j=1}^{n} A_{ij}^{\alpha\beta}P_\alpha^i P_\beta^j = \sum_{\alpha,\beta=1}^{n}\sum_{i,j=1}^{n} \delta_{\alpha\beta}P_\alpha^i P_\beta^j \delta_{ij} + \lambda \sum_{\alpha,\beta=1}^{n}\sum_{i,j=1}^{n} \delta_{i\alpha}P_\alpha^i P_\beta^j \delta_{j\beta}$$
$$= \sum_{\alpha,i=1}^{n} P_\alpha^i P_\alpha^i + \lambda \sum_{\alpha=1}^{n} P_\alpha^\alpha \sum_{\beta=1}^{n} P_\beta^\beta = \sum_{\alpha,i=1}^{n} (P_\alpha^i)^2$$
$$+\lambda(tr(P))^2 \geqslant \|P\|_2^2.$$

Hence, the desired result.

□

2.2. Classical Weak Solutions of the Elasticity System with Standard Data

Using these formulations and Theorem 1.1 and the regularity theorem of elliptic problems (see [11]), we have,

Theorem 2.1. *Let Ω be an open bounded domain of class $C^{1,1}$ subset of $\mathbb{R}^n$, we suppose that $\vec{f} \in L^p(\Omega)^n$ with $p \in]1, +\infty[$. Then, there exists a unique solution $\vec{u}$ of elasticity system (P) which belongs to $W_0^{1,p^*}(\Omega)^n$ with $p^* = \dfrac{np}{n-p}$, if $p < n$ and $p^* < +\infty$ if $p \geqslant n$, such that, noting that $\nabla\vec{u} : \nabla\vec{\varphi} = \sum_{i,j=1}^{n} \dfrac{\partial u^i}{\partial u_j}\dfrac{\partial \varphi^i}{\partial x_j}$, we have*

$$a(\vec{u},\vec{\varphi}) := \int_\Omega \nabla\vec{u} : \nabla\vec{\varphi}\,\mathrm{d}x + \lambda \int_\Omega \mathrm{div}(\vec{u})\,\mathrm{div}(\vec{\varphi})\mathrm{d}x$$
$$= \int_\Omega \vec{f}\cdot\ \vec{\varphi}\,\mathrm{d}x \quad \forall \vec{\varphi} \in W_0^{1,p'}(\Omega)^n, \qquad (P_{var})$$

Furthermore,

$$\vec{u} \in W^2L^p(\Omega), \quad ||\vec{u}||_{W^2L^p(\Omega)} \leqslant c||\vec{f}||_{L^p(\Omega)}.$$

Idea of the proof. Since (P) is a linear elliptic system, then,

- For $p = 2$, it suffices to verify the Lax–Milgram conditions and use the regularity property of the elliptic problem to obtain the existence of a unique solution $\vec{u} \in H_0^1(\Omega)^n \cap H^2(\Omega)^n$.
- For all $p \in]1, +\infty[$, $\vec{f} \in L^p(\Omega)^n$, using the approximation of $\vec{f}$ by a bounded sequence of functions, and using Morrey's Theorem 1.1, we obtain the solution
$$\vec{u} \in W_0^{1,p^*}(\Omega)^n \quad \text{and} \quad \|\nabla\vec{u}\|_{L^{p^*}(\Omega)^{n^2}} \leqslant C_p'\|\vec{f}\|_{L^p(\Omega)^n}.$$

□

Remark 2.1. A weak solution is equivalent to a weak variatinal solution $(P)_{var}$.

Lemma 2.1. *Suppose that $\vec{f} \in L^{p,q}(\Omega)^n$, for some $p \in]1, +\infty[$ and $q \in [1, +\infty[$. Then the solution $\vec{u}$ of the elasticity system belongs to $W^2L^{p,q}(\Omega)^n \cap W_0^{1,1}(\Omega)^n$ and in addition, there exists a positive constant $C > 0$ such that,*

$$\| \vec{u} \|_{W^2L^{p,q}(\Omega)^n} \leqslant C\| \vec{f} \|_{L^{p,q}(\Omega)^n}.$$

Proof. We note that throughout the proof C_0, C_1, C_2, C and C' denote positive constants which may vary from line to line. Owing to the results of Theorem 2.1 and regularity theorem of elliptic problems in [11], we can construct a linear and bounded mapping on $L^p(\Omega)^n$ into $L^p(\Omega)$ for all $p \in]1, \infty]$,

$$T_0 : L^p(\Omega)^n \longrightarrow L^p(\Omega),$$
$$\vec{f} \longrightarrow T_0\vec{f} = \max_\alpha \|D^\alpha \vec{u}\|_1,$$

where

$$T_0\vec{f}(x) = \max_\alpha \|D^\alpha \vec{u}(x)\|_1.$$
$$||D^\alpha u(x)||_1 = \sum_{i=1}^n |D^\alpha u^i(x)|.$$

It is bounded in the following sense

$$\| T_0\vec{f} \|_{L^p(\Omega)} = \| \vec{u} \|_{W^2L^p(\Omega)^n} \leqslant C' \| \vec{f} \|_{L^p(\Omega)^n}.$$

To extend this situation to the Lorentz space, we use the interpolation of Marcinkiewicz (see [2]). For all $1 < p_0 \leqslant p_1 \leqslant \infty$, we will verify the conditions of this interpolation, then we will show that T_0 is of weak type (p_0, p_0) and (p_1, p_1);

$$\| T_0\vec{f} \|_{L^{p_0,\infty}(\Omega)} \leqslant C_0 \| \vec{f} \|_{L^{p_0,1}(\Omega)^n}$$
$$\| T_0\vec{f} \|_{L^{p_1,\infty}(\Omega)} \leqslant C_1 \| \vec{f} \|_{L^{p_1,1}(\Omega)^n}.$$

In fact, the injection into the Lorentz space (see [6,7,15]) gives us

$$L^{p_i,1} \hookrightarrow L^{p_i,p_i} = L^{p_i} \hookrightarrow L^{p_i,\infty}, i = 0, 1.$$

Combining these injections with the regularity theorem (see, Theorem 9.15, page 241 in [11]), we obtain,

$$\| T_0\vec{f} \|_{L^{p_0,\infty}(\Omega)} \leqslant \| T_0\vec{f} \|_{L^{p_0}(\Omega)} \leqslant C_0 \| \vec{f} \|_{L^{p_0}(\Omega)^n} \leqslant C_0 \| \vec{f} \|_{L^{p_0,1}(\Omega)^n}.$$

Similarly,

$$\| T_0\vec{f} \|_{L^{p_1,\infty}(\Omega)} \leqslant \| T_0\vec{f} \|_{L^{p_1}(\Omega)} \leqslant C_1 \| \vec{f} \|_{L^{p_1}(\Omega)^n} \leqslant C_2 \| \vec{f} \|_{L^{p_1,1}(\Omega)^n}.$$

Therefore, T_0 is of weak type (p_0, p_0) and (p_1, p_1). Let $0 < \theta < 1$ and $p \in]1, +\infty]$ such that $\frac{1}{p} = \dfrac{1-\theta}{p_0} + \dfrac{\theta}{p_1}$, then T_0 maps $L^{p,q}$ into $L^{p,q}$, and is bounded.

This implies in particular the estimate which controls $\| \vec{u} \|_{W^2L^{p,q}(\Omega^n)}$. i.e.,

$$\| T_0\vec{f} \|_{L^{p,q}(\Omega)} = \| \vec{u} \|_{W^2L^{p,q}(\Omega)^n}$$
$$\leqslant C(\theta) \max\{C_0, C_1\} \| \vec{f} \|_{L^{p,q}(\Omega)^n}.$$

□

3. Existence and Regularity of the Solution for Integrable Data with Respect to Distance

The object of this work is to study the elasticity system when the source function $\vec{f}$ is in the space $L^1(\Omega,\delta)^n$ of integrable functions with respect to the distance function δ. Therefore, $\vec{f}$ is more singular than in Theorem 2.1. As in the case of the linear problem $-\Delta u = f \ \in L^1(\Omega,\delta),\ u = 0$ treated by Brézis [2], Diaz and Rakotoson [6, Theorem 1, 2 and 3 page 2 and 3], [16, Proposition 2 page 2900], [17, Proposition 2 page 1137]. We note that the solution $\vec{u}$ can not generally have its gradient in $L^1(\Omega)^n$.

Our aim is then to determine the regularity of the gradient of the solution under these conditions. Since the existence and the uniqueness of the solution $\vec{u}$ does not follow from the previous theorems, and this solution is not necessarily in $W^{1,1}(\Omega)^n$, we go as in the case of the equations, using the notion of very weak solution introduced in the framework of the data $L^1(\Omega,\delta)$ by H. Brezis. Hereafter, Ω is a bounded domain atleast of class $C^{1,1}$ subset of $\mathbb{R}^n$ and $\delta(x) = \mathrm{dist}(\mathrm{x},\partial\Omega)$.

3.1. Definition of Very Weak Solution Given by Brézis.

Definition 3.1 (*see [6, definition 3.1 page 47], [2]*). Let $E = -\Delta - \lambda\nabla\,\mathrm{div}$ be the second-order elliptic operator and $\vec{f} \in L^1(\Omega,\delta)$ such that,

$$(P) : \begin{cases} E\vec{u} = \vec{f} & \text{in } \ \Omega, \\ \vec{u} = 0 & \text{on } \ \partial\Omega. \end{cases}$$

We say that, $\vec{u}$ is a very weak solution of the problem if

$$(P)_{vw} : \begin{cases} \vec{u} \in L^1(\Omega)^n, \forall\vec{\varphi} \in C^2(\bar{\Omega})^n \text{ and } \vec{\varphi} = \vec{0} \text{ on } \partial\Omega, \\ \displaystyle\int_\Omega \vec{u}\cdot E^*\vec{\varphi}(x)\,\mathrm{d}x = \int_\Omega \vec{f}\cdot\vec{\varphi(x)}\,\mathrm{d}x, \end{cases}$$

with E^* is the adjoint operator of E, noting that $E^* = E$.

Remark 3.1. Since we do not know the regularity of $\vec{u}$, we transport the information on the derivatives of the test functions by making integrations by parts. Noting that each integral of the problem $(P)_{vw}$ has a meaning.

Proposition 3.1. Uniqueness of the very weak solution
If the very weak solution exists, then it is unique.

Proof. Let $\vec{g} \in C_c^\infty(\Omega)^n$. According to the Schauder estimations, there exists a function $\vec{\varphi} \in C^2(\overline{\Omega})^n$ such that $\vec{\varphi} = 0$ is a weak solution for $E\vec{\varphi} = \vec{g}$. Thus, if $\vec{u}$ is the difference of two very weak solutions, then :

$$0 = \int_\Omega \vec{u}\cdot E\vec{\varphi}\mathrm{d}x = \int_\Omega \vec{u}\cdot\vec{g}\mathrm{d}x \quad \forall\,\vec{g} \in C_c^\infty(\Omega)^n,$$

hence

$$\vec{u} = 0.$$

□

Theorem 3.1 (see [18, Lemma 1 page 3]). *The set*

$$\left\{\vec{\varphi} \in C^2(\bar{\Omega})^n : \vec{\varphi} = \vec{0} \text{ on } \partial\Omega\right\},$$

is dense in

$$W^2L^{p,q}(\Omega) \cap W_0^1L^{p,q}(\Omega)^n,\ 1 < p < +\infty,\ q \in [1, +\infty[.$$

When $p = q$ in this theorem, the result states without detailed proof in [11, page 254 exercise 9.6].

The following lemma results from existence for the linear problems essentially due to Diaz and Rakotoson (see Theorem 1 page 812 in [7], Theorem 3.8 page 50 in [6]).

Lemma 3.1. *Let $\vec{f} = (f^i)_{i=1,..,n}$ and the function $f^i \in L^1(\Omega, \delta^{\alpha_i}((1+|\log\delta|)^{\beta_i})$ with $0 < \alpha_i, \beta_i \leqslant 1$.*
Then, there exists a matrix that is a second order $\mathcal{F} = (\vec{F}^1, \vec{F}^2, \dots, \vec{F}^n)^t$ defined over Ω and a positive constant $C_\Omega > 0$ independent of Ω, α_i and β_i such that,

1. $\operatorname{div}(\mathcal{F}) = \vec{f}$, *this equality is defined in the distributional sense with* $\mathcal{F} \in \mathcal{M}_{n\times n}(\Omega; \mathbb{R})$.
2. *If* $\alpha_i \in]0, 1[$ *and* $\beta_i = 0$ *then* $F^i \in V_{\alpha_i\beta_i} = L^{\frac{n}{n-1+\alpha_i}}(\Omega)^n$.
3. *If* $\alpha_i = \beta_i = 0$ *then* $F^i \in V_{\alpha_i\beta_i} = L^{\frac{n}{n-1},\infty}(\Omega)^n$.
4. *If* $\alpha_i = \beta_i = 1$ *then* $F^i \in V_{\alpha_i\beta_i} = L^1(\Omega)^n$.
5. *If* $\alpha_i = 1$ *and* $\beta_i = 0$ *then* $F^i \in V_{\alpha_i\beta_i} = L^{1+\frac{1}{n}}(\Omega, \delta)^n$.

Thus, the matrix $\mathcal{F} \in \prod_{i=1}^n V_{\alpha_i\beta_i} = W$ and $\forall i = 1, \dots, n$ we have,

$$\|F^i\|_{V_{\alpha_i\beta_i}} \leqslant C_\Omega \|f^i\|_{L^1(\Omega,\delta^{\alpha_i}(1+|\log\delta|)^{\beta_i})}.$$

Equivalent to

$$\|\mathcal{F}\|_W \leqslant C_\Omega \sum_{i=1}^n \|f^i\|_{L^1(\Omega,\delta^{\alpha_i}(1+|\log\delta|)^{\beta_i})}.$$

Idea of the proof: According to Díaz and Rakotoson [6] page 50, if the function $f^i \in L^1(\Omega, \delta^{\alpha_i}(1 + |\log\delta|)^{\beta_i})$ and under one of the conditions $2, 3, 4$ and 5, there exists a function $v^i \in L^1(\Omega)$ verified in a very weak sense of the following problem:

$$\begin{cases} -\Delta v^i = f^i & \text{in } \Omega \\ v^i = 0 & \text{on } \partial\Omega, \end{cases}$$

and

$$\|\nabla v^i\|_{V_{\alpha_i\beta_i}(\Omega)^n} \leqslant C(\Omega)\|f^i\|_{L^1(\Omega,\delta^{\alpha_i}(1+|\log\delta|)^{\beta_i})}.$$

Thus,

$$F^i = \nabla v^i \in V_{\alpha_i\beta_i}(\Omega)^n$$

which makes it appropriate. □

3.2. Existence and Uniqueness of the Solution for the Data in $L^1(\Omega,\delta)^n$.

Theorem 3.2. *Let $\vec{f}\in L^1(\Omega,\delta)^n$. Then, there exists the one and only very weak solution of $(P)_{vw}$. Furthemore, $\vec{u}\in L^{n',\infty}(\Omega)^n$ with $n'=\dfrac{n}{n-1}$, such that,*

$$\int_\Omega \vec{u}\cdot E^*\vec{\varphi}\,\mathrm{d}x=\int_\Omega \vec{f}\cdot\vec{\varphi}\,\mathrm{d}x,\quad \forall\vec{\varphi}\in W^2L^{n,1}(\Omega)^n\cap H_0^1(\Omega)^n,$$

where $E^*=-\Delta-\lambda\nabla\,\mathrm{div}$ and there exists a constant $C(\Omega,E^*)>0$ such that,

$$\|\ \vec{u}\ \|_{L^{n',\infty}(\Omega)^n}\leqslant C(\Omega,E^*)\|\ \vec{f}\ \|_{L^1(\Omega,\delta)^n}.$$

Proof. For $k\geqslant 1$, we define the following truncation:

$$T_k(\vec{f})=\vec{f_k}=\big(T_k(f^1),T_k(f^2),\ldots,T_k(f^n)\big)^t$$

with

$$T_k(f^i):=\begin{cases} f^i & \text{if } |f^i|\leqslant k,\\ k\ \mathrm{sign}(f^i) & \text{elsewhere.}\end{cases}$$

By this truncation, we have constructed a function $\vec{f_k}$ that belongs to $L^\infty(\Omega)^n$ $\cap\, L^1(\Omega,\delta^{\alpha_i}(1+|\log\delta|^{\beta_i})^n$, which converges to $\vec{f}$ in $L^1(\Omega,\delta^{\alpha_i}(1+|\log\delta|^{\beta_i})^n$ as k goes to infinity and the following problem:

$$(P_k):\begin{cases} -\Delta\vec{u_k}-\lambda\nabla\,\mathrm{div}\,\vec{u_k}=\vec{f_k} & \text{in } \Omega,\\ \vec{u_k}=\vec{0} & \text{on } \partial\Omega.\end{cases}$$

According to the Theorem 1.1, (P_k) is well-posed and admits a unique solution $\vec{u_k}\in W_0^{1,p}(\Omega)^n,\forall p\in]1,+\infty[$. In particular, $\forall\vec{\varphi}\in C^2(\bar{\Omega}),\ \vec{\varphi}=0$ on $\partial\Omega$,

$$\int_\Omega \vec{u_k}\cdot E^*\vec{\varphi}\,\mathrm{d}x=\int_\Omega \vec{f_k}\cdot\vec{\varphi}\,\mathrm{d}x. \tag{3.1}$$

Thanks to the completeness of the space $L^{n',\infty}$, it suffices to prove that the sequence $\vec{u_k}$ verifies the Cauchy creteria to converge in this space. In fact, for $k\geqslant 1$ and $n\geqslant 1$, we define,

$$\vec{u_{kn}}=\vec{u_k}-\vec{u_n}\quad\text{and}\quad \vec{f_{kn}}=\vec{f_n}-\vec{f_k}.$$

Then, from the relation (3.1) and the density Lemma 3.1, we have $\forall\vec{\varphi}\in H^2(\Omega)^n\cap H_0^1(\Omega)^n$,

$$\int_\Omega \vec{u_{kn}}\cdot E^*\vec{\varphi}\,\mathrm{d}x=\int_\Omega \vec{f_{kn}}\cdot\vec{\varphi}\,\mathrm{d}x. \tag{3.2}$$

The idea is to construct and control the norm $\|\ \vec{u_{kn}}\ \|_{L^{n',+\infty}(\Omega)^n}$ from Eq. (3.2).

Let B be a measurable subspace of Ω, according to Morrey's Theorem 1.1 and Lemma 2.1 there exists $\vec{\varphi_B}\in W^2L^{p,q}(\Omega)^n\cap H_0^1(\Omega)^n$ verifying,

$$\begin{cases} E^*\vec{\varphi_B}=\dfrac{\vec{u_{kn}}}{|\vec{u_{kn}}|}\,\chi_B & \text{in } \Omega\\ \vec{\varphi_B}=\vec{0} & \text{on } \partial\Omega.\end{cases}$$

It is conveniant that if $\vec{u}_{kn}=0$, the fonction $\dfrac{\vec{u}_{kn}}{|\vec{u}_{kn}|}=0$. Using the Hardy–Littlewood inequality (see [15, section 1.2 page 11], [2, chapter 2 page 43]), we estimate the function
$g_{\vec{kn}B}=\dfrac{u_{\vec{kn}}}{|u_{\vec{kn}}|}\chi_B \in L^{n,1}(\Omega)^n$. We have,

$$\begin{aligned}
\mid g_{\vec{kn}B}\mid_{L^{n,1}(\Omega)^n} \leqslant \mid \chi_B \mid_{L^{n,1}(\Omega)^n} &\leqslant \int_0^{|\Omega|}\left(t^{\frac{1}{n}}|\chi_B|_{**}(t)\right)\frac{\mathrm{d}t}{t}\\
&\leqslant \int_0^{|\Omega|} t^{\frac{1}{n}}\left(\frac{1}{t}\int_0^t (\chi_B)_*(s)\,\mathrm{d}s\right)\frac{\mathrm{d}t}{t}\\
&= \int_0^{|\Omega|} t^{\frac{1}{n}-2}\min(t,|B|)\,\mathrm{d}t\\
&\leqslant C_\Omega |B|^{\frac{1}{n}}.
\end{aligned}$$

Thus,

$$\|\vec{\varphi}_B\|_{W^2L^{n,1}(\Omega)^n} \leqslant C\|\vec{g}_{knB}\|_{L^{n,1}(\Omega)^n} \leqslant C|B|^{\frac{1}{n}}. \tag{3.3}$$

We choose $\vec{\varphi}_B$ as a test function in (3.2) and we deduce that,

$$\int_B |u_{\vec{kn}}|\,\mathrm{d}x = \int_\Omega f_{\vec{kn}}\cdot\varphi_{\vec{B}}\,\mathrm{d}x = \int_\Omega f_{\vec{kn}}\cdot\varphi_{\vec{B}}\frac{\delta(x)}{\delta(x)}\,\mathrm{d}x.$$

On the other hand, with the injection of Sobolev associated with the Lorentz space (see [16, Lemma 1 page 1133], [7, Theorem 1 page 813]), we know that,

$$\varphi_{\vec{B}} \in W^2L^{n,1}(\Omega)^n \hookrightarrow W^1L^\infty(\Omega)^n. \tag{3.4}$$

As $\varphi_{\vec{B}} \in W_0^1L^\infty(\Omega)^n$, we can apply the Hardy inequality (see [6,13]) whence,

$$\frac{|\varphi_{\vec{B}}|(x)}{\delta(x)} \leqslant \|\nabla\varphi_{\vec{B}}\|_{L^\infty(\Omega)}. \tag{3.5}$$

Combining (3.3), (3.4) and (3.5), we obtain

$$\frac{|\varphi_{\vec{B}}|}{\delta(x)} \leqslant \|\nabla\varphi_{\vec{B}}\|_{L^\infty(\Omega)} \leqslant |C\varphi_{\vec{B}}|_{W^2L^{n,1}(\Omega)} \leqslant C|B|^{1/n}.$$

In fact,

$$\int_\Omega |u_{\vec{kn}}|\,\mathrm{d}x \leqslant \left\|\frac{\varphi_{\vec{B}}}{\delta(x)}\right\|_{L^\infty(\Omega)}\int_\Omega |f_{\vec{kn}}|\delta(x)\,\mathrm{d}x \leqslant C|B|^{1/n}\|f_{\vec{kn}}\|_{L^1(\Omega,\delta)^n}.$$

Multiplying the two members of the inequality by $|B|^{-1/n}$, we have,

$$|B|^{-1/n}\int_B |u_{\vec{kn}}|\,\mathrm{d}x \leqslant C|f_{\vec{kn}}|_{L^1(\Omega,\delta)^n}.$$

Applying the supremum on the previous inequality, then the inequality becomes,

$$\sup_{|B|=t}|B|^{-1/n}\int_B |u_{\vec{kn}}|\,\mathrm{d}x \leqslant C\|f_{\vec{kn}}\|_{L^1(\Omega,\delta)^n} \tag{3.6}$$

By Hardy–Littlewood equality (see [15], [2, chapter 2 page 43]), the relation (3.6) becomes,

$$t^{1-\frac{1}{n}}\frac{1}{t}\int_0^t |\vec{u_{kn}}|_*(s)\,\mathrm{d}s \leqslant C|\vec{f_{kn}}|_{L^1(\Omega,\delta)^n}. \tag{3.7}$$

Thanks to the definition of the maximum function (see [15], [2, section 2.3 page 52]), the estimation (3.7) is written as,

$$t^{1/n'}|\vec{u_{kn}}|_{**}(t) \leqslant C\|\vec{f_{kn}}\|_{L^1(\Omega,\delta)^n}. \tag{3.8}$$

Consequently,

$$\sup_{|t|\leqslant|\Omega|} t^{1/n'}|\vec{u_{kn}}|_{**}(t) \leqslant C\|\vec{f_{kn}}\|_{L^1(\Omega,\delta)^n} \longrightarrow 0 \ \text{ if } k,n \longrightarrow +\infty.$$

Finally, we pass by the estimation of $\vec{u}$ in $L^{n',\infty}(\Omega)^n$. For that, we repeat the same procedure, we replace $\vec{u}_{kn}$ and $\vec{f}_{kn}$, respectively, by $\vec{u}_k$ and $\vec{f}_k$. We will then have,

$$\sup_{|t|\leqslant|\Omega|} t^{1/n'}|\ \vec{u_k}\ |_{**}(t) \leqslant C\|\ \vec{f_k}\ \|_{L^1(\Omega,\delta)^n}.$$

and we pass by the limit $k \longrightarrow +\infty$, we obtain finally that,

$$\|\vec{u}\|_{L^{n',\infty}(\Omega)^n} = \sup_{|t|\leqslant|\Omega|} t^{1/n'}|\vec{u}|_{**}(t) \leqslant C\|\ \vec{f}\ \|_{L^1(\Omega,\delta)^n}.$$

Hence the desired result. □

3.3. Theorem of the Regularity of Very Weak Solution

Proposition 3.2. *Let* $\vec{f} \in L^1(\Omega;\delta)^n \cap W^{-1,p'}(\Omega)^n$, $1 < p' < +\infty$. *If* $\vec{u}$ *is a weak solution associated to the data* $\vec{f}$, *then* $\vec{u}$ *is the unique, very weak solution.*

Proof. As the problem is linear, we can assume that $\vec{f} = (f^1,\ldots,f^n)$, $f^i \geqslant 0$, $\forall\, i = 1,\ldots,n$. Since $f^i \in L^1(\Omega;\delta) \cap W^{-1,p'}(\Omega)$, then using Brézis–Browder Theorem, we have

$$< f^i,\varphi^i >_{W^{-1,p'},W_0^{1,p}} = \int_\Omega f^i\varphi^i\,\mathrm{d}x,$$

$\forall\varphi^i \in W_0^{1,p}(\Omega)$ and $f^i\varphi^i \in L^1(\Omega)$. As $\vec{u}$ is a weak solution, then $\vec{u} \in W^{1,p'}(\Omega)$ and

$$a(\vec{u};\vec{\varphi}) = \int_\Omega \vec{f}\cdot\vec{\varphi}\mathrm{d}x \quad \forall\,\vec{\varphi} \in C_c^\infty(\Omega)^n.$$

Therefore, by density, we conclude

$$a(\vec{u};\vec{\varphi}) = \int_\Omega \vec{f}\cdot\vec{\varphi} \quad \forall\,\vec{\varphi} \in W_0^{1,p}(\Omega)^n.$$

Noting that

$$\left\{\vec{\varphi} \in C^2(\vec{\Omega}) : \vec{\varphi} = 0 \text{ on } \partial\Omega\right\} \subset W_0^{1,p}(\Omega)^n,$$

we then integrate by parts and obtain the desired result. □

As a corollary of this proposition 3.2, we have the following existence and uniqueness theorem:

Theorem 3.3. *Let* $\vec{f}$ *be a function belonging to* $\prod_{i=1}^{n} L^1(\Omega; \delta^{\alpha_i})$ *with* $0 < \alpha_i < 1$*. Then, there exists a very weak solution of* $(P)_{vw}$ *which belongs to* $W_0^1 L^{\frac{n}{n-1+\overline{\alpha}}}(\Omega)$*, where* $\overline{\alpha} = \max_{1\leqslant i\leqslant n} \alpha_i$.
Moreover, this solution is also the weak solution of (P) *associated to* $\vec{f}$.

Proof. As shown by Díaz and Rakotoson (see [6, lemma 3.7 page 49]), the weighted space $L^1(\Omega, \delta^{\alpha_i})$ is injected into the dual space of $W_0^1 L^{\frac{n}{1-\alpha_i}}(\Omega)$, this means that,

$$L^1(\Omega, \delta^{\alpha_i}) \subset (W_0^1 L^{\frac{n}{1-\alpha_i}}(\Omega))^* = (W^{-1} L^{\frac{n}{n-1+\alpha_i}})(\Omega).$$

Moreover, according to Lemma 3.1, there exists a matrix function $\mathcal{F}$ in $\prod_{i=1}^{n} L^{\frac{n}{n-1+\alpha_i}}(\Omega)^n \subset L^{\frac{n}{n-1+\overline{\alpha}}}(\Omega)^{n^2}$, such that $\mathrm{div}(\mathcal{F}) = \vec{f}$. We can write (P) in the divergence form:

$$(P)_{df} : \begin{cases} -D_\alpha \left(A_{ij}^{\alpha\beta}(D_\beta u^j) \right) = D_\alpha F_\alpha^i & \text{in } \Omega, \\ u^j = 0 & \text{on } \partial\Omega. \end{cases}$$

Applying Morrey's Theorem 1.1, we deduce then that the problem is well-posed and admits a unique solution $\vec{v} \in W_0^1 L^{\frac{n}{n-1+\overline{\alpha}}}(\Omega)^n$. We deduce from Proposition 3.2 that $\vec{v}$ is the very weak solution and $\vec{v} = \vec{u}$. □

Now the question is what happens when $\bar{\alpha} = \max_i \alpha_i$ tends to 0 or 1. Knowing that

$$L^1(\Omega, \delta^0) = L^1(\Omega) \quad \text{and} \quad L^1(\Omega, \delta^1) = L^1(\Omega, \delta),$$

When $\bar{\alpha}$ goes to 1, this means that if one of the functions $f_i \in L^1(\Omega, \delta)$, would the solution $\vec{u}$ be in $W^1 L^1(\Omega)^n$? Similarly, if one of the functions $f_i \in L^1(\Omega)$, then would $\vec{u}$ belong to $W^1 L^{\frac{n}{n-1}}(\Omega)^n$?
As in the case of equations, if $\alpha_i = 0\ \forall i$, this means that if $\vec{f} \in L^1(\Omega)^n$, $\vec{u}$ is in a larger space that is $W^1 L^{\frac{n}{n-1},\infty}(\Omega)^n$ and if $\max_I \alpha_i = 1$, then $\nabla \vec{u}$ is not necessarily in $L^1(\Omega)^n$.

Theorem 3.4. *For a source function* $\vec{f} \in L^1(\Omega)^n$*, the solution* $\vec{u}$ *of* (P) *belongs to* $W^1 L^{n',\infty}(\Omega)^n$*. Furthermore, there exists a positive constant* C *such that,*

$$\|\vec{u}\|_{W^1 L^{n',\infty}(\Omega)^n} \leqslant C \|\vec{f}\|_{L^1(\Omega)^n}.$$

Proof. From Lemma 3.1, part 3, there exists a function $\vec{F}^i \in L^{n',\infty}(\Omega)^n$ such that $\mathrm{div}\, \vec{F}^i = f^i$ and consequently,

$$\|\vec{F}^i\|_{L^{n',\infty}(\Omega)^n} \leqslant C \|f^i\|_{L^1(\Omega)}.$$

Then, we can consider the following divergence form of the elasticity problem:

$$(P)_{df} : \begin{cases} -D_\alpha \left(A_{ij}^{\alpha\beta}(D_\beta u^j) \right) = D_\alpha F_\alpha^i & \text{in } \Omega \\ u^j = 0 & \text{on } \partial\Omega. \end{cases}$$

We have $\mathcal{F} \in L^{n',\infty}(\Omega)^{n^2} \subset L^{n'-\varepsilon}(\Omega)^{n^2}\ \forall\ 0 < \varepsilon < n' - 1$. Then, applying the result of Theorem 1.1, we obtain the existence of a unique weak solution $\vec{u}$ in $W^1L^{n'-\varepsilon}(\Omega)^n$, where $\varepsilon \in]0, n'-1[$. In other words, the weak solution $\vec{u} \in \bigcap_{p<n'} W^1L^p(\Omega)^n$. To prove a finer result of the theorem, we can introduce the linear and continuous operator T which operates from L^p into itself with $1 < p < +\infty$ such that,

$$T : L^p(\Omega)^{n^2} \longrightarrow L^p(\Omega)^{n^2},$$
$$\mathcal{F} \longrightarrow T\mathcal{F} = \nabla\vec{u}.$$

According to Morrey's result Theorem 1.1 there exists a constant $C_p > 0$ such that,

$$\| \nabla\vec{u} \|_{L^p(\Omega)^{n^2}} \leqslant C_p \| \mathcal{F} \|_{L^p(\Omega)^{n^2}}\ \forall\ p_0 < n' \leqslant p < n' + 1 = p_1.$$

We apply Marcinkiewicz interpolation (see [2]) which extends the regularity of the Lebesgue space to the Lorentz space. Then, T is continuous from $L^{p,q}$ into $L^{p,q}$ for all $q \in [1, \infty]$. In particular, we obtain the continuity of T of $L^{n',\infty}$ into itself. Consequently,

$$\| \vec{u} \|_{W^1L^{n',\infty}(\Omega)^n} \leqslant C \| \mathcal{F} \|_{L^{n',\infty}(\Omega)^{n^2}} \leqslant C \|\vec{f}\|_{L^1(\Omega)^n}.$$

Thus, the weak solution $\vec{u}$ associated to $\vec{f} \in L^1(\Omega)^n \cap W^{-1}L^{n'-\varepsilon}(\Omega)^n$ and according to Proposition 3.2, $\vec{u}$ is a very weak solution. □

Theorem 3.5. *Let $\vec{f} \in L^1(\Omega, \delta(1 + |\ln\delta|))^n$, then the solution $\vec{u}$ of $(P)_{vw}$ is in $W_0^{1,1}(\Omega)^n$.*

Preuve. For $k \geqslant 1$ and $n \geqslant 1$, we define,

$$\vec{u_{kn}} = \vec{u_k} - \vec{u_n} \quad \text{and} \quad \vec{f_{kn}} = \vec{f_n} - \vec{f_k},$$

with $\vec{u}_k$ already defined in the Theorem 3.2. Since the space $W^{1,1}(\Omega)$ is complete, it suffices to prove that the sequence $\vec{u_k}$ verifies the Cauchy criteria in this space.

The procedure consists by constructing and controlling the norm $\| \vec{u_{kn}} \|_{W^{1,1}(\Omega)^n}$. Recalling that $E^* = -\Delta - \lambda\nabla\,\mathrm{div}$ and supposing the following problem:

$$(P^*_{kn}) : \begin{cases} E^*\vec{\varphi} = \mathrm{div}\left(\dfrac{\nabla\vec{u}_{kn}}{|\nabla\vec{u}_{kn}|}\right) & \text{in } \Omega. \\ \vec{\varphi} = \vec{0} & \text{on } \partial\Omega. \end{cases}$$

We note that $\vec{g}_{kn} = \dfrac{\nabla\vec{u}_{kn}}{|\nabla\vec{u}_{kn}|}$. As $\vec{u}_{kn} \in W^{1,p}(\Omega)^n$, then $\vec{g}_{kn} \in L^P(\Omega)^n$, it follows that according to Morrey Theorem 1.1, this problem is well-posed, in particular admits a unique solution $\vec{\varphi} \in W^{1,p}(\Omega)^n$. Moreover, using Campanato [3] and [1] technique, we show that the solution $\vec{\varphi}$ of such system belongs to $W_0^1\mathrm{bmo}_r(\Omega)^n$ and there exists a constant $C > 0$ such that,

$$\|\nabla\vec{\varphi}\|_{\mathrm{bmo}_r(\Omega)^n} \leqslant C \left\| \frac{\nabla\vec{u}_{kn}}{|\nabla\vec{u}_{kn}|} \right\|_{L^\infty(\Omega)^n} \leqslant C(\Omega).$$

Furthermore, thanks to the injection between bmo and $L_{\exp}$ (see [6]), we obtain that,

$$\vec{\varphi} \in W_0^1 \mathrm{bmo}_r(\Omega)^n \hookrightarrow W_0^1 L_{\exp}(\Omega)^n.$$

Combining the relation (3.10) and the problem (P^*_{kn}), we have,

$$\begin{aligned}\int_\Omega |\nabla \vec{u}_{kn}| \,\mathrm{d}x &= \int_\Omega \vec{f}_{kn}.\vec{\varphi} \,\mathrm{d}x \\ &\leqslant \int_\Omega \vec{f}_{kn}.\vec{\varphi}\ \frac{\delta(1+|\ln\delta|)}{\delta(1+|\ln\delta|)} \,\mathrm{d}x. \qquad (3.9)\end{aligned}$$

AS $\vec{\varphi} \in W_0^1 L_{\exp}(\Omega)^n$ then, the Hardy inequality (see [6,13]) is applicable, whence

$$\frac{|\vec{\varphi}(x)|}{\delta(x)(1+|\ln\delta|)} \leqslant C(\Omega)\|\nabla\vec{\varphi}\|_{L_{\exp}(\Omega)^n}. \qquad (3.10)$$

Then, the relations (3.9) and (3.10) leads to,

$$\begin{aligned}\int_\Omega |\nabla \vec{u}_{kn}| \,\mathrm{d}x &\leqslant \frac{|\vec{\varphi}(x)|}{\delta(x)(1+|\ln\delta|)} \int_\Omega |\vec{f}_{kn}\ |\delta(x)\ (1+|\ln\delta|) \,\mathrm{d}x \\ &\leqslant C(\Omega)\|\nabla\vec{\varphi}\|_{L_{\exp}(\Omega)^n} \int_\Omega |\vec{f}_{kn}\ |\delta(x)\ (1+|\ln\delta|) \,\mathrm{d}x \\ &\leqslant C(\Omega) \int_\Omega |\vec{f}_{kn}|\ \delta(x)\ (1+|\ln\delta|) \,\mathrm{d}x \underset{k,n\to\infty}{\longrightarrow} 0.\end{aligned}$$

□

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