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Some Relations on the Double $\mathcal{L}_{22}$-Integral Transform and Their Applications

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Abstract. In this paper, the authors introduce the double Laplace-type integral transform $\mathcal{L}_{22}$ and its properties. Several simple theorems dealing with general properties of the $\mathcal{L}_{22}$-integral transform are proved. The convolution, its properties and convolution theorem are given. The main focus of this paper is to develop a method for the $\mathcal{L}_{22}$-integral transform to solve problems in applied mathematics which involve partial differential and integral equations.

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1. Introduction

J. Fourier provided the modern mathematical theory of heat conduction, Fourier series and Fourier integrals with applications on La Théorie Analytique de la Chaleur. He discovered a double integral representation of a non-periodic function $f(x)$ for all real x which is universally known as the Fourier Integral Theorem in the form

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} \left[\int_{-\infty}^{\infty} f(\xi) e^{-ik\xi} \mathrm{d}\xi \right] \mathrm{d}k. \tag{1.1}$$

This theorem is regarded as one of the most fundamental representation theorems of mathematical analysis. The Fourier integral theorem was used by Fourier to introduce the Fourier transform and the inverse Fourier transform as follows:

$$\mathcal{F}\{f(x)\} = F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f(x) \mathrm{d}x, \tag{1.2}$$

$$\mathcal{F}^{-1}\{F(k)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} F(k)\mathrm{d}k. \tag{1.3}$$

The Laplace transform is a special case of the Fourier transform. The Laplace transform was introduced by the following formula:

$$\mathcal{L}\{f(x); y\} = \int_{0}^{\infty} e^{-xy} f(x)\mathrm{d}x = \bar{F}(y), \ Re(y) > 0. \tag{1.4}$$

O. Heaviside made the Laplace transform very popular by applying it to solve ordinary differential equations and to develop modern operational method.

At present, there is a very extensive literature available for the Laplace transform of a function $f(x)$ of one variable and its applications [2,4,8,9].

But there is a very little work available for the double Laplace transform of $f(x,y)$ of two positive real variables x and y and their properties. The double Laplace transform of $f(x,y)$ is defined in [3] by the formula:

$$\begin{aligned} \bar{\bar{F}}(p,q) &= L_2\{f(x,y);(p,q)\} = \mathcal{L}\{\bar{F}(p,y);q\} \\ &= \int_{0}^{\infty}\int_{0}^{\infty} f(x,y)e^{-px-qy}\mathrm{d}x\mathrm{d}y. \end{aligned} \tag{1.5}$$

Debnath [4] presented a study of interest by the double Laplace transform, its properties with examples and applications to ordinary, partial differential equations and integral equations.

The following Laplace-type integral transform, $\mathcal{L}_2$, was introduced by Yürekli and Sadek [12] as follows:

$$\mathcal{L}_2\{f(x); y\} = \int_{0}^{\infty} x e^{-x^2y^2} f(x)\mathrm{d}x = \tilde{F}(y), \ Re(y) > 0. \tag{1.6}$$

Yürekli and et al. gave a lot of properties of the $\mathcal{L}_2$-transform and they applied this transform to solve some special differential equations [5,6,10,11,13].

The following inverse $\mathcal{L}_2$-transform was introduced by Aghili, Ansari and Sedghi [1] as follows:

$$\begin{aligned} \mathcal{L}_2^{-1}\{F(y);t\} = f(x) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} 2F(\sqrt{y})e^{x^2y}\mathrm{d}y \\ &= \sum_{k=1}^{m}[Res\{2F(\sqrt{y})e^{x^2y}, x = x_k\}]. \end{aligned} \tag{1.7}$$

In the present paper, the authors introduce the double $\mathcal{L}_{22}$-transform, its properties with examples and applications. Several simple theorems dealing with general properties of the double Laplace transform are proved. The convolution of $f(x,y)$ and $g(x,y)$, its properties and convolution theorem with proof are given. Also, the authors introduce the method of the double Laplace-type $\mathcal{L}_{22}$-transform to solve problems in applied mathematics.

2. The Main Definitions and Theorems

Definition 2.1. The double Laplace-type transform $\mathcal{L}_{22}$ of a function $f(x,y)$ of two variables defined in the first quadrant of the xy-plane is defined by the following double integral form:

$$\tilde{\tilde{F}}(p,q) = \mathcal{L}_{22}\{f(x,y);(p,q)\} = \int_0^\infty\int_0^\infty xye^{-x^2p^2-y^2q^2}f(x,y)\mathrm{d}x\mathrm{d}y, \tag{2.1}$$

provided the integral exists.
The double $\mathcal{L}_{22}$-transform is related to the double Laplace transform L_2-transform (1.5) by means of the identity:

$$4\mathcal{L}_{22}\{f(x,y);(p,q)\} = L_2\{f(x^{1/2},y^{1/2});(p^2,q^2)\} \tag{2.2}$$

or equivalently

$$4\tilde{\tilde{F}}(p,q) = \bar{\bar{F}}(p^2,q^2).$$

Definition 2.2. The inverse of the $\mathcal{L}_{22}\{f(x,y);(p,q)\} = \tilde{\tilde{F}}(p,q)$ function is defined as follows:

$$\mathcal{L}_{22}^{-1}\{\tilde{\tilde{F}}(p,q);(x,y)\} = f(x,y) = -\frac{1}{\pi^2}\int_{c-i\infty}^{c+i\infty}\int_{d-i\infty}^{d+i\infty} e^{px^2+qy^2}\tilde{\tilde{F}}(p^{1/2}q^{1/2})\mathrm{d}p\mathrm{d}q, \tag{2.3}$$

where $\tilde{\tilde{F}}(p,q)$ must be an analytic function for all p and q in the region defined by the inequalities $Re(p) \geq c$ and $Re(q) \geq d$ which c and d are real constants to be chosen suitably. We obtain the formula (2.3) from (2.2). Substituting $p^2 = p$ and $q^2 = q$ into the relation (2.2), we have

$$\mathcal{L}_{22}\{f(x^{1/2},y^{1/2});(p,q)\} = 4\tilde{\tilde{F}}(p^{1/2},q^{1/2}). \tag{2.4}$$

Applying the $\mathcal{L}_{22}^{-1}$-transform to both sides of the relation (2.4), we get

$$f(x^{1/2},y^{1/2}) = -\frac{1}{\pi^2}\int_{c-i\infty}^{c+i\infty} e^{px}\mathrm{d}p\int_{d-i\infty}^{d+i\infty} e^{qy}\tilde{\tilde{F}}(p^{1/2},q^{1/2})\mathrm{d}q, \tag{2.5}$$

and then, we obtain (2.3).

Corollary 2.3. *The double $\mathcal{L}_{22}$-integral transform and its inverse $\mathcal{L}_{22}^{-1}$-integral transform satisfy the linear property.*

Proof. Using the definitions of the $\mathcal{L}_{22}$ and $\mathcal{L}_{22}^{-1}$-integral transforms and linearity of the integrals, we arrive at the linear properties for this transformations. □

Example.

$$\mathcal{L}_{22}^{-1}\left\{\frac{k!n!}{p^{k+1}q^{n+1}};(x,y)\right\} = x^{2k}y^{2n}. \tag{2.6}$$

Proof. We know from [3],

$$\mathcal{L}_{22}\{x^{2k}y^{2n};(p,q)\} = \frac{1}{4}L_2\{x^k y^n;(p^2,q^2)\} = \frac{1}{4}\frac{k!n!}{p^{2k+2}q^{2n+2}} = \tilde{\tilde{F}}(p,q) \quad (2.7)$$

and from (2.7), we could write

$$4\tilde{\tilde{F}}(p^{1/2},q^{1/2}) = \frac{k!n!}{p^{k+1}q^{n+1}}. \quad (2.8)$$

Using the definition (2.3), we obtain

$$\begin{aligned} f(x,y) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{px^2}\mathrm{d}p\frac{1}{2\pi i}\int_{d-i\infty}^{d+i\infty} 4e^{qy^2}\tilde{\tilde{F}}(p^{1/2},q^{1/2})\mathrm{d}q \\ &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{px^2}\mathrm{d}p\frac{1}{2\pi i}\int_{d-i\infty}^{d+i\infty} e^{qy^2}\frac{k!n!}{p^{k+1}q^{n+1}}\mathrm{d}q, \end{aligned} \quad (2.9)$$

where for $n=m$ and $k=j$, we know

$$Res\left\{e^{qy^2}\frac{k!n!}{p^{k+1}q^{n+1}};q=q_m\right\} = y^{2n}\frac{k!}{p^{k+1}} \quad (2.10)$$

and

$$Res\left\{e^{px^2}\frac{k!y^{2n}}{p^{k+1}};p=p_m\right\} = y^{2n}x^{2k}. \quad (2.11)$$

Using (2.10) and (2.11), we arrive at the relation (2.6). □

Corollary 2.4. *If the following relation,*

$$\mathcal{L}_{22}^{-1}\{\tilde{\tilde{F}}(p,q);(x,y)\} = \mathcal{L}_2^{-1}\{A(p);x\}\mathcal{L}_2^{-1}\{B(q);y\}, \quad (2.12)$$

holds true, we have

$$\begin{aligned} &\mathcal{L}_{22}^{-1}\{\tilde{\tilde{F}}(p,q);(x,y)\} \\ &\quad = \sum_{m=1}^{\infty} Res\{e^{px^2}A(\sqrt{p});p=p_m\} \\ &\quad\quad \times\sum_{j=1}^{\infty} Res\{e^{qy^2}B(\sqrt{q});q=q_j\}. \end{aligned} \quad (2.13)$$

Definition 2.5. If a positive constant K exists, such that for all $x>X$ and $y>Y$

$$|f(x,y)| \le Ke^{ax+by} \quad (2.14)$$

then, $f(x,y)$ is called exponential order as $x\to\infty$, $y\to\infty$.
And this property is considered as follows:

$$f(x,y) = \mathcal{O}(e^{ax+by}) \quad \text{as } x\to\infty.y\to\infty. \quad (2.15)$$

Or, equivalently, for $\alpha > a$, $\beta > b$,

$$\lim_{x\to\infty,\ y\to\infty} e^{-\alpha^2x^2-\beta^2y^2}|f(x,y)| = K \lim_{x\to\infty,\ y\to\infty} e^{-(\alpha^2-a^2)x}e^{-(\beta^2-b^2)y} = 0. \tag{2.16}$$

Theorem 2.6. *If a function $f(x,y)$ is a continuous function in every finite intervals $(0,X)$, $(0,Y)$ and of exponential order $e^{\alpha^2x^2+b^2y^2}$, then the double Laplace-type transform of $f(x,y)$ exists for all p and q, provided that $Re(p^2) > a^2$ and $Re(q^2) > b^2$.*

Proof. We have from the definition (2.1),

$$\begin{aligned}|\tilde{\tilde{F}}(p,q)| &= \left|\int_0^\infty\int_0^\infty xye^{-p^2x^2-q^2y^2}f(x,y)\mathrm{d}x\mathrm{d}y\right| \\ &\le K\int_0^\infty xe^{-x^2(p^2-a^2)}\mathrm{d}x\int_0^\infty ye^{-y^2(q^2-b^2)}\mathrm{d}y \\ &= \frac{K}{4}\left[\frac{1}{(p^2-a^2)(q^2-b^2)}\right],\end{aligned} \tag{2.17}$$

where $Re(p^2) > a^2$, $Re(q^2) > b^2$.
It follows from (2.17),

$$\lim_{p\to\infty,\ q\to\infty} |\tilde{\tilde{F}}(p,q)| = 0 \text{ or } \lim_{p\to\infty,\ q\to\infty} \tilde{\tilde{F}}(p,q) = 0. \tag{2.18}$$

□

Corollary 2.7. *The relation (2.18) can be regarded as the limit property of the $\mathcal{L}_{22}$-transform. $\tilde{\tilde{F}}(p,q) = p^2q^2$ or $\tilde{\tilde{F}}(p,q) = p^2+q^2$ are not the $\mathcal{L}_{22}$-transform of any function $f(x,y)$, because $\tilde{\tilde{F}}(p,q)$ does not tend to zero as $p\to\infty$ and $q\to\infty$.*

Theorem 2.8. *The following properties hold true under suitable conditions on $f(x,y)$:*

$$\mathcal{L}_{22}\{e^{-a^2x^2-b^2y^2}f(x,y);(p,q)\} = \tilde{\tilde{F}}(\sqrt{p^2+a^2},\sqrt{q^2+b^2}), \tag{2.19}$$

$$\mathcal{L}_{22}\{f(ax)g(by);(p,q)\} = \frac{1}{a^2b^2}\tilde{F}\left(\frac{p}{a}\right)\tilde{F}\left(\frac{q}{b}\right),\ a,b>0, \tag{2.20}$$

$$\mathcal{L}_{22}\{f(x);(p,q)\} = \frac{1}{2q^2}\tilde{F}(p),$$

$$\mathcal{L}_{22}\{g(y);(p,q)\} = \frac{1}{2p^2}\tilde{G}(q),\ p,q\neq 0, \tag{2.21}$$

$$\mathcal{L}_{22}\{f(x^2+y^2);(p,q)\} = \frac{1}{q^2-p^2}[\bar{F}(p^2)-\bar{F}(q^2)],\ p\neq q, \tag{2.22}$$

$$\mathcal{L}_{22}\{f(x^2-y^2);(p,q)\} = \frac{1}{4(p^2+q^2)}[\bar{F}(p^2)+\bar{F}(q^2)],\ \textit{when } f \textit{ is even}, \tag{2.23}$$

$$= \frac{1}{4(p^2+q^2)}[\bar{F}(p^2) - \bar{F}(q^2)], \ \textit{when } f \textit{ is odd}, \tag{2.24}$$

$$\mathcal{L}_{22}\{f(x)H(x-y);(p,q)\} = \frac{1}{2q^2}[\tilde{F}(p) - \tilde{F}(\sqrt{p^2+q^2})], \ x \neq y, \ x > y, \tag{2.25}$$

$$\mathcal{L}_{22}\{f(x)H(y-x);(p,q)\} = \frac{1}{2q^2}\tilde{F}(\sqrt{p^2+q^2}), \ x \neq y, \ y > x, \tag{2.26}$$

$$\mathcal{L}_{22}\{f(x)H(x+y);(p,q)\} = \frac{1}{2q^2}\tilde{F}(p), \ q \neq 0, \tag{2.27}$$

$$\mathcal{L}_{22}\{H(x-y);(p,q)\} = \frac{1}{4}\frac{1}{p^2(p^2+q^2)}, \ p \neq 0, \tag{2.28}$$

where the integrals involved converge absolutely.

Proof. Using the definitions of the $\mathcal{L}_{22}$-integral transform, the L_2-integral transform and [4], we can prove the properties in Theorem 2.8. □

Theorem 2.9. *We have the following relation between $\mathcal{L}_{22}\{f(x,y);(p,q)\}$ and $L_2\{f(x,y);(p,q)\}$:*

$$\begin{aligned} &4\mathcal{L}_{22}\{f(x^2-\xi^2, y^2-\eta^2)H(x^2-\xi^2, y^2-\eta^2);(p,q)\} \\ &\quad = e^{-\xi^2p^2-\eta^2q^2}\mathcal{L}_{22}\{f(x,y);(p^2,q^2)\}, \end{aligned} \tag{2.29}$$

where $H(x^2-\xi^2, y^2-\eta^2)$ is the Heaviside function [4,7].

Proof. Using the definition of the $\mathcal{L}_{22}$-transform (2.1) and Heaviside function, we get

$$\begin{aligned} &\mathcal{L}_{22}\{f(x^2-\xi^2, y^2-\eta^2)H(x^2-\xi^2, y^2-\eta^2);(p,q)\} \\ &\int_{\eta^2}^{\infty}\int_{\xi^2}^{\infty} xye^{-x^2p^2-y^2q^2} f(x^2-\xi^2, y^2-\eta^2)\mathrm{d}x\mathrm{d}y. \end{aligned} \tag{2.30}$$

Making the change variables $x^2-\xi^2 = u^2$ and $y^2-\eta^2 = v^2$, we find

$$\begin{aligned} \mathcal{L}_{22}\{f(u^2,v^2);(p,q)\} &= \int_0^{\infty}\int_0^{\infty} uve^{-(u^2+\xi^2)p^2-(v^2+\eta^2)q^2} f(u^2,v^2)\mathrm{d}u\mathrm{d}v \\ &= e^{-\xi^2p^2-\eta^2q^2}\int_0^{\infty}\int_0^{\infty} uve^{-u^2p^2-v^2q^2} f(u^2,v^2)\mathrm{d}u\mathrm{d}v. \end{aligned} \tag{2.31}$$

Now, by setting $u^2 = x$ and $v^2 = y$ into (2.31), we arrive at the relation (2.29). □

Theorem 2.10. *If* $f(x,y) = f(\sqrt{x^2+a^2}, \sqrt{y^2+b^2})$ *for all* x, y *and* $a, b \in \mathcal{R}$ *and if* $\mathcal{L}_{22}\{f(x,y);(p,q)\}$ *exists, then*

$$\mathcal{L}_{22}\{f(x,y);(p,q)\} = [1 - e^{-p^2a^2-q^2b^2}]^{-1} \int_0^{a^2}\int_0^{b^2} xye^{-p^2x^2-q^2y^2} f(x,y)\mathrm{d}x\mathrm{d}y \tag{2.32}$$

holds true.

Proof. Using the definition of the $\mathcal{L}_{22}$-transform (2.1), we have

$$\begin{aligned}\tilde{\tilde{F}}(p,q) = \mathcal{L}_{22}\{f(x,y);(p,q)\} &= \int_0^{\infty}\int_0^{\infty} xye^{-p^2x^2-q^2y^2} f(x,y)\mathrm{d}x\mathrm{d}y \\ &= \int_0^{a^2}\int_0^{b^2} xye^{-p^2x^2-q^2y^2} f(x,y)\mathrm{d}x\mathrm{d}y \\ &\quad + \int_{a^2}^{\infty}\int_{b^2}^{\infty} xye^{-p^2x^2-q^2y^2} f(x,y)\mathrm{d}x\mathrm{d}y. \end{aligned} \tag{2.33}$$

Changing the variables on the second double integral to $x^2 = u^2 + a^2$ and $y^2 = v^2 + b^2$ and using the hypothesis, we obtain the relation (2.32):

$$\begin{aligned}\tilde{\tilde{F}}(p,q) &= \int_0^{a^2}\int_0^{b^2} xye^{-p^2x^2-q^2y^2} f(x,y)\mathrm{d}x\mathrm{d}y \\ &\quad + \int_0^{\infty}\int_0^{\infty} uve^{-p^2u^2-q^2v^2} f(u,v)\mathrm{d}u\mathrm{d}v. \end{aligned} \tag{2.34}$$

□

Definition 2.11. The convolution of $f(x,y)$ and $g(x,y)$ is denoted by

$$(f**g)(x,y) = \int_0^{x}\int_0^{y} \xi\eta f((x^2-\xi^2)^{1/2},(y^2-\eta^2)^{1/2})g(\xi,\eta)\mathrm{d}\xi\mathrm{d}\eta. \tag{2.35}$$

Using the definition (2.35), it can easily be verified that the following properties of convolution hold true:

$$[f**(g**h)](x,y) = [(f**g)**h](x,y), \text{ (Associative)} \tag{2.36}$$

$$(f**g)(x,y) = (g**f)(x,y), \text{ (Commutative)} \tag{2.37}$$

$$[f**(ag+bh)](x,y) = a(f**g)(x,y) + b(f**h)(x,y), \text{ (Distributive)} \tag{2.38}$$

$$(f**\delta)(x,y) = f(x,y) = (\delta**f)(x,y), \text{ (Identity)} \tag{2.39}$$

where $\delta(x,y)$ is the Dirac delta function of x and y.

Corollary 2.12. *By virtue of these convolution properties, it is clear that the set of all* $\mathcal{L}_{22}$*-transformable functions forms a commutative semigroup with* *respect to the convolution operation* $**$*. In general, this set does not form a group, because* $f**g^{-1}$ *does not have a* $\mathcal{L}_{22}$*-transform.*

Theorem 2.13. (Convolution Theorem) *If* $\mathcal{L}_{22}\{f(x,y);(p,q)\} = \tilde{\tilde{F}}(p,q)$ *and* $\mathcal{L}_{22}\{g(x,y);(p,q)\} = \tilde{\tilde{G}}(p,q)$*, then*

$$\begin{aligned}\mathcal{L}_{22}\{[(f**g)(x,y)];(p,q)\} &= \mathcal{L}_{22}\{f(x,y);(p,q)\}\mathcal{L}_{22}\{g(x,y);(p,q)\}\\ &= \tilde{\tilde{F}}(p,q)\tilde{\tilde{G}}(p,q).\end{aligned} \tag{2.40}$$

Or equivalently,

$$\mathcal{L}_{22}^{-1}\{\tilde{\tilde{F}}(p,q)\tilde{\tilde{G}}(p,q);(x,y)\} = (f**g)(x,y), \tag{2.41}$$

where $(f**g)(x,y)$ *is defined by the double integral* (2.35)*.*

Proof. Using the definitions (2.35),(2.1), we have

$$\begin{aligned}\mathcal{L}_{22}\{(f**g)(x,y);(p,q)\} &= \int_0^\infty\int_0^\infty xye^{-p^2x^2-q^2y^2}(f**g)(x,y)\mathrm{d}x\mathrm{d}y\\ &= \int_0^\infty\int_0^\infty xye^{-p^2x^2-q^2y^2}\\ &\quad\times\left[\int_0^x\int_0^y \xi\eta f((x^2-\xi^2)^{1/2},(y^2-\eta^2)^{1/2})g(\xi,\eta)\mathrm{d}\xi\mathrm{d}\eta\right]\mathrm{d}x\mathrm{d}y.\end{aligned} \tag{2.42}$$

Using the Heaviside unit step function [see 1] and changing the order of integration, we obtain for $0<\xi<x$, $0<x<\infty$ and $0<\eta<y$, $0<y<\infty$,

$$\begin{aligned}\mathcal{L}_{22}\{(f**g)(x,y);(p,q)\} &= \int_0^\infty\int_0^\infty xye^{-p^2x^2-q^2y^2}\mathrm{d}x\mathrm{d}y\\ &\quad\times\int_0^\infty\int_0^\infty \xi\eta f((x^2-\xi^2)^{1/2},(y^2-\eta^2)^{1/2})g(\xi,\eta)H(x-\xi,y-\eta)\mathrm{d}\xi\mathrm{d}\eta\\ &= \int_0^\infty\int_0^\infty \xi\eta g(\xi,\eta)\mathrm{d}\xi\mathrm{d}\eta\\ &\quad\times\int_0^\infty\int_0^\infty xye^{-p^2x^2-q^2y^2}f((x^2-\xi^2)^{1/2},(y^2-\eta^2)^{1/2})H(x-\xi,y-\eta)\mathrm{d}x\mathrm{d}y\\ &= \int_0^\infty\int_0^\infty \xi\eta e^{-p^2\xi^2-q^2\eta^2}g(\xi,\eta)\tilde{\tilde{F}}(p,q)\mathrm{d}\xi\mathrm{d}\eta = \tilde{\tilde{F}}(p,q)\tilde{\tilde{G}}(p,q).\end{aligned} \tag{2.43}$$

□

Remark 2.14. We have the following relation:

$$\mathcal{L}_{22}\{f(x,y)H(x-a,y-b);(p,q)\} = e^{-p^2a^2-q^2b^2}\mathcal{L}_{22}\{f(x,y);(p,q)\}. \tag{2.44}$$

Using the definition of $\mathcal{L}_{22}$-transform (2.1) and changing the variables of integration from x^2-a^2 to x^2 and from y^2-b^2 to y^2, we obtain the relation (2.44).

Corollary 2.15. *If $f(x,y)=k(x)l(y)$ and $g(x,y)=m(x)n(y)$, then we have*

$$\mathcal{L}_{22}\{(f**g)(x,y);(p,q)\}=\mathcal{L}_2\{(k*m)(x);p\}\mathcal{L}_2\{(l*n)(y),q\}$$
$$\tilde{\tilde{F}}(p,q)\tilde{\tilde{G}}(p,q)=K(p)M(p)L(q)N(q). \tag{2.45}$$

Proof. Using the definitions of $\mathcal{L}_{22}$-integral transform (2.1) and $\mathcal{L}_2$-integral transform (1.6), we get

$$\begin{aligned}\mathcal{L}_{22}\{(f**g)(x,y);(p,q)\}&=\int_0^\infty\int_0^\infty xye^{-px^2-qy^2}\mathrm{d}x\mathrm{d}y\\ &\quad\times\int_0^x\int_0^y \xi\eta f((x^2-\xi^2)^{1/2},(y^2-\eta^2)^{1/2})g(\xi,\eta)\mathrm{d}\xi\mathrm{d}\eta\\ &=\int_0^\infty\int_0^\infty xye^{-p^2x^2-q^2y^2}\int_0^x\int_0^y\xi\eta k((x^2-\xi^2)^{1/2})l((y^2-\eta^2)^{1/2})m(\xi)n(\eta)\mathrm{d}\xi\mathrm{d}\eta\mathrm{d}x\mathrm{d}y.\end{aligned} \tag{2.46}$$

Changing the order of integration and using the following known relation [1]:

$$\begin{aligned}\mathcal{L}_{22}\{(f**g)(x,y);(p,q)\}&=\int_0^\infty xe^{-p^2x^2}\mathrm{d}x\int_0^x\xi k((x^2-\xi^2)^{1/2})m(\xi)\mathrm{d}\xi\\ &\quad\times\int_0^\infty ye^{-q^2y^2}\mathrm{d}y\int_0^y\eta l((y^2-\eta^2)^{1/2})n(\eta)\mathrm{d}\eta\\ &=\mathcal{L}_2\{(k*m)(x);p\}\mathcal{L}_2\{(l*n)(y);q\},\end{aligned} \tag{2.47}$$

we obtain the relation (2.45). □

Example. We calculate the inverse $\mathcal{L}_{22}$-transform of the function $\frac{1}{p^2q^2}e^{-ap-bq}$ using Theorem 2.13.

Proof. Using Theorem 2.13 and (2.45), we get

$$\begin{aligned}&\mathcal{L}_{22}^{-1}\left(\frac{1}{p^2q^2}e^{-ap-bq};(x,t)\right)\\ &\quad=\mathcal{L}_2^{-1}\left(\frac{1}{p^2}e^{-ap};x\right)\mathcal{L}_2^{-1}\left(\frac{1}{q^2}e^{-bq};t\right).\end{aligned} \tag{2.48}$$

Then, we know the following well-known formulas from [6, p.146 Entry 28],

$$\mathcal{L}_2^{-1}\left(\frac{1}{p^2};x\right)=2,\ \mathcal{L}_2^{-1}\left(\frac{1}{q^2};t\right)=2, \tag{2.49}$$

$$\mathcal{L}_2^{-1}\{e^{-ap};x\}=\frac{a}{\sqrt{\pi}}x^{-3}e^{-\frac{a^2}{4x^2}},\ \mathcal{L}_2^{-1}\{e^{-bq};t\}=\frac{b}{\sqrt{\pi}}t^{-3}e^{-\frac{b^2}{4t^2}}. \tag{2.50}$$

Using the identities (2.49) and (2.50), we obtain

$$\mathcal{L}_{22}^{-1}\Big(\frac{1}{p^2q^2}e^{-ap-bq};(x,t)\Big) = 2*\frac{a}{\sqrt{\pi}}x^{-3}e^{-\frac{a^2}{4x^2}} \times 2*\frac{b}{\sqrt{\pi}}t^{-3}e^{-\frac{b^2}{4t^2}}. \quad (2.51)$$

Using (2.45) and (2.51), we get

$$\mathcal{L}_{22}^{-1}\Big(\frac{1}{p^2q^2}e^{-ap-bq};(x,t)\Big) = \frac{2a}{\sqrt{\pi}}\int\limits_0^x \xi^{-2}e^{-\frac{a^2}{4\xi^2}}\mathrm{d}\xi \times \frac{2b}{\sqrt{\pi}}\int\limits_0^t \eta^{-2}e^{-\frac{b^2}{4\eta^2}}\mathrm{d}\eta. \quad (2.52)$$

Then, applying the transformations $\xi^{-1} = \frac{2}{a}u$ and $\eta^{-1} = \frac{2}{b}v$ to the variables ξ and η of (2.52), we get

$$\begin{aligned}\mathcal{L}_{22}^{-1}\Big(\frac{1}{p^2q^2}e^{-ap-bq};(x,t)\Big) &= 2\frac{2}{\sqrt{\pi}}\int\limits_{\frac{a}{2x}}^{\infty} e^{-u^2}\mathrm{d}u \times 2\frac{2}{\sqrt{\pi}}\int\limits_{\frac{b}{2t}}^{\infty} e^{-v^2}\mathrm{d}v \\ &= 4\mathrm{erfc}\Big(\frac{a}{2x}\Big)\mathrm{erfc}\Big(\frac{b}{2t}\Big). \quad (2.53)\end{aligned}$$

□

Theorem 2.16. *If we have the following identity,*

$$\mathcal{L}_{22}\{\Phi(x,y,t,z);(p,q)\} = \phi(p,q)tze^{-t^2k^2(p)-z^2m^2(q)}, \quad (2.54)$$

where $\phi(p,q)$, $k^2(p)$ and $m^2(q)$ are analytic functions, then the following relation holds true:

$$\mathcal{L}_{22}\Bigg\{\int\limits_0^\infty\int\limits_0^\infty f(t,z)\Phi(x,y,t,z)\mathrm{d}t\mathrm{d}z;(p,q)\Bigg\} = \phi(p,q)\tilde{\tilde{F}}(k(p),m(q)). \quad (2.55)$$

Proof. Using the definition of $\mathcal{L}_{22}$-integral transform (2.1) and changing the order of integration, we find

$$\begin{aligned}&\mathcal{L}_{22}\Bigg\{\int\limits_0^\infty\int\limits_0^\infty f(t,z)\Phi(x,y,t,z)\mathrm{d}t\mathrm{d}z;(p,q)\Bigg\} \\ &= \int\limits_0^\infty\int\limits_0^\infty xye^{-p^2x^2-q^2y^2}\Bigg[\int\limits_0^\infty\int\limits_0^\infty f(t,z)\Phi(x,y,t,z)\mathrm{d}t\mathrm{d}z\Bigg]\mathrm{d}x\mathrm{d}y \\ &= \int\limits_0^\infty\int\limits_0^\infty f(t,z)\mathrm{d}t\mathrm{d}z\Bigg[\int\limits_0^\infty\int\limits_0^\infty xye^{-(p^2x^2+q^2y^2)}\Phi(x,y,t,z)\mathrm{d}x\mathrm{d}y\Bigg]. \quad (2.56)\end{aligned}$$

By the hypothesis relation (2.54), we obtain

$$\begin{aligned}&\mathcal{L}_{22}\Bigg\{\int\limits_0^\infty\int\limits_0^\infty f(t,z)\Phi(x,y,t,z)\mathrm{d}t\mathrm{d}z;(p,q)\Bigg\} \\ &= \phi(p,q)\int\limits_0^\infty\int\limits_0^\infty tze^{-t^2k^2(p)-z^2m^2(q)}f(t,z)\mathrm{d}t\mathrm{d}z\end{aligned}$$

$$= \phi(p,q)\tilde{\tilde{F}}(k(p), m(q)). \tag{2.57}$$

□

Example. We can solve the following integral equation using Theorem 2.16,

$$\frac{4}{\pi^2}\int_0^\infty\int_0^\infty f(t,z)\sin(xt)\sin(yz)\mathrm{d}t\mathrm{d}z = \mathrm{erf}\left(\frac{x}{a}\right)\mathrm{erf}\left(\frac{y}{b}\right). \tag{2.58}$$

Proof. Applying the $\mathcal{L}_{22}$-transform to both sides of (2.58), we get

$$\mathcal{L}_{22}\{\sin(xt)\sin(yz);(p,q)\} = \int_0^\infty\int_0^\infty xye^{-p^2x^2-q^2y^2}\sin(xt)\sin(yz)\mathrm{d}x\mathrm{d}y. \tag{2.59}$$

Using the following relations and the known formula [6, Vol1. p.153 Entry 32], we have

$$\mathcal{L}_2\{\sin(xt);p\} = \frac{1}{2}\mathcal{L}\{\sin(x^{1/2}t);p^2\} = \frac{\sqrt{\pi}}{4}\frac{t}{p^3}e^{-t^2/4p^2} \tag{2.60}$$

and

$$\mathcal{L}_2\{\sin(yz);q\} = \frac{1}{2}\mathcal{L}\{\sin(y^{1/2}z);q^2\} = \frac{\sqrt{\pi}}{4}\frac{z}{q^3}e^{-z^2/4q^2}. \tag{2.61}$$

Substituting (2.60) and (2.61) into (2.59), we obtain

$$\mathcal{L}_{22}\{\sin(xt)\sin(yz);(p,q)\} = \frac{\pi}{16}\frac{1}{p^3q^3}tze^{-\frac{t^2}{4p^2}-\frac{z^2}{4q^2}}, \tag{2.62}$$

where with respect to Theorem 2.16, we can take

$$\phi(p,q) = \frac{\pi}{16}\frac{1}{p^3q^3}. \tag{2.63}$$

Also, from the right-hand side of (2.58), we have

$$\begin{aligned}\mathcal{L}_{22}\{\mathrm{erf}(\frac{x}{a})\mathrm{erf}(\frac{y}{b});(p,q)\} &= \int_0^\infty\int_0^\infty xye^{-p^2x^2-q^2y^2}\mathrm{erf}(\frac{x}{a})\mathrm{erf}(\frac{y}{b})\mathrm{d}x\mathrm{d}y\\ &= \mathcal{L}_2\{\mathrm{erf}(\frac{x}{a});p\}\mathcal{L}_2\{\mathrm{erf}(\frac{y}{b});q\}.\end{aligned} \tag{2.64}$$

Making use of the following identities and formulas [6, Vol1. p.176 Entry 6], we obtain

$$\mathcal{L}_2\{\mathrm{erf}(\frac{x}{a});p\} = \frac{1}{2}\mathcal{L}\{\mathrm{erf}(\frac{x^{1/2}}{a});p^2\} = \frac{1}{2a}\frac{1}{p^2}\frac{1}{(p^2+\frac{1}{a^2})^{1/2}}, \tag{2.65}$$

$$\mathcal{L}_2\{\mathrm{erf}(\frac{y}{b});q\} = \frac{1}{2}\mathcal{L}\{\mathrm{erf}(\frac{y^{1/2}}{b});q^2\} = \frac{1}{2b}\frac{1}{q^2}\frac{1}{(q^2+\frac{1}{b^2})^{1/2}}. \tag{2.66}$$

Now, applying Theorem 2.16 to the equation, we have

$$\tilde{\tilde{F}}\left(\frac{1}{2p},\frac{1}{2q}\right) = \frac{\pi}{ab}pq\frac{1}{(p^2+\frac{1}{a^2})^{1/2}}\frac{1}{(q^2+\frac{1}{b^2})^{1/2}}. \tag{2.67}$$

Substituting $p = \frac{1}{2p}$, $q = \frac{1}{2q}$ into (2.67), we get

$$\tilde{\tilde{F}}(p,q) = \frac{\pi^{1/2}}{\sqrt{a^2+4p^2}}\frac{\pi^{1/2}}{\sqrt{b^2+4q^2}} = \pi A(p)B(q). \tag{2.68}$$

Applying the inverse $\mathcal{L}_{22}$-transform to both sides of (2.68) and using the formula [6, Vol.1 p.144 Entry 3], we have

$$\begin{aligned} f(x,y) &= \pi\mathcal{L}_2^{-1}\{\tilde{A}(p)\}\mathcal{L}_2^{-1}\{\tilde{B}(q)\} \\ &= \frac{4}{x}e^{-a^2\frac{x^2}{4}}\frac{4}{y}e^{-b^2\frac{y^2}{4}} \\ &= \frac{16}{xy}e^{-(a^2x^2+b^2y^2)/4}. \end{aligned} \tag{2.69}$$

□

Example. The following integral equation could be solved using Theorem 2.16:

$$\frac{4}{\pi^2}\int_0^\infty\int_0^\infty f(t,z)\sin(xt)\sin(yz)\mathrm{d}t\mathrm{d}z = 1. \tag{2.70}$$

Proof. Applying the $\mathcal{L}_{22}$-transform to the both sides of 2.70, we get

$$\frac{4}{\pi^2}\mathcal{L}_{22}\left\{\int_0^\infty\int_0^\infty f(t,z)\sin(xt)\sin(yz)\mathrm{d}t\mathrm{d}z;(p,q)\right\} = \mathcal{L}\{1;(p,q)\}. \tag{2.71}$$

Using Theorem 2.16, the identity (2.62) and the following identity,

$$\mathcal{L}_{22}\{1;(p,q)\} = \frac{1}{4p^2q^2}, \tag{2.72}$$

we have

$$\begin{aligned} \frac{4}{\pi^2}\frac{\pi}{16}\frac{1}{p^3q^3}\tilde{\tilde{F}}\left(\frac{1}{2p},\frac{1}{2q}\right) &= \frac{1}{4p^2q^2}, \\ \tilde{\tilde{F}}(p,q) &= \frac{\pi}{4pq}. \end{aligned} \tag{2.73}$$

Using the definition of the $\mathcal{L}_2$-transform and the following identity [6, Vol1. p.137 Entry 1], we have

$$\mathcal{L}_2\{t^{-1};p\} = \frac{1}{2}\frac{\Gamma(\frac{1}{2})}{p} = \frac{\sqrt{\pi}}{2p}, \tag{2.74}$$

$$\mathcal{L}_2\{z^{-1};q\} = \frac{\sqrt{\pi}}{2q}. \tag{2.75}$$

Applying the inverse $\mathcal{L}_{22}$-transform to the both sides of (2.73),using the identities (2.74), (2.75) and the relation (2.12), we obtain

$$f(t,z) = \frac{1}{tz}. \tag{2.76}$$

□

Example. We can solve the double Fresnel integral using Theorem 2.16:

$$\int_0^\infty \int_0^\infty \sin(t^2)\sin(z^2)\mathrm{d}t\mathrm{d}z = \frac{\pi}{8}. \tag{2.77}$$

Proof. We consider the following integral:

$$I(x,y) = \int_0^\infty \int_0^\infty \sin(xt^2)\sin(yz^2)\mathrm{d}t\mathrm{d}z. \tag{2.78}$$

Changing the variables of the integral (2.78) by the transformation $t^2 = u$, $z^2 = v$, we have

$$I(x,y) = \frac{1}{4}\int_0^\infty \int_0^\infty \frac{1}{\sqrt{uv}}\sin(xu)\sin(yv)\mathrm{d}u\mathrm{d}v. \tag{2.79}$$

We could apply the $\mathcal{L}_{22}$-transform to both sides of (2.79). Thus, we need the following calculations:

$$\mathcal{L}_{22}\{\sin(xu)\sin(yv);(p,q)\} = \frac{\pi}{16}\frac{1}{p^3q^3}tze^{-\frac{t^2}{4p^2}-\frac{z^2}{4q^2}}, \tag{2.80}$$

$$\begin{aligned}\mathcal{L}_{22}\left\{\frac{1}{\sqrt{uv}};(p,q)\right\} &= \mathcal{L}_2\left\{u^{-1/2};p\right\}\mathcal{L}_2\{v^{-1/2};q\} \\ &= \frac{1}{4}\Gamma\left(\frac{3}{4}\right)^2\frac{1}{p^{3/2}q^{3/2}}.\end{aligned} \tag{2.81}$$

According to Theorem 2.16, we can consider the following identities:

$$\phi(p,q) = \frac{\pi}{16}\frac{1}{p^3q^3},\ \tilde{\tilde{F}}(k(p),m(q)) = 2\Gamma\left(\frac{3}{4}\right)^2 p^{3/2}q^{3/2}, \tag{2.82}$$

where $k(p) = \frac{1}{2p}$, $m(q) = \frac{1}{2q}$.
Then, we have

$$\tilde{\tilde{I}}(p,q) = \frac{\pi}{32}\Gamma\left(\frac{3}{4}\right)^2\frac{1}{p^{3/2}q^{3/2}}. \tag{2.83}$$

Applying the inverse $\mathcal{L}_{22}$-transform to both sides of (2.83) and using the identity [6, Vol1 p. 137 Entry 1], we have

$$I(x,y) = \frac{\pi}{8}\frac{1}{\sqrt{xy}}. \tag{2.84}$$

We can see the solution of the double Fresnel integral from the following relation:

$$I(1,1) = \int_0^\infty \int_0^\infty \sin(t^2)\sin(z^2)\mathrm{d}t\mathrm{d}z = \frac{\pi}{8}. \tag{2.85}$$

□

Example. We calculate the following integral using the Theorem 2.16:

$$\int_0^\infty\int_0^\infty \frac{\cos(mx)}{m}\frac{\cos(ny)}{n}\sin(xt)\sin(yz)\mathrm{d}x\mathrm{d}y = \frac{\pi^2}{4}H(t-m)H(z-n). \quad (2.86)$$

Proof. Applying the $\mathcal{L}_{22}$-transform to the integral in the right-hand side of the (2.86), we get

$$\tilde{\tilde{I}}(x,y,p,q) = \mathcal{L}_{22}\left\{\int_0^\infty\int_0^\infty \frac{\cos(mx)}{m}\frac{\cos(ny)}{n}\sin(xt)\sin(yz)\mathrm{d}x\mathrm{d}y;(p,q)\right\}. \quad (2.87)$$

According to the Theorem 2.16, we have

$$f(x,y) = \frac{\cos(mx)}{m}\frac{\cos(ny)}{n},\ \Phi(x,y,z,t) = \sin(xt)\sin(yz). \quad (2.88)$$

To apply the Theorem 2.16 to the (2.87), we need some following calculations. Using the identities [6, Vol1 153 Entry 32], [6, Vol1 p.158 Entry 67], we obtain

$$\mathcal{L}_{22}\{\Phi(x,y,z,t);(p,q)\} = \frac{\pi}{16}\frac{1}{p^3q^3}tze^{-\frac{t^2}{4p^2}-\frac{z^2}{4q^2}}, \quad (2.89)$$

where $\phi(p,q) = \frac{\pi}{16}\frac{1}{p^3q^3}$ and

$$\begin{aligned}\tilde{\tilde{F}}(p,q) &= \mathcal{L}_{22}\{f(x,y);(p,q)\} = \mathcal{L}_2\left\{\frac{\cos(mx)}{m};p\right\}\mathcal{L}_2\left\{\frac{\cos(ny)}{n};q\right\}\\ &= \frac{\pi}{4pq}e^{-\frac{m^2}{4p^2}-\frac{n^2}{4q^2}}. \end{aligned} \quad (2.90)$$

According to the Theorem 2.16, we get

$$\mathcal{L}_{22}\{I(x,y,z,t);(p,q)\} = \phi(p,q)\tilde{\tilde{F}}(\frac{1}{2p},\frac{1}{2q}) = \frac{\pi^2}{16}\frac{1}{p^2q^2}e^{-m^2p^2-n^2q^2}. \quad (2.91)$$

Applying the inverse $\mathcal{L}_{22}$-transform to both sides of (2.91) and using the following identity:

$$\begin{aligned}\mathcal{L}_{22}\{H(t-m)H(z-n);(p,q)\} &= \mathcal{L}_2\{H(t-m);p\}\mathcal{L}_2\{H(z-n);q\}\\ &= \int_m^\infty te^{-t^2p^2}\mathrm{d}t\int_n^\infty ze^{-z^2q^2}\mathrm{d}z\\ &= \frac{1}{4p^2q^2}e^{-m^2p^2-n^2q^2}, \end{aligned} \quad (2.92)$$

we obtain

$$I(x,y,z,t) = \frac{\pi^2}{4}H(t-m)H(z-n). \quad (2.93)$$

□

3. Some Properties of the $\mathcal{L}_{22}$-Transform and Differential Operator δ

Definition 3.1. The δ-derivative operator is defined as in [13]

$$\delta_x = \frac{1}{x}\frac{d}{\mathrm{d}x} \tag{3.1}$$

and

$$\delta_x^2 = \frac{1}{x^2}\frac{d^2}{\mathrm{d}x^2} - \frac{1}{x^3}\frac{d}{\mathrm{d}x}. \tag{3.2}$$

The δ-derivative operator can be successively applied in a similar fashion for any positive integer power.

Theorem 3.2. *If $f, f', \ldots, f^{(n-1)}$ are all continuous functions with a piecewise continuous derivative $f^{(n)}$ on the interval $x \geq 0$, and if all functions are of exponential order* $\exp(c^2x^2)$ *as $x \to \infty$ for some constant c, then*

$$\begin{aligned}\mathcal{L}_2\{\delta_x^n f(x); s\} &= 2^n s^{2n}\mathcal{L}_2\{f(t); s\} - 2^{n-1}s^{2(n-1)}f(0^+)\\ &\quad -2^{n-2}s^{2(n-2)}(\delta_x f)(0^+) - \cdots - (\delta_x^{n-1}f)(0^+),\end{aligned} \tag{3.3}$$

for $n = 1, 2, \ldots$.

Theorem 3.3. *If f is piecewise continuous on $x \geq 0$ and is of exponential order* $\exp(c^2x^2)$ *as $x \to \infty$, then*

$$\mathcal{L}_2\{x^{2n}f(x); s\} = \frac{(-1)^n}{2^n}\delta_s^n\mathcal{L}_2\{f(x); s\}, \tag{3.4}$$

for $n = 1, 2, \ldots$.

The proofs of Theorems 3.2 and 3.3 could be found in [13].

Theorem 3.4. *If $f, \frac{\partial f}{\partial x}, \ldots, \frac{\partial^{n-1} f}{\partial x^{n-1}}, \frac{\partial f}{\partial y}, \ldots, \frac{\partial^{n-1} f}{\partial y^{n-1}}$ are all continuous functions with piecewise continuous derivatives $\frac{\partial^n f}{\partial x^n}, \frac{\partial^n f}{\partial y^n}$ in the intervals first quadrant and if all functions are of exponential order* $\exp(a^2x^2 + b^2y^2)$ *as $x \to \infty, y \to \infty$ for constants a, b then the following identities:*

$$\begin{aligned}\mathcal{L}_{22}\{\delta_x^k f(x,y); (p,q)\} &= (2p^2)^k\mathcal{L}_{22}\{f(x,y); (p,q)\} - (2p^2)^{k-1}\tilde{f}(0^+, q)\\ &\quad -(2p^2)^{k-2}(\delta_x\tilde{f})(0^+, q) - \cdots - 2p^2(\delta_x^{k-2}\tilde{f})(0^+, q)\\ &\quad -(\delta_x^{k-1}\tilde{f})(0^+, q),\end{aligned} \tag{3.5}$$

$$\begin{aligned}\mathcal{L}_{22}\{\delta_y^k f(x,y); (p,q)\} &= (2q^2)^k\mathcal{L}_{22}\{f(x,y); (p,q)\} - (2q^2)^{k-1}\tilde{f}(p, 0^+)\\ &\quad -(2q^2)^{k-2}(\delta_y\tilde{f})(p, 0^+) - \cdots - 2q^2(\delta_y^{k-2}\tilde{f})(p, 0^+)\\ &\quad -(\delta_y^{k-1}\tilde{f})(p, 0^+),\end{aligned} \tag{3.6}$$

hold true, where $k \in \mathbb{N}$.

Proof. Under the hypothesis of Theorem 3.4, if we use the definitions of the $\mathcal{L}_{22}$-transform (5), the $\mathcal{L}_2$-transform (2) and the δ_x,δ_y derivatives (10), then use integration by parts, we obtain

$$\mathcal{L}_{22}\{\delta_x f(x,y);(p,q)\} = \int_0^\infty y e^{-y^2q^2} \int_0^\infty x e^{-x^2p^2} \delta_x f(x,y)\mathrm{d}x\mathrm{d}y$$

$$\int_0^\infty y e^{-y^2q^2} \mathcal{L}_2\{\delta_x f(x,y);p\}\mathrm{d}y = 2p^2\tilde{\tilde{f}}(p,q) - \tilde{f}(0^+,q). \tag{3.7}$$

And we have

$$\mathcal{L}_{22}\{\delta_y f(x,y);(p,q)\} = 2q^2\tilde{\tilde{f}}(p,q) - \tilde{f}(p,0^+), \tag{3.8}$$

where f is of exponential order $e^{a^2x^2+b^2y^2}$ as $x \to \infty$, $y \to \infty$.
Similarly, f and $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ are continuous functions with a piecewise continuous derivatives $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y^2}$ on the first quadrant and if all functions are of exponential order $e^{a^2x^2+b^2y^2}$ as $x \to \infty$, $y \to \infty$, we can use (102) and (103) to obtain

$$\mathcal{L}_{22}\{\delta_x^2 f(x,y);(p,q)\} = 4p^4\tilde{\tilde{f}}(p,q) - 2p^2\tilde{f}(0^+,q) - (\delta_x\tilde{f})(0^+,q), \tag{3.9}$$

$$\mathcal{L}_{22}\{\delta_y^2 f(x,y);(p,q)\} = 4q^4\tilde{\tilde{f}}(p,q) - 2q^2\tilde{f}(p,0^+) - (\delta_y\tilde{f})(p,0^+), \tag{3.10}$$

then using the known formula [13],

$$\mathcal{L}_2\{\delta_x^2 f(x,y);p\} = 4p^4\mathcal{L}_2\{f(x,y);p\} - 2p^2 f(0^+,y) - (\delta_x f)(0^+,y) \tag{3.11}$$

and repeating applications of (106),(107),(108) and (109), we arrive at (104) and (105) of Theorem 3.4. □

Remark 3.5. Under the hypothesis of Theorem 3.4, we have the following relation:

$$\mathcal{L}_{22}\{\delta_x\delta_y f(x,y);(p,q)\} = 4p^2q^2\tilde{\tilde{f}}(p,q) - 2q^2\tilde{f}(0^+,q) \\ -2p^2\tilde{f}(p,0^+) + f(0^+,0^+). \tag{3.12}$$

Theorem 3.6. *Under the hypothesis of Theorem 3.4, we have*

$$\mathcal{L}_{22}\{x^{2n}y^{2m}f(x,y);(p,q)\} = \frac{(-1)^{n+m}}{2^{n+m}}\delta_p^n\delta_q^m\mathcal{L}_{22}\{f(x,y);(p,q)\}, \tag{3.13}$$

where $n, m \in \mathbb{N}$.

Proof. Using the definitions of the $\mathcal{L}_{22}$-integral transform (7) and δ-operator (100) and taking the δ_p-derivative, then taking δ_q-derivative of (7) with respect to p and with respect to q, respectively, we arrive at (112). □

4. Illustrative Examples

Example. We solve partial differential equation with the following conditions:

$$xt^3u_{xx} + x^3tu_{tt} - t^3u_x - x^3u_t = 0, \tag{4.1}$$

$$\text{(i) } u_x(0,t) = 0,\ u_t(x,0) = 0, \tag{4.2}$$

$$\text{(ii) } u(0,t) = e^{-t^2},\ u(x,0) = 0, \tag{4.3}$$

$$\text{(iii) } u(0,t) = t^2,\ u(x,0^+) = 0, \tag{4.4}$$

$$\text{(iv)}\ u(0,t)=e^{\eta^2t^2},\ u(x,0)=0. \tag{4.5}$$

Proof. Dividing the Eq. (4.1) by x^3t^3 and using the definition (11), we obtain

$$\delta_x^2 u+\delta_t^2 u=0. \tag{4.6}$$

Applying the $\mathcal{L}_{22}$-transform to both sides of (4.2) and using Theorem 3.4, we have

$$\begin{aligned}\mathcal{L}_{22}\{\delta_x^2u(x,t);(p,q)\} &= \int_0^\infty te^{-q^2t^2}\Big[\int_0^\infty xe^{-p^2x^2}\delta_x^2u\mathrm{d}x\Big]\mathrm{d}t\\ &= \int_0^\infty te^{-q^2t^2}\mathcal{L}_2\{\delta_x^2u(x,y);p\}\mathrm{d}t\\ &= \int_0^\infty te^{-q^2t^2}(4p^4\tilde{u}(p,t)-2p^2u(0,t)-(\delta_xu)(0,t))\mathrm{d}t\\ &= 4p^4\tilde{\tilde{u}}(p,q)-2p^2\tilde{u}(0,q)-(\delta_x\tilde{u})(0,q).\end{aligned} \tag{4.7}$$

Similarly, we have

$$\mathcal{L}_{22}\{\delta_t^2u(x,t);(p,q)\}=4q^4\tilde{\tilde{u}}(p,q)-2q^2\tilde{u}(p,0)-(\delta_t\tilde{u})(p,0). \tag{4.8}$$

and

$$\tilde{u}_t(p,0)=0,\ \tilde{u}_x(0,q)=0. \tag{4.9}$$

Setting (4.7) and (4.8) into (4.6), we have

$$\tilde{\tilde{u}}(p,q)=\frac{2p^2}{4p^4+4q^4}\tilde{u}(0,q)+\frac{2q^2}{4p^4+4q^4}\tilde{u}(p,0) \tag{4.10}$$

and applying $\mathcal{L}_2^{-1}$-transform to (4.10) with respect to q and p, respectively, we get

$$\mathcal{L}_2^{-1}\left\{\frac{2p^2}{4p^4+4q^4};t\right\}=\sin(p^2t^2), \tag{4.11}$$

$$\mathcal{L}_2^{-1}\left\{\frac{2q^2}{4p^4+4q^4};t\right\}=\cos(p^2t^2). \tag{4.12}$$

Substituting (4.11) and (4.12) into (4.10), we obtain

$$\begin{aligned}\tilde{u}(p,t)&=\sin(p^2t^2)*u(0,t)+\cos(p^2t^2)\tilde{u}(p,0)\\ &=\int_0^t\xi\sin(p^2\xi^2)u(0,(t^2-\xi^2)^{1/2})\mathrm{d}\xi+\cos(p^2t^2)\tilde{u}(p,0).\end{aligned} \tag{4.13}$$

Now, applying the $\mathcal{L}_2^{-1}$-transform (3) to (4.10) with respect to p, we shall consider the following series representations:

$$\sin(p^2\xi^2)=\sum_{n=0}^\infty\frac{(-1)^n}{(2n+1)!}(p^2\xi^2)^{2n+1}, \tag{4.14}$$

$$\cos(p^2t^2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(p^2t^2)^{2n}. \tag{4.15}$$

Using the following known formulas([7], Vol1 p.137 Entry 1),

$$\mathcal{L}_2^{-1}\{p^{4n+2};x\} = \frac{2x^{-4n-4}}{(-2n-2)!},\ \mathcal{L}_2^{-1}\{p^{4n};x\} = \frac{2x^{-4n-2}}{(-2n-1)!}, \tag{4.16}$$

$$\mathcal{L}_2^{-1}\{\sin(p^2\xi^2);x\} = 2\sum_{n=0}^{\infty} \frac{(-1)^n\xi^{4n+2}}{(2n+1)!(-2n-2)!x^{4n+4}}, \tag{4.17}$$

$$\mathcal{L}_2^{-1}\{\cos(p^2t^2);x\} = 2\sum_{n=0}^{\infty} \frac{(-1)^n t^{4n}}{(2n)!(-2n-1)!x^{4n+2}} \tag{4.18}$$

and the following identities

$$\begin{aligned}(2n+1)!(-2n-2)! = B(2n+2,-2n-1),\ (2n)!(-2n-1)!\\ = B(2n+1,-2n),\end{aligned} \tag{4.19}$$

we obtain the formal solution of the problem (4.1),(4.2) as follows:

$$\begin{aligned} u(x,t) &= 2\sum_{n=0}^{\infty}(-1)^n \frac{x^{-4n-4}}{B(2n+2,-2n-1)} \int_0^t \xi^{4n+3} u(0,\sqrt{t^2-\xi^2})\mathrm{d}\xi \\ &\quad +2\sum_{n=0}^{\infty}(-1)^n \frac{t^{4n}}{B(2n+1,-2n)} \int_0^x \xi^{-4n-1} u(\sqrt{x^2-\xi^2},0)\mathrm{d}\xi. \end{aligned} \tag{4.20}$$

If we consider the partial differential equation (4.1) and the condition (4.3), then the solution of this problem is obtained as follows:

$$\begin{aligned} u(x,t) &= 2\sum_{n=0}^{\infty}(-1)^n \frac{x^{-4n-4}}{B(2n+2,-2n-1)} \int_0^t \xi^{4n+3} e^{-t^2+\xi^2}\mathrm{d}\xi \\ &= 2e^{-t^2}\sum_{n=0}^{\infty}(-1)^n \frac{x^{-4n-4}}{B(2n+2,-2n-1)} \int_0^t \xi^{4n+3} e^{\xi^2}\mathrm{d}\xi, \end{aligned} \tag{4.21}$$

where setting $\xi^2 = u$, we get

$$\begin{aligned} \int_0^t \xi^{4n+3} e^{\xi^2}\mathrm{d}\xi &= \frac{1}{2}\int_0^{t^2} u^{2n+1}e^u \mathrm{d}u \\ &= \left[e^{t^2}\sum_{k=0}^{2n+1}(-1)^{k+1}\frac{(2n+1)!}{k!}t^{2k} + (2n+1)!\right]\frac{1}{2}e^{-t^2}. \end{aligned} \tag{4.22}$$

Substituting (4.22) into (4.21), we obtain the solution of the problem (4.1)-(4.3) as follows:

$$u(x,t) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{-4n-4}}{\Gamma(-2n-2)}\left[e^{-t^2}\sum_{k=0}^{2n+1}(-1)^{k+1}\frac{t^{2k}}{k!}\right]. \tag{4.23}$$

iii) Similarly, the solution of the problem (4.1)–(4.4) is obtained by the following series:

$$u(x,t) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{-4n-4}t^{4n+6}}{\Gamma(2n+2)\Gamma(-2n-2)(2n+2)(2n+3)}$$
$$= \frac{1}{6}\left(\frac{t^3}{x^2}\right)^2 \sum_{0}^{\infty}(-1)^n \frac{1}{B(2n+4,-2n-2)}\left(\frac{t}{x}\right)^{4n}, \tag{4.24}$$

where

$$\int_0^t \xi^{4n+3}(t^2-\xi^2)\mathrm{d}\xi = \frac{1}{2}t^{4n+6}\frac{1}{(2n+2)(2n+3)}. \tag{4.25}$$

iv) If we consider the problem (4.1)–(4.5), we obtain

$$u(x,t) = 2\sum_{n=0}^{\infty}(-1)^n \frac{x^{-4n-4}}{B(2n+2,-2n-1)} \int_0^t \xi^{4n+3}e^{\eta^2(t^2-\xi^2)}\mathrm{d}\xi, \tag{4.26}$$

where setting $\xi^2 = u$ and using integration by part, we get

$$\int_0^t \xi^{4n+3}e^{\eta^2(t^2-\xi^2)}\mathrm{d}\xi = \frac{1}{2}e^{\eta^2t^2}\int_0^{t^2} u^{2n+1}e^{-\eta^2 u}\mathrm{d}u$$
$$= \frac{x}{2}\Bigg[\Gamma(2n+2)(-\eta^2)^{2n+2}e^{\eta^2t^2}$$
$$-(-\eta^2)^{2n+2}\Gamma(2n+2)\sum_{k=0}^{2n+1}\frac{(t/\eta)^{2k}}{k!}\Bigg]. \tag{4.27}$$

Substituting (4.27) into (4.26), we obtain the following solution:

$$u(x,t) = \sum_{n=0}^{\infty}\frac{(-1)^n}{\Gamma(-2n-1)}\left(\frac{x}{\eta}\right)^{-4n-4}\left[e^{\eta^2t^2} - \sum_{k=0}^{2n+1}\frac{(t/\eta)^{2k}}{k!}\right]. \tag{4.28}$$

□

Example. We solve the following partial differential equation:

$$tu_x + xu_t = xt \tag{4.29}$$

with the conditions

$$u(x,0) = \frac{x^2}{4},\ u(0,t) = \frac{t^2}{4}. \tag{4.30}$$

Proof. Dividing the Eq. (4.29) by xt, we have

$$\delta_x u + \delta_t u = 1. \tag{4.31}$$

Applying the $\mathcal{L}_{22}$-transform to the both sides of (4.31) and using Theorem 3.2, we get

$$\mathcal{L}_{22}\{\delta_x u\} + \mathcal{L}_{22}\{\delta_t u\} = \mathcal{L}_{22}\{1\}, \tag{4.32}$$

$$\begin{aligned}\mathcal{L}_{22}\{\delta_x u(x,t);(s,k)\} &= \int_0^\infty te^{-s^2t^2}\int_0^\infty xe^{-k^2x^2}\delta_x u\mathrm{d}x\mathrm{d}t \\ &= \int_0^\infty te^{-s^2t^2}\mathcal{L}_2\{\delta_x u;k\}\mathrm{d}t \\ &= 2k^2\tilde{\tilde{u}}(k,s) - \tilde{u}(0^+,s), \end{aligned}\tag{4.33}$$

$$\mathcal{L}_{22}\{\delta_t u(x,t);(s,k)\} = 2s^2\tilde{\tilde{u}}(k,s) - \tilde{u}(k,0^+), \tag{4.34}$$

and

$$\mathcal{L}_{22}\{1;(s,k)\} = \frac{1}{4k^2s^2}. \tag{4.35}$$

Putting (4.33),(4.34) and (4.35) back at (4.32), we find

$$(2k^2+2s^2)\tilde{\tilde{u}}(k,s) - \tilde{u}(0^+,s) - \tilde{u}(k,0^+) = \frac{1}{4k^2s^2}. \tag{4.36}$$

Using (4.30) boundary conditions, we have $\bar{u}(0^+,s) = \frac{1}{8s^4}$, $\bar{u}(k,0^+) = \frac{1}{8k^4}$ thus,

$$\tilde{\tilde{u}}(k,s) = \frac{1}{16}\left[\frac{1}{k^2s^4} + \frac{1}{k^4s^2}\right]. \tag{4.37}$$

Applying the inverse $\mathcal{L}_{22}$-transform to the both sides of (4.37), we get

$$u(x,t) = \frac{1}{16}\left[\mathcal{L}_{22}^{-1}\left\{\frac{1}{k^2s^4};(x,t)\right\} + \mathcal{L}_{22}^{-1}\left\{\frac{1}{k^4s^2};(x,t)\right\}\right]. \tag{4.38}$$

We know the following identities [6, Vol.1 p.137 Entry 1],

$$\mathcal{L}_2\{x^{2k};y\} = \frac{1}{2}\mathcal{L}\{x^k;y^2\} = \frac{1}{2}\frac{\Gamma(k+1)}{y^{2(k+1)}}. \tag{4.39}$$

Using (4.39),(2.12) and the definition of $\mathcal{L}_2$-transform, we have

$$\mathcal{L}_{22}^{-1}\left\{\frac{1}{k^2s^4};(x,t)\right\} = \mathcal{L}_2^{-1}\left\{\frac{1}{k^2};x\right\}\mathcal{L}_2^{-1}\left\{\frac{1}{s^4};t\right\} = 4t^2 \tag{4.40}$$

and

$$\mathcal{L}_{22}^{-1}\left\{\frac{1}{k^4s^2};(x,t)\right\} = \mathcal{L}_2^{-1}\left\{\frac{1}{k^4};x\right\}\mathcal{L}_2^{-1}\left\{\frac{1}{s^2};t\right\} = 4x^2. \tag{4.41}$$

Substituting the identities (4.40) and (4.41) to (4.38), we obtain

$$u(x,t) = \frac{1}{4}(x^2+t^2). \tag{4.42}$$

□

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