Mediterr. J. Math. (2018) 15:27
https://doi.org/10.1007/s00009-018-1074-3
1660-5446/18/010001-33
published online January 30, 2018

Mediterranean Journal
of Mathematics

Legendre Wavelets Direct Method for the Numerical Solution of Time-Fractional Order Telegraph Equations

Xiaoyong Xu and Da Xu

Abstract. In this paper, a Legendre wavelet collocation method for solving a class of time-fractional order telegraph equation defined by Caputo sense is discussed. Fractional integral formula of a single Legendre wavelet in the Riemann–Liouville sense is derived by means of shifted Legendre polynomials. The main characteristic behind this approach is that it reduces equations to those of solving a system of algebraic equations which greatly simplifies the problem. The convergence analysis and error analysis of the proposed method are investigated. Several examples are presented to show the applicability and accuracy of the proposed method.

Mathematics Subject Classification. 65T60, 26A33.

Keywords. Legendre wavelets, time-fractional order telegraph equation, Riemann–Liouville integral, shifted Legendre polynomials, collocation method.

1. Introduction

Fractional order ordinary and partial differential equations, as generalization of classical integer order differential equations, play an important role in modelling various phenomena of physics, chemistry, engineering, aerodynamics, etc. and have become the focus of many researchers in recent years. Comparing with integer order differential equations, fractional order ordinary and partial differential equations can describe natural physical process and dynamic system more accurately. Telegraph equations belonging to hyperbolic partial differential equations are applicable in wave propagation ([1]),

Supported by the National Natural Science Foundation of China (Grant Nos. 11601076, 11671131), the Youth Science Foundation of Jiangxi Province (Grant Nos. 20151BAB211004, 20151BAB211012), the Construct Program of the Key Discipline in Hunan Province and the Science and Technology Project of Jiangxi Provincial Education Department (Grant No. GJJ170445).

Birkhäuser

random walk theory ([2]), signal analysis ([3]), etc. (see [4] and the references therein). The time-fractional order telegraph equations have recently been considered by many authors.

It is noted that most fractional order differential equations do not have closed form solutions. Many researchers have proposed various methods to solve the telegraph equations. To mention a few, the analytical solution of the time-fractional telegraph equation with three kinds of boundary conditions is derived by using the method of separation of variables ([5]). Dehghan and his group applied Radial basis function ([6]), Chebyshev cardinal functions ([7]) and Chebyshev tau method ([8]) to tackle the telegraph equation. In [9], Das and Gupta used homotopy analysis method for solving fractional hyperbolic partial differential equation. In [10], Mollahasani applied hybrid functions of Legendre polynomials and Block pulse functions to obtain the solution of telegraph equation of fractional order. Differential transform method ([11]), Variational iteration method ([12]), Adomian decomposition method ([13]) and He's homotopy perturbation ([14]) are used to achieve closed form solutions of the problem. In [15], fully discrete local discontinuous Galerkin method is used to solve the fractional telegraph equation. In [16], Sweilam introduced a numerical method based on Sinc–Legendre collocation method for solving the time-fractional telegraph equation. Heydari applied two-dimensional Legendre wavelets and block pulse function to solve the problem ([17]).

This paper focuses on the time-fractional order telegraph equation of order α $(1 < \alpha \leq 2)$ as

$$\frac{\partial^{\alpha}\mu(x,t)}{\partial t^{\alpha}} + \lambda_1 \frac{\partial^{\alpha-1}\mu(x,t)}{\partial t^{\alpha-1}} + \lambda_2 \mu(x,t) = \lambda_3 \frac{\partial^2 \mu(x,t)}{\partial x^2} + f(x,t),$$
$$0 < x < 1,\ 0 < t \leq 1, \tag{1}$$

with the initial conditions

$$\mu(x,0) = f_1(x), \frac{\partial \mu(x,0)}{\partial t} = f_2(x),\ \ 0 \leq x \leq 1, \tag{2}$$

and the Dirichlet boundary conditions

$$\mu(0,t) = g_0(t),\ \mu(1,t) = g_1(t),\ 0 < t \leq 1, \tag{3}$$

where $f_1(x), f_2(x), g_0(t), g_1(t)$ are given functions with second-order continuous derivatives, $\frac{\partial \mu(x,t)}{\partial t}$ represents the Caputo fractional derivative, and λ_1, λ_2, λ_3 are constants.

Spectral methods are widely used in seeking numerical solutions of fractional order differential equations, due to their excellent error properties and exponential rates of convergence for smooth problems. There are three most common spectral schemes, namely, the collocation method, Galerkin and Tau methods. Collocation methods have been applied successfully to numerical simulations of many problems in science and engineering, see [18–21].

Wavelets, as another basis set and very well-localized functions, are considerably useful for solving differential and integral equations. Particularly, orthogonal wavelets are widely used in dealing with various types of differential equations in the relevant literatures. Recently, the operational matrices for Legendre wavelet, Chebyshev wavelet and Bernoulli wavelet have been extensively used to cope with different problems ([20–22]). It is observed that most papers using orthogonal wavelets methods to solve fractional order differential equations are based on the operational matrix of fractional integration or differentiation ([23–26]). The approximation error is inevitably generated during the construction of operational matrix. In [27], some descriptions are given to show some disadvantages of using the operational matrix of Legendre and Chebyshev wavelet.

Inspired by the work mentioned above, the main aim of this paper is to extend the Legendre wavelets for solving the time-fractional order equations (1) with the initial condition (2) and Dirichlet boundary conditions (3). To reduce the approximation error at most during the calculation process, Riemann–Liouville fractional integral formula of a single Legendre wavelet is derived. The proposed method is based on reducing the equations into a linear system of algebraic equations. The proposed method is very convenient for solving such problems, since the initial and boundary conditions are taken into account automatically.

The organization of the rest part is as follows: Section 2 describes some necessary definitions and preliminaries of calculus. Section 3 gives some properties of Legendre wavelets. And the convergence analysis and error analysis of the proposed method are given. In Sect. 4, the fractional integral formula for a sinlge Legendre wavelet is derived. The proposed method is described for solving time-fractional order telegraph equations in Sect. 5. In Sect. 6, the numerical results are presented. Finally, a brief conclusion is stated in Sect. 7.

2. Definitions and Preliminaries

In this section, we present some necessary definitions and preliminaries of the fractional calculus theory which will be used later.

Definition 2.1. A real function $u(x)$, $x > 0$, is said to be in the space C_σ, $\sigma \in \mathbb{R}$, if there is a real number ρ with $\rho > \sigma$ such that $u(x) = x^\rho u_0(x)$, where $u_0(x) \in C[0, \infty)$, and $u(x) \in C_\sigma^n$ if $u^{(n)}(x) \in C_\sigma$, $n \in \mathbb{N}$.

Definition 2.2. ([28]) The Riemann–Liouville fractional integral operator I^α of order α ($\alpha \geq 0$) for a function $u(x) \in C_\sigma$ ($\sigma \geq -1$) is defined as

$$I^\alpha u(x) = \begin{cases} u(x), & \alpha = 0, \\ \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} u(t) \mathrm{d}t, & \alpha > 0. \end{cases}$$

Definition 2.3. ([28]) The Caputo fractional derivative operator D^α of order α ($\alpha \geq 0$) for a function $h(x) \in C_1^n$ is defined as

$$D^\alpha u(x) = \begin{cases} u^{(n)}(x), & \alpha = n \in \mathbb{N}, \\ \frac{1}{\Gamma(n-\alpha)} \int_0^x \frac{u^{(n)}(t)}{(x-t)^{\alpha+1-n}} \mathrm{d}t, & n-1 < \alpha < n. \end{cases}$$

The relations between the Riemann–Liouville fractional integral operator I^α and the Caputo fractional derivative operator D^α are given by the following expressions:

$$D^\alpha I^\alpha u(x) = u(x), D^\beta I^\alpha u(x) = I^{\alpha-\beta} u(x), \quad \alpha > \beta,$$

$$I^\alpha D^\alpha u(x) = u(x) - \sum_{k=0}^{\lceil\alpha\rceil-1} u^{(k)}(0^+)\frac{x^k}{k!}, \quad x > 0, \tag{4}$$

where the ceiling function $\lceil\alpha\rceil$ denotes the smallest integer than or equal to α and $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$. The following properties of the operators I^α and D^α are needed in this paper:

1. $I^{\alpha_1} I^{\alpha_2} u(x) = I^{\alpha_1+\alpha_2} u(x)$ for $\alpha_1, \alpha_2 > 0$;
2. $I^\alpha(\lambda_1 u(x) + \lambda_2 v(x)) = \lambda_1 I^\alpha u(x) + \lambda_2 I^\alpha v(x)$ for constants λ_1, λ_2;
3. $D^\alpha(\lambda_1 u(x) + \lambda_2 v(x)) = \lambda_1 D^\alpha u(x) + \lambda_2 D^\alpha v(x)$ for constants λ_1, λ_2;
4. $D^\alpha C = 0$ for constant C;
5. $D^\alpha x^n = \begin{cases} 0, & n \in \mathbb{N}_0 \text{ with } n < \lceil\alpha\rceil, \\ \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)} x^{n-\alpha}, & n \in \mathbb{N}_0 \text{ with } n \geq \lceil\alpha\rceil. \end{cases}$

3. Legendre Wavelets and Their Properties

Legendre wavelets $\psi_{n,m}(x) = \psi(k, \hat{n}, m, x)$ have four arguments: k is any arbitrary positive integer, $n = 1, 2, \ldots, 2^{k-1}$, $\hat{n} = 2n - 1$, m is the order of Legendre polynomial and x is normalized time. They are defined on the interval $[0, 1]$ as ([29])

$$\psi_{n,m}(x) = \begin{cases} \sqrt{m + \frac{1}{2}}\, 2^{\frac{k}{2}} L_m(2^k x - \hat{n}), & \frac{\hat{n}-1}{2^k} \leq x \leq \frac{\hat{n}}{2^k}, \\ 0, & \text{otherwise}, \end{cases} \tag{5}$$

where $n = 1, 2, \ldots$, $m = 0, 1, \ldots$. The coefficient $\sqrt{m + \frac{1}{2}}$ is for orthonomality. Here $L_m(x)$ is Legendre polynomial of order m defined on the interval $[-1, 1]$ and can be determined from the following recurrence formulae:

$$L_0(x) = 1, L_1(x) = x, L_{m+1}(x) = \left(\frac{2m+1}{m+1}\right) x L_m(x) - \left(\frac{m}{m+1}\right) L_{m-1}(x), \quad m = 1, 2, 3, \ldots.$$

The shifted Legendre polynomials $L_m^*(x)$ are defined on $[0, 1]$ as:

$$L_m^*(x) = \sum_{i=0}^{m} (-1)^{m-i} \frac{(m+i)!\, x^i}{(m-i)!(i!)^2}. \tag{6}$$

By using the shifted Legendre polynomials, Legendre wavelets can be written as follows:

$$\psi_{n,m}(x) = \begin{cases} \sqrt{m + \frac{1}{2}}\, 2^{\frac{k}{2}} L_m^*(2^{k-1} x - n + 1), & \frac{n-1}{2^{k-1}} \leq x \leq \frac{n}{2^{k-1}}, \\ 0, & \text{otherwise}, \end{cases} \tag{7}$$

A function $f(x) \in L^2(\mathbb{R})$ defined on the interval $[0, 1)$ may be expanded by the Legendre wavelets as

$$f(x) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{n,m} \psi_{n,m}(x), \tag{8}$$

where $c_{n,m} = \langle f(x), \psi_{n,m}(x)\rangle = \int_0^1 f(x)\psi_{n,m}(x)\mathrm{d}x$, in which $\langle \cdot, \cdot\rangle$ denotes the inner product in $L^2[0,1)$. If the infinite series in Eq. (8) is truncated, then it can be written as

$$f(x) \cong \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}(x) = C^T\Psi(x), \tag{9}$$

where C and $\Psi(x)$ are $2^{k-1}M \times 1$ matrices given by

$$C = \left(c_{1,0}c_{1,1}\cdots c_{1,M-1}c_{2,0}c_{2,1}\cdots c_{2,M-1}\cdots c_{2^{k-1},0}c_{2^{k-1},1}\cdots c_{2^{k-1},M-1}\right)^T,$$
$$\Psi(x) = \left(\psi_{1,0}\psi_{1,1}\cdots\psi_{1,M-1}\psi_{2,0}\psi_{2,1}\cdots\psi_{2,M-1}\cdots\psi_{2^{k-1},0}\psi_{2^{k-1},1}\cdots\psi_{2^{k-1},M-1}\right)^T. \tag{10}$$

The two-dimensional Legendre wavelets are defined as ([30])

$$\psi_{n_1,m_1,n_2,m_2}(x,t) = \begin{cases} \sqrt{m_1+\frac{1}{2}}\sqrt{m_2+\frac{1}{2}}\,2^{\frac{k_1+k2}{2}}L_{m_1}(2^{k_1}x-2n_1+1) \\ \quad L_{m_2}(2^{k_2}t-2n_2+1), \\ \frac{n_1-1}{2^{k_1-1}} \le x < \frac{n_1}{2^{k_1-1}}, \frac{n_2-1}{2^{k_2-1}} \le t < \frac{n_2}{2^{k_2-1}}, \\ 0, \qquad otherwise \end{cases} \tag{11}$$

where n_1 and n_2 are defined similarly to n, k_1 and k_2 are any positive integers, m_1 and m_2 are the orders of Legendre polynomials and $\psi_{n_1,m_1,n_2,m_2}(x,t)$ forms a basis for $L^2([0,1)\times[0,1))$. A function $\mu(x,t)$ defined over $[0,1)\times[0,1)$ can be expanded by Legendre wavelets as follows

$$\mu(x,t) = \sum_{n_1=1}^{\infty}\sum_{m_1=0}^{\infty}\sum_{n_2=1}^{\infty}\sum_{m_2=0}^{\infty} d_{n_1,m_1,n_2,m_2}\psi_{n_1,m_1}(x)\psi_{n_2,m_2}(t), \tag{12}$$

where $\Psi(x)$ and $\Psi(t)$ are $2^{k_1-1}M_1 \times 1$ and $2^{k_2-1}M_2 \times 1$ matrices and are defined in Eq. (10). If the infinite series in Eq. (12) is truncated, then it can be written as

$$\mu(x,t) \cong \sum_{n_1=1}^{2^{k_1-1}}\sum_{m_1=0}^{M_1-1}\sum_{n_2=1}^{2^{k_2-1}}\sum_{m_2=0}^{M_2-1} d_{n_1,m_1,n_2,m_2}\psi_{n_1,m_1}(x)\psi_{n_2,m_2}(t) = \Psi(x)U\Psi(t). \tag{13}$$

Moreover, U is $2^{k_1-1}M_1\times 2^{k_2-1}M_2$ matrix and its elements can be calculated from the formula

$$d_{n_1,m_1,n_2,m_2} = \int_0^1\int_0^1 \mu(x,t)\psi_{n_1,m_1}(x)\psi_{n_2,m_2}(t)\mathrm{d}x\mathrm{d}t,$$

where $n_1 = 1,\ldots,2^{k_1-1}, m_1 = 0,\ldots,M_1-1, n_2 = 1,\ldots,2^{k_2-1}, m_2 = 0,\ldots,M_2-1$.

We investigate the convergence analysis and error analysis of the proposed method in the following theorems:

Theorem 1. *Let* $\left(\frac{\partial^\alpha\mu(x,t)}{\partial t^\alpha}\right)_{k_1,M_1,k_2,M_2}$ *be the approximation of* $\frac{\partial^\alpha\mu(x,t)}{\partial t^\alpha}$ $(1 < \alpha \le 2)$ *and suppose that the mixed fourth partial derivative of* $\frac{\partial^\alpha\mu(x,t)}{\partial t^\alpha}$ *is* *bounded by a constant* B_1, *i.e.* $\left|\frac{\partial^{\alpha+4}\mu(x,t)}{\partial x^2\partial t^{\alpha+2}}\right| < B_1$ *then we have the following upper bound error*

$$\Big|\Big|\frac{\partial^{\alpha}\mu(x,t)}{\partial t^{\alpha}} - \Big(\frac{\partial^{\alpha}\mu(x,t)}{\partial t^{\alpha}}\Big)_{k_1,M_1,k_2,M_2}\Big|\Big|_{L^2}$$
$$< B_1\frac{1}{2^4}\frac{1}{2^{2k_1+2k_2}}\left(\left(\frac{\Gamma'(M_1-3/2)}{\Gamma(M_1-3/2)}\right)'''\right)^{\frac{1}{2}}\left(\left(\frac{\Gamma'(M_2-3/2)}{\Gamma(M_2-3/2)}\right)'''\right)^{\frac{1}{2}},$$

where $\Gamma'(x)/\Gamma(x)$ *is digamma function.*

Proof.

$$\frac{\partial^{\alpha}\mu(x,t)}{\partial t^{\alpha}} = \sum_{n_1=1}^{\infty}\sum_{m_1=0}^{\infty}\sum_{n_2=1}^{\infty}\sum_{m_2=0}^{\infty} d_{n_1,m_1,n_2,m_2}\psi_{n_1,m_1}(x)\psi_{n_2,m_2}(t),$$

and

$$\Big(\frac{\partial^{\alpha}\mu(x,t)}{\partial t^{\alpha}}\Big)_{k_1,M_1,k_2,M_2}$$
$$= \sum_{n_1=1}^{2^{k_1-1}}\sum_{m_1=0}^{M_1-1}\sum_{n_2=1}^{2^{k_2-1}}\sum_{m_2=0}^{M_2-1} d_{n_1,m_1,n_2,m_2}\psi_{n_1,m_1}(x)\psi_{n_2,m_2}(t).$$

Then we have

$$\Big|\Big|\frac{\partial^{\alpha}\mu(x,t)}{\partial t^{\alpha}} - \Big(\frac{\partial^{\alpha}\mu(x,t)}{\partial t^{\alpha}}\Big)_{k_1,M_1,k_2,M_2}\Big|\Big|^2_{L^2}$$
$$= \int_0^1\int_0^1\left(\sum_{n_1=2^{k_1-1}+1}^{\infty}\sum_{m_1=M_1}^{\infty}\sum_{n_2=2^{k_2-1}+1}^{\infty}\sum_{m_2=M_2}^{\infty} d_{n_1,m_1,n_2,m_2}\psi_{n_1,m_1}(x)\psi_{n_2,m_2}(t)\right)^2 \mathrm{d}x\mathrm{d}t$$
$$= \sum_{n_1=2^{k_1-1}+1}^{\infty}\sum_{m_1=M_1}^{\infty}\sum_{n_2=2^{k_2-1}+1}^{\infty}\sum_{m_2=M_2}^{\infty} d^2_{n_1,m_1,n_2,m_2},$$

where $d_{n_1,m_1,n_2,m_2} = \int_0^1\int_0^1 \frac{\partial^{\alpha}\mu(x,t)}{\partial t^{\alpha}}\psi_{n_1,m_1}(x)\psi_{n_2,m_2}(t)\mathrm{d}x\mathrm{d}t$. Let $A_{n_1,m_1}(t) = \int_0^1 \frac{\partial^{\alpha}\mu(x,t)}{\partial t^{\alpha}}\psi_{n_1,m_1}(x)\mathrm{d}x$ Then

$$A_{n_1,m_1}(t) = \int_{\frac{n_1-1}{2^{k_1-1}}}^{\frac{n_1}{2^{k_1-1}}} \frac{\partial^{\alpha}\mu(x,t)}{\partial t^{\alpha}}\Big(\frac{2m_1+1}{2}\Big)^{\frac{1}{2}} 2^{\frac{k_1}{2}} L_{m_1}(2^{k_1}x-2n_1+1)\mathrm{d}x.$$

Now by change of variable $2^{k_1}x-2n_1+1 = s$ and $\mathrm{d}x = \frac{1}{2^{k_1}}\mathrm{d}s$, we obtain

$$A_{n_1,m_1}(t) = \int_{-1}^{1}\frac{\partial^{\alpha}\mu(\frac{s+2n_1-1}{2^{k_1}},t)}{\partial t^{\alpha}}\left(\frac{2m_1+1}{2}\right)^{\frac{1}{2}} 2^{\frac{k_1}{2}} L_{m_1}(s)\frac{1}{2^{k_1}}\mathrm{d}x\mathrm{d}t$$
$$= \left(\frac{1}{2^{k_1+1}(2m_1+1)}\right)^{\frac{1}{2}}\int_{-1}^{1}\frac{\partial^{\alpha}\mu(\frac{s+2n_1-1}{2^{k_1}},t)}{\partial t^{\alpha}} d\left(L_{m_1+1}(s) - L_{m_1-1}(s)\right)$$
$$= -\left(\frac{1}{2^{3k_1+1}(2m_1+1)}\right)^{\frac{1}{2}}$$
$$\int_{-1}^{1}\frac{\partial^{\alpha+1}\mu(\frac{s+2n_1-1}{2^{k_1}},t)}{\partial s\partial t^{\alpha}}\left(L_{m_1+1}(s) - L_{m_1-1}(s)\right)\mathrm{d}s$$
$$= -\left(\frac{1}{2^{3k_1+1}(2m_1+1)}\right)^{\frac{1}{2}}\int_{-1}^{1}\frac{\partial^{\alpha+1}\mu(\frac{s+2n_1-1}{2^{k_1}},t)}{\partial s\partial t^{\alpha}}$$

$$d\left(\frac{L_{m_1+2}(s)-L_{m_1}(s)}{2m_1+3}-\frac{L_{m_1}(s)-L_{m_1-2}(s)}{2m_1-1}\right)$$
$$=\left(\frac{1}{2^{5k_1+1}(2m_1+1)}\right)^{\frac{1}{2}}\int_{-1}^{1}\frac{\partial^{\alpha+2}\mu(\frac{s+2n_1-1}{2^{k_1}},t)}{\partial s^2\partial t^\alpha}$$
$$\left(\frac{L_{m_1+2}(s)-L_{m_1}(s)}{2m_1+3}-\frac{L_{m_1}(s)-L_{m_1-2}(s)}{2m_1-1}\right)\mathrm{d}s.$$

So

$$d_{n_1,m_1,n_2,m_2}=\left(\frac{1}{2^{5k_1+1}(2m_1+1)}\right)^{\frac{1}{2}}\int_0^1\psi_{n_2,m_2}(t)\int_{-1}^{1}\frac{\partial^{\alpha+2}\mu(\frac{s+2n_1-1}{2^{k_1}},t)}{\partial s^2\partial t^\alpha}\cdot$$
$$\left(\frac{L_{m_1+2}(s)-L_{m_1}(s)}{2m_1+3}-\frac{L_{m_1}(s)-L_{m_1-2}(s)}{2m_1-1}\right)dsdt.$$

Now let $\tau_{n_1,m_1}(s)=(2m_1-1)L_{m_1+2}(s)-2(2m_1+1)L_{m_1}(s)+(2m_1+3)L_{m_1-2}(s)$, then we have

$$d_{n_1,m_1,n_2,m_2}=\left(\frac{1}{2^{5k_1+1}(2m_1+1)}\right)^{\frac{1}{2}}\frac{1}{(2m_1-1)(2m_1+3)}\int_0^1\psi_{n_2,m_2}(t)$$
$$\int_{-1}^{1}\frac{\partial^{\alpha+2}\mu(\frac{s+2n_1-1}{2^{k_1}},t)}{\partial s^2\partial t^\alpha}\tau_{n_1,m_1}(s)dsdt.$$

Putting $2^{k_2}t-2n_2+1=r$ and doing the same operations as above, it gives

$$d_{n_1,m_1,n_2,m_2}$$
$$=\left(\frac{1}{2^{5k_1+1}(2m_1+1)}\right)^{\frac{1}{2}}\frac{1}{(2m_1-1)(2m_1+3)}\left(\frac{1}{2^{5k_2+1}(2m_2+1)}\right)^{\frac{1}{2}}$$
$$\frac{1}{(2m_2-1)(2m_2+3)}\cdot$$
$$\int_{-1}^{1}\int_{-1}^{1}\frac{\partial^{\alpha+4}\mu(\frac{s+2n_1-1}{2^{k_1}},\frac{r+2n_2-1}{2^{k_2}})}{\partial s^2\partial r^{\alpha+2}}\tau_{n_1,m_1}(s)\tau_{n_2,m_2}(r)dsdr,$$

where $\tau_{n_2,m_2}(r)=(2m_2-1)L_{m_2+2}(r)-2(2m_2+1)L_{m_2}(r)+(2m_2+3)L_{m_2-2}(r)$. Due to the orthogonality of Legendre polynomial, we have ([29])

$$\left(\int_{-1}^{1}\tau_{n_1,m_1}(s)\mathrm{d}s\right)^2<\frac{24(2m_1+3)^2}{2m_1-3},$$
$$\left(\int_{-1}^{1}\tau_{n_2,m_2}(r)\mathrm{d}r\right)^2<\frac{24(2m_2+3)^2}{2m_2-3}.$$

Thus

$$d^2_{n_1,m_1,n_2,m_2}<B_1^2\frac{12}{2^{5k_1(2m_1-3)^4}}\frac{12}{2^{5k_2(2m_2-3)^4}},$$

which implies

$$\sum_{n_1=2^{k_1-1}+1}^{\infty}\sum_{m_1=M_1}^{\infty}\sum_{n_2=2^{k_2-1}+1}^{\infty}\sum_{m_2=M_2}^{\infty}d^2_{n_1,m_1,n_2,m_2}$$
$$<\sum_{n_1=2^{k_1-1}+1}^{\infty}\sum_{m_1=M_1}^{\infty}\sum_{n_2=2^{k_2-1}+1}^{\infty}\sum_{m_2=M_2}^{\infty}B_1^2\frac{12}{2^{5k_1(2m_1-3)^4}}\frac{12}{2^{5k_2(2m_2-3)^4}}$$

$$
\begin{aligned}
&< 12B_1^2\Big(\sum_{n_1=2^{k_1-1}+1}^{\infty} \frac{1}{(2n_1)^5} \cdot \frac{1}{6} \cdot \frac{1}{2^4} \sum_{m_1=M_1}^{\infty} \frac{6}{(m_1-3/2)^4} \Big) \\
&\quad \cdot 12\Big(\sum_{n_2=2^{k_2-1}+1}^{\infty} \frac{1}{(2n_2)^5} \cdot \frac{1}{6} \cdot \frac{1}{2^4} \sum_{m_2=M_2}^{\infty} \frac{6}{(m_2-3/2)^4} \Big) \\
&< 12B_1^2 \frac{1}{2^5} \cdot \frac{1}{6} \cdot \frac{1}{2^4} \cdot \frac{1}{(2^{k_1-1}+1)^4} \cdot \Big(\frac{\Gamma'(M_1-3/2)}{\Gamma(M_1-3/2)} \Big)''' \\
&\quad \cdot 12 \frac{1}{2^5} \cdot \frac{1}{6} \cdot \frac{1}{2^4} \cdot \frac{1}{(2^{k_2-1}+1)^4} \cdot \Big(\frac{\Gamma'(M_2-3/2)}{\Gamma(M_2-3/2)} \Big)''' \\
&< \frac{B_1^2}{2^8} \frac{1}{2^{4(k_1+k_2)}} \Big(\frac{\Gamma'(M_1-3/2)}{\Gamma(M_1-3/2)} \Big)''' \Big(\frac{\Gamma'(M_2-3/2)}{\Gamma(M_2-3/2)} \Big)''' .
\end{aligned}
$$

The proof is completed. □

Remark 1. In the calculation, we usually take $k = k_1 = k_2$ and $M = M_1 = M_2$. Therefore, the inequality in the Theorem 1 can be rewritten as

$$\Big\| \frac{\partial^\alpha \mu(x,t)}{\partial t^\alpha} - \Big(\frac{\partial^\alpha \mu(x,t)}{\partial t^\alpha} \Big)_{k,M} \Big\|_{L^2} < \frac{B_1}{2^4} \frac{1}{2^{4k}} \Big(\frac{\Gamma'(M-3/2)}{\Gamma(M-3/2)} \Big)''' .$$

From this theorem, we can see clearly that $\Big\| \frac{\partial^\alpha \mu(x,t)}{\partial t^\alpha} - \Big(\frac{\partial^\alpha \mu(x,t)}{\partial t^\alpha} \Big)_{k,M} \Big\|_{L^2} \longrightarrow 0$ when M is fixed and $k \longrightarrow \infty$. Similarly we have the following lemmas:

Lemma 1. *Let* $\frac{\partial^{\alpha-1} \mu(x,t)}{\partial t^{\alpha-1}}$ *be a continuous function defined on* $[0,1) \times [0,1)$ *and* $\Big(\frac{\partial^{\alpha-1} \mu(x,t)}{\partial t^{\alpha-1}} \Big)_{k,M}$ *be the approximation of* $\frac{\partial^{\alpha-1} \mu(x,t)}{\partial t^{\alpha-1}}$*. Suppose that the mixed fourth partial derivative of* $\frac{\partial^{\alpha-1} \mu(x,t)}{\partial t^{\alpha-1}}$ *is bounded by a constant* B_2*,i.e.* $\big| \frac{\partial^{\alpha+3} \mu(x,t)}{\partial x^2 \partial t^{\alpha+1}} \big| < B_2$*; then we have the following bound of error:*

$$\Big\| \frac{\partial^{\alpha-1} \mu(x,t)}{\partial t^{\alpha-1}} - \Big(\frac{\partial^{\alpha-1} \mu(x,t)}{\partial t^{\alpha-1}} \Big)_{k,M} \Big\|_{L^2} < \frac{B_2}{2^4} \frac{1}{2^{4k}} \Big(\frac{\Gamma'(M-3/2)}{\Gamma(M-3/2)} \Big)''' .$$

Lemma 2. *Let* $\frac{\partial^2 \mu(x,t)}{\partial x^2}$ *be a continuous function defined on* $[0,1) \times [0,1)$ *and* $\Big(\frac{\partial^2 \mu(x,t)}{\partial x^2} \Big)_{k,M}$ *be the approximation of* $\frac{\partial^2 \mu(x,t)}{\partial x^2}$*. Suppose that the mixed fourth partial derivative of* $\frac{\partial^2 \mu(x,t)}{\partial x^2}$ *is bounded by a constant* B_3*,i.e.* $\big| \frac{\partial^6 \mu(x,t)}{\partial x^4 \partial t^2} \big| < B_3$*; then we have the following bound of error:*

$$\Big\| \frac{\partial^2 \mu(x,t)}{\partial x^2} - \Big(\frac{\partial^2 \mu(x,t)}{\partial x^2} \Big)_{k,M} \Big\|_{L^2} < \frac{B_3}{2^4} \frac{1}{2^{4k}} \Big(\frac{\Gamma'(M-3/2)}{\Gamma(M-3/2)} \Big)''' .$$

Remark 2. Using the conditions of Theorem 1 and Lemmas 1-2, we can conclude that $f(x,t)$ has mixed fourth partial derivative and can be bounded by a constant B_4, i.e. $\big| \frac{\partial^4 f(x,t)}{\partial x^2 \partial t^2} \big| < B_4$. Let $\Big(f(x,t) \Big)_{k,M}$ be the approximation of $f(x,t)$; then we have the following bound of error:

$$\Big\| f(x,t) - \Big(f(x,t) \Big)_{k,M} \Big\|_{L^2} < \frac{B_4}{2^4} \frac{1}{2^{4k}} \Big(\frac{\Gamma'(M-3/2)}{\Gamma(M-3/2)} \Big)''' .$$

Theorem 2. *Let* $\left(\frac{\partial^\alpha \mu(x,t)}{\partial t^\alpha}\right)_{k,M}$, $\left(\frac{\partial^{\alpha-1} \mu(x,t)}{\partial t^{\alpha-1}}\right)_{k,M}$, $\left(\frac{\partial^2 \mu(x,t)}{\partial x^2}\right)_{k,M}$, $\Big(f(x,t)\Big)_{k,M}$ *be the approximations of* $\frac{\partial^\alpha \mu(x,t)}{\partial t^\alpha}$, $\frac{\partial^{\alpha-1} \mu(x,t)}{\partial t^{\alpha-1}}$, $\frac{\partial^2 \mu(x,t)}{\partial x^2}$, $f(x,t)$ $(1 < \alpha \leq 2)$ *and assume that the mixed fourth partial derivative of* $\frac{\partial^\alpha \mu(x,t)}{\partial t^\alpha}$, $\frac{\partial^{\alpha-1} \mu(x,t)}{\partial t^{\alpha-1}}$ *are bounded by constants* B_1,B_2,B_3 *and* B_4*; then for Eq. (1), we have the following upper bound error:*

$$\left\|\Big|\mu(x,t) - (\mu(x,t))_{k,M}\Big|\right\|_{L^2} < \left(\left|\frac{\lambda_3}{\lambda_2}\right| + \left|\frac{\lambda_1}{\lambda_2}\right| + \left|\frac{2}{\lambda_2}\right|\right)\left(\frac{B}{2^4}\frac{1}{2^{4k}}\left(\frac{\Gamma'(M-3/2)}{\Gamma(M-3/2)}\right)'''\right),$$

where $\Gamma'(x)/\Gamma(x)$ *is digamma function,* $B = max\{B_1, B_2, B_3, B_4\}$.

Proof. The problem under consideration is as follows:

$$\frac{\partial^\alpha \mu(x,t)}{\partial t^\alpha} + \lambda_1 \frac{\partial^{\alpha-1} \mu(x,t)}{\partial t^{\alpha-1}} + \lambda_2 \mu(x,t) = \lambda_3 \frac{\partial^2 \mu(x,t)}{\partial x^2} + f(x,t), \tag{14}$$

and approximation solution of Eq. (14) is as follows

$$\begin{aligned} &\left(\frac{\partial^\alpha \mu(x,t)}{\partial t^\alpha}\right)_{k,M} + \lambda_1\left(\frac{\partial^{\alpha-1} \mu(x,t)}{\partial t^{\alpha-1}}\right)_{k,M} + \lambda_2\Big(\mu(x,t)\Big)_{k,M} \\ &\quad = \lambda_3\left(\frac{\partial^2 \mu(x,t)}{\partial x^2}\right)_{k,M} + \Big(f(x,t)\Big)_{k,M}. \end{aligned} \tag{15}$$

Let $(e(x,t))_{k,M} = \mu(x,t) - (\mu(x,t))_{k,M}$ be the bounded error function, where $\mu(x,t)$ is the exact solution of Eq. (14). Subtracting Eq. (15) from Eq. (14), we get the following error equation

$$\begin{aligned} (\mu(x,t) - (\mu(x,t))_{k,M}) = &\frac{\lambda_3}{\lambda_2}\left(\frac{\partial^2 \mu(x,t)}{\partial x^2} - \left(\frac{\partial^2 \mu(x,t)}{\partial x^2}\right)_{k,M}\right) \\ &- \frac{1}{\lambda_2}\left(\frac{\partial^\alpha \mu(x,t)}{\partial t^\alpha} - \left(\frac{\partial^\alpha \mu(x,t)}{\partial t^\alpha}\right)_{k,M}\right) \\ &- \frac{\lambda_1}{\lambda_2}\left(\frac{\partial^{\alpha-1} \mu(x,t)}{\partial t^{\alpha-1}} - \left(\frac{\partial^{\alpha-1} \mu(x,t)}{\partial t^{\alpha-1}}\right)_{k,M}\right) \\ &+ \frac{1}{\lambda_2}(f(x,t) - (f(x,t))_{k,M}). \end{aligned} \tag{16}$$

Taking L^2-norm on both sides of Eq. (16) and using the results of Theorem 1, Lemmas 1–2 and Remark 2, we obtain

$$\begin{aligned} \|e(x,t)\|_{k,M} = &\left\|\frac{\lambda_3}{\lambda_2}\left(\frac{\partial^2 \mu(x,t)}{\partial x^2} - \left(\frac{\partial^2 \mu(x,t)}{\partial x^2}\right)_{k,M}\right)\right. \\ &- \frac{1}{\lambda_2}\left(\frac{\partial^\alpha \mu(x,t)}{\partial t^\alpha} - \left(\frac{\partial^\alpha \mu(x,t)}{\partial t^\alpha}\right)_{k,M}\right) \\ &- \frac{\lambda_1}{\lambda_2}\left(\frac{\partial^{\alpha-1} \mu(x,t)}{\partial t^{\alpha-1}} - \left(\frac{\partial^{\alpha-1} \mu(x,t)}{\partial t^{\alpha-1}}\right)_{k,M}\right) \\ &\left.+ \frac{1}{\lambda_2}(f(x,t) - (f(x,t)_{k,M})\right\| \end{aligned}$$

$$\leq \left|\frac{\lambda_3}{\lambda_2}\right| \left\| \left(\frac{\partial^2 \mu(x,t)}{\partial x^2} - \left(\frac{\partial^2 \mu(x,t)}{\partial x^2} \right)_{k,M} \right) \right\|$$
$$+ \left|\frac{1}{\lambda_2}\right| \left\| \left(\frac{\partial^\alpha \mu(x,t)}{\partial t^\alpha} - \left(\frac{\partial^\alpha \mu(x,t)}{\partial t^\alpha} \right)_{k,M} \right) \right\|$$
$$+ \left|\frac{\lambda_1}{\lambda_2}\right| \left\| \left(\frac{\partial^{\alpha-1} \mu(x,t)}{\partial t^{\alpha-1}} - \left(\frac{\partial^{\alpha-1} \mu(x,t)}{\partial t^{\alpha-1}} \right)_{k,M} \right) \right\|$$
$$+ \left|\frac{1}{\lambda_2}\right| \left\| (f(x,t) - (f(x,t))_{k,M}) \right\|$$
$$\leq \left(\left|\frac{\lambda_3}{\lambda_2}\right| + \left|\frac{\lambda_1}{\lambda_2}\right| + \left|\frac{2}{\lambda_2}\right| \right) \left(\frac{B}{2^4} \frac{1}{2^{4k}} \left(\frac{\Gamma'(M-3/2)}{\Gamma(M-3/2)} \right)''' \right).$$

The proof is completed. □

4. The Fractional Integral of a Single Legendre Wavelet

In this section, we will derive the fractional integral formula of a single Legendre wavelet in the Riemann–Liouville sense.

Theorem 3. *The fractional integral of a single Legendre wavelet function defined on the interval* $[0,1]$ *with compact support* $\left[\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}}\right]$ *is given by*

$$I^\alpha \psi_{n,m}(x) = \begin{cases} 0, & x < \frac{n-1}{2^{k-1}}, \\ \frac{\sqrt{m+\frac{1}{2}}}{\Gamma(\alpha)} 2^{\frac{k}{2}} \sum_{r=0}^{m} \sum_{i=r}^{m} \sum_{j=0}^{r} S_{i,i-r}^{m,n,k} \frac{(-1)^j}{j+\alpha} C_r^j x^{r-j} (x - \frac{n-1}{2^{k-1}})^{j+\alpha}, & \frac{n-1}{2^{k-1}} \leq x \leq \frac{n}{2^{k-1}}, \\ \frac{\sqrt{m+\frac{1}{2}}}{\Gamma(\alpha)} 2^{\frac{k}{2}} \sum_{r=0}^{m} \sum_{i=r}^{m} \sum_{j=0}^{r} S_{i,i-r}^{m,n,k} \frac{(-1)^j}{j+\alpha} C_r^j x^{r-j} \\ \quad \left[(x - \frac{n-1}{2^{k-1}})^{j+\alpha} - (x - \frac{n}{2^{k-1}})^{j+\alpha}\right], & x > \frac{n}{2^{k-1}}, \end{cases}$$

where $S_{i,i-r}^{m,n,k} = (-1)^{m-r} 2^{r(k-1)} (n-1)^{i-r} \frac{(m+i)!}{(m-i)!i!r!(i-r)!}$, $C_r^j = \frac{r!}{j!(r-j)!}$.

Proof. According to the analytical form of the shifted Legendre polynomials, we have

$$\begin{aligned} L_m^*(2^{k-1}x - n + 1) &= \sum_{i=0}^{m} (-1)^{m-i} \frac{(m+i)!}{(m-i)!(i!)^2} (2^{k-1}x - n + 1)^i \\ &= \sum_{i=0}^{m} (-1)^{m-i} \frac{(m+i)!}{(m-i)!(i!)^2} 2^{i(k-1)} \sum_{r=0}^{i} C_i^r x^{i-r} (-\frac{n-1}{2^{k-1}})^r \\ &= \sum_{i=0}^{m} \sum_{r=0}^{i} (-1)^{m-i+r} 2^{(i-r)(k-1)} (n-1)^r \frac{(m+i)!}{(m-i)!i!} \frac{1}{r!(i-r)!} x^{i-r}. \end{aligned}$$

By interchanging the summation and substituting $i-r$ with r, $L_m^*(2^{k-1}x - n + 1)$ can be written as

$$L_m^*(2^{k-1}x - n + 1) = \sum_{r=0}^{m} \sum_{i=r}^{m} (-1)^{m-r} 2^{r(k-1)} (n-1)^{i-r} \frac{(m+i)!}{(m-i)!i!r!(i-r)!} x^r. \tag{17}$$

Let $S_{i,i-r}^{m,n,k} = (-1)^{m-r} 2^{r(k-1)} (n-1)^{i-r} \frac{(m+i)!}{(m-i)!i!r!(i-r)!}$; then

$$L_m^*(2^{k-1}x - n + 1) = \sum_{r=0}^{m} \sum_{i=r}^{m} S_{i,i-r}^{m,n,k} x^r. \tag{18}$$

Therefore,

$$\psi_{n,m}(x) = \begin{cases} \sqrt{m+\frac{1}{2}} 2^{\frac{k}{2}} \sum\limits_{r=0}^{m} \sum\limits_{i=r}^{m} S_{i,i-r}^{m,n,k} x^r, & \frac{n-1}{2^{k-1}} \leq x < \frac{n}{2^{k-1}}. \\ 0, & \text{otherwise.} \end{cases} \tag{19}$$

Next, we calculate the integrals $I_1 = \frac{1}{\Gamma(\alpha)} \int_{\frac{n-1}{2^{k-1}}}^{x} (x-t)^{\alpha-1} t^r \mathrm{d}t$ and $I_2 = \frac{1}{\Gamma(\alpha)} \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} (x-t)^{\alpha-1} t^r \mathrm{d}t$. Let $u = x - t$, then

$$\begin{aligned} I_1 &= \frac{1}{\Gamma(\alpha)} \int_{\frac{n-1}{2^{k-1}}}^{x} (x-t)^{\alpha-1} t^r \mathrm{d}t \\ &= \frac{1}{\Gamma(\alpha)} \int_{0}^{x-\frac{n-1}{2^{k-1}}} u^{\alpha-1} (x-u)^r \mathrm{d}u \\ &= \frac{1}{\Gamma(\alpha)} \int_{0}^{x-\frac{n-1}{2^{k-1}}} u^{\alpha-1} \sum_{j=0}^{r} C_r^j x^{r-j} (-u)^j \mathrm{d}u \\ &= \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{r} (-1)^j C_r^j x^{r-j} \int_{0}^{x-\frac{n-1}{2^{k-1}}} u^{j+\alpha-1} \mathrm{d}u \\ &= \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{r} \frac{(-1)^j}{j+\alpha} C_r^j x^{r-j} (x - \frac{n-1}{2^{k-1}})^{j+\alpha}. \end{aligned}$$

Similarly, we get

$$\begin{aligned} I_2 &= \frac{1}{\Gamma(\alpha)} \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} (x-t)^{\alpha-1} t^r \mathrm{d}t \\ &= \frac{1}{\Gamma(\alpha)} \int_{x-\frac{n}{2^{k-1}}}^{x-\frac{n-1}{2^{k-1}}} u^{\alpha-1} (x-u)^r \mathrm{d}u \\ &= \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{r} \frac{(-1)^j}{j+\alpha} C_r^j x^{r-j} \big((x - \frac{n-1}{2^{k-1}})^{j+\alpha} - (x - \frac{n}{2^{k-1}})^{j+\alpha} \big). \end{aligned}$$

Applying the Riemann–Liouville fractional integral of order α with respect to x on $\psi_{n,m}(x)$, we obtain

$$\begin{aligned} & I^\alpha \psi_{n,m}(x) \\ &= \begin{cases} 0, & x < \frac{n-1}{2^{k-1}}, \\ \frac{1}{\Gamma(\alpha)} \int_{\frac{n-1}{2^{k-1}}}^{x} (x-t)^{\alpha-1} \psi_{n,m}(t) \mathrm{d}t, & \frac{n-1}{2^{k-1}} \leq x \leq \frac{n}{2^{k-1}}, \\ \frac{1}{\Gamma(\alpha)} \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} (x-t)^{\alpha-1} \psi_{n,m}(t) \mathrm{d}t, & x > \frac{n}{2^{k-1}}, \end{cases} \end{aligned}$$

$$
= \begin{cases} 0, & x < \frac{n-1}{2^{k-1}}, \\ \frac{\sqrt{m+\frac{1}{2}}}{\Gamma(\alpha)} 2^{\frac{k}{2}} \sum\limits_{r=0}^{m} \sum\limits_{i=r}^{m} S_{i,i-r}^{m,n,k} \int_{\frac{n-1}{2^{k-1}}}^{x} (x-t)^{\alpha-1} t^r \mathrm{d}t, & \frac{n-1}{2^{k-1}} \le x \le \frac{n}{2^{k-1}}, \\ \frac{\sqrt{m+\frac{1}{2}}}{\Gamma(\alpha)} 2^{\frac{k}{2}} \sum\limits_{r=0}^{m} \sum\limits_{i=r}^{m} S_{i,i-r}^{m,n,k} \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} (x-t)^{\alpha-1} t^r \mathrm{d}t, & x > \frac{n}{2^{k-1}}, \end{cases}
$$

$$
= \begin{cases} 0, & x < \frac{n-1}{2^{k-1}}, \\ \frac{\sqrt{m+\frac{1}{2}}}{\Gamma(\alpha)} 2^{\frac{k}{2}} \sum\limits_{r=0}^{m} \sum\limits_{i=r}^{m} S_{i,i-r}^{m,n,k} \sum\limits_{j=0}^{r} \frac{(-1)^j}{j+\alpha} C_r^j x^{r-j} (x - \frac{n-1}{2^{k-1}})^{j+\alpha}, & \frac{n-1}{2^{k-1}} \le x \le \frac{n}{2^{k-1}}, \\ \frac{\sqrt{m+\frac{1}{2}}}{\Gamma(\alpha)} 2^{\frac{k}{2}} \sum\limits_{r=0}^{m} \sum\limits_{i=r}^{m} S_{i,i-r}^{m,n,k} \sum\limits_{j=0}^{r} \frac{(-1)^j}{j+\alpha} C_r^j x^{r-j} & \\ \quad \left[(x - \frac{n-1}{2^{k-1}})^{j+\alpha} - (x - \frac{n}{2^{k-1}})^{j+\alpha}\right], & x > \frac{n}{2^{k-1}}. \end{cases}
$$

Thus, we have

$$
I^{\alpha}\psi_{n,m}(x) = \begin{cases} 0, & x < \frac{n-1}{2^{k-1}}, \\ \frac{\sqrt{m+\frac{1}{2}}}{\Gamma(\alpha)} 2^{\frac{k}{2}} \sum\limits_{r=0}^{m} \sum\limits_{i=r}^{m} \sum\limits_{j=0}^{r} S_{i,i-r}^{m,n,k} \frac{(-1)^j}{j+\alpha} & \\ \quad C_r^j x^{r-j} (x - \frac{n-1}{2^{k-1}})^{j+\alpha}, & \frac{n-1}{2^{k-1}} \le x \le \frac{n}{2^{k-1}}, \\ \frac{\sqrt{m+\frac{1}{2}}}{\Gamma(\alpha)} 2^{\frac{k}{2}} \sum\limits_{r=0}^{m} \sum\limits_{i=r}^{m} \sum\limits_{j=0}^{r} S_{i,i-r}^{m,n,k} \frac{(-1)^j}{j+\alpha} C_r^j x^{r-j} & \\ \quad \left[(x - \frac{n-1}{2^{k-1}})^{j+\alpha} - (x - \frac{n}{2^{k-1}})^{j+\alpha}\right], & x > \frac{n}{2^{k-1}}. \end{cases}
$$

The proof is complete. □

For example, in the case of $k = 2, M = 3, x = 0.55, \alpha = 1.75$, we obtain

$$
I^{\alpha}\Psi_{6\times 1}(x) = \begin{pmatrix} 0.304214828678896 \\ -0.115630783697839 \\ -0.00725854452471297 \\ 0.00464867728189508 \\ -0.00746616376821649 \\ 0.008247768307552 \end{pmatrix},
$$

where $\Psi_{6\times 1}(x) = \left(\psi_{1,0}\ \psi_{1,1}(x)\ \psi_{1,2}(x)\ \psi_{2,0}(x)\ \psi_{2,1}(x)\ \psi_{2,2}(x)\right)^{\mathrm{T}}$.

5. Description of the proposed method

Consider the time-fractional order telegraph equation with the following form ([31]):

$$
\frac{\partial^{\alpha}\mu(x,t)}{\partial t^{\alpha}} + \lambda_1 \frac{\partial^{\alpha-1}\mu(x,t)}{\partial t^{\alpha-1}} + \lambda_2\mu(x,t) = \lambda_3 \frac{\partial\mu^2(x,t)}{\partial x^2} + f(x,t),
$$
$$
0 < x < 1,\ 0 \le t \le 1,\ 1 < \alpha \le 2, \tag{20}
$$

with initial conditions

$$
\mu(x,0) = f_1(x), \mu_t(x,0) = f_2(x)\ \ 0 \le x \le 1, \tag{21}
$$

and boundary conditions

$$
\mu(0,t) = g_0(t),\ \mu(1,t) = g_1(t),\ 0 < t \le 1, \tag{22}
$$

where $f_1(\cdot), f_2(\cdot), g_0(\cdot), g_1(\cdot)$ are given functions with second-order continuous derivatives in $L^2[0,1)$ and $f(\cdot,\cdot)$ is a given function in $L^2([0,1)\times[0,1))$. To solve this problem, we suppose

$$\frac{\partial^4\mu(x,t)}{\partial x^2\partial t^2} \approx \Psi^T(x)\cdot U\cdot\Psi(t), \tag{23}$$

where $U = \big(u_{i,j}\big)_{2^{k-1}M\times 2^{k-1}M}$ is an unknown matrix which should be determined and $\Psi(\cdot)$ is as in (10). By integrating two times with respect to t on both sides of (23) and together with (4), we have

$$\frac{\partial^2\mu(x,t)}{\partial x^2} \approx f_1''(x) + tf_2''(x) + \Psi^T(x)\cdot U\cdot(I^2\Psi(t)), \tag{24}$$

Also by integrating Eq. (24) two times with respect to x, we obtain

$$\begin{aligned}\mu(x,t) \approx{}& \mu(0,t) + x\frac{\partial\mu(x,t)}{\partial x}\Big|_{x=0} + (f_1(x) - f_1(0) - xf_1^{'}(0))\\ &+ t(f_2(x) - f_2(0) - xf_2^{'}(0))\\ &+ (I^2\Psi(x))^T\cdot U\cdot(I^2\Psi(t)).\end{aligned} \tag{25}$$

Putting $x = 1$ in (25) and considering the boundary conditions (22), we obtain

$$\begin{aligned}\mu(1,t) \approx{}& \mu(0,t) + \frac{\partial\mu(x,t)}{\partial x}\Big|_{x=0} + (f_1(1) - f_1(0) - f_1^{'}(0))\\ &+ t(f_2(1) - f_2(0) - f_2^{'}(0))\\ &+ (I^2\Psi(1))^T\cdot U\cdot(I^2\Psi(t)).\end{aligned} \tag{26}$$

Thus, we have

$$\begin{aligned}\frac{\partial\mu(x,t)}{\partial x}\Big|_{x=0} \approx{}& g_1(t) - g_0(t) - (f_1(1) - f_1(0) - f_1^{'}(0))\\ &- t(f_2(1) - f_2(0) - f_2^{'}(0))\\ &- (I^2\Psi(1))^T\cdot U\cdot(I^2\Psi(t)).\end{aligned} \tag{27}$$

Write

$$\begin{aligned}H(t) ={}& g_1(t) - g_0(t) - (f_1(1) - f_1(0) - f_1^{'}(0)) - t(f_2(1) - f_2(0) - f_2^{'}(0))\\ &- (I^2\Psi(1))^T\cdot U\cdot(I^2\Psi(t)).\end{aligned} \tag{28}$$

So

$$\begin{aligned}\mu(x,t) \approx{}& g_0(t) + xH(t) + (f_1(x) - f_1(0) - xf_1^{'}(0))\\ &+ t(f_2(x) - f_2(0) - xf_2^{'}(0))\\ &+ (I^2\Psi(x))^T\cdot U\cdot(I^2\Psi(t)).\end{aligned} \tag{29}$$

Now by fractional differentiation of order α and order $(\alpha-1)$ of (29) with respect to t, we get

$$\begin{aligned}\frac{\partial\mu^\alpha(x,t)}{\partial t^\alpha} \approx{}& D^\alpha g_0(t) + xD^\alpha H(t)\\ &+ \big(I^2\Psi(x)\big)^T\cdot U\cdot\big(I^{2-\alpha}\Psi(t)\big),\end{aligned} \tag{30}$$

$$\begin{aligned}\frac{\partial\mu^{\alpha-1}(x,t)}{\partial t^{\alpha-1}} \approx\; & D^{\alpha-1}g_0(t)+xD^{\alpha-1}H(t)+(f_2(x)-f_2(0)\\ & -xf_2^{'}(0))\frac{\Gamma(2)}{\Gamma(3-\alpha)}t^{2-\alpha}\\ & +\left(I^2\Psi(x)\right)^T\cdot U\cdot\left(I^{3-\alpha}\Psi(t)\right),\end{aligned} \tag{31}$$

where

$$D^{\alpha}H(t)=D^{\alpha}g_1(t)-D^{\alpha}g_0(t)-\left(I^2\Psi(1)\right)^T\cdot U\cdot\left(I^{2-\alpha}\Psi(t)\right), \tag{32}$$

$$\begin{aligned}D^{\alpha-1}H(t)=\;& D^{\alpha-1}g_1(t)-D^{\alpha-1}g_0(t)\\ & -(f_2(1)-f_2(0)-f_2^{'}(0))\frac{\Gamma(2)}{\Gamma(3-\alpha)}t^{2-\alpha}\\ & -\left(I^2\Psi(1)\right)^T\cdot U\cdot\left(I^{3-\alpha}\Psi(t)\right).\end{aligned} \tag{33}$$

Now by substituting (24), (29), (30) and (31) into Eq. (20), replacing $\approx$ by $=$, and taking collocation points $x_i=\frac{2i-1}{2^kM}, t_j=\frac{2j-1}{2^kM}$, $i,j=1,2,\cdots,2^{k-1}M$, we obtain the following linear system of algebraic equations:

$$\begin{aligned}& D^{\alpha}g_0(t_j)+x_iD^{\alpha}H(t_j)+\left(I^2\Psi(x_i)\right)^T\cdot U\cdot\left(I^{2-\alpha}\Psi(t_j)\right)\\ & +\lambda_1\Big(D^{\alpha-1}g_0(t_j)+x_iD^{\alpha-1}H(t_j)\\ & +(f_2(x_i)-f_2(0)-x_if_2^{'}(0))\frac{\Gamma(2)}{\Gamma(3-\alpha)}t_j^{2-\alpha}+\left(I^2\Psi(x_i)\right)^T\\ & \cdot U\cdot\left(I^{3-\alpha}\Psi(t_j)\right)\Big)+\lambda_2\Big(g_0(t_j)+x_iH(t_j)\\ & +(f_1(x_i)-f_1(0)-x_if_1^{'}(0))+t_j(f_2(x_i)-f_2(0)-x_if_2^{'}(0))\\ & +(I^2\Psi(x_i))^T\cdot U\cdot(I^2\Psi(t_j))\Big)\\ & -\lambda_3\Big(f_1^{''}(x_i)+t_jf_2^{''}(x_i)+\Psi^T(x_i)\cdot U\cdot I^2\Psi(t_j)\Big)=f(x_i,t_j),\end{aligned}$$

$i,j=1,2,\ldots,2^{k-1}M$. By solving this system to determine the unknown matrix U, we can achieve an approximate solution for the problem by substituting U into (29).

6. Numerical examples

In this section, we give some numerical examples to demonstrate the efficiency and reliability of the proposed method.

Example 1. Consider the time-fractional telegraph equations of order α $(1<\alpha\leq 2)$

$$\begin{aligned}&\frac{\partial^{\alpha}\mu(x,t)}{\partial t^{\alpha}}+\frac{\partial^{\alpha-1}\mu(x,t)}{\partial t^{\alpha-1}}+\mu(x,t)=\frac{\partial^2\mu(x,t)}{\partial x^2}+f(x,t),\\ & 0<x<1,\ 0<t\leq 1,\end{aligned}$$

with the initial conditions

$$\mu(x,0)=0,\ \mu_t(x,0)=x(x-1),\ 0\leq x\leq 1,$$

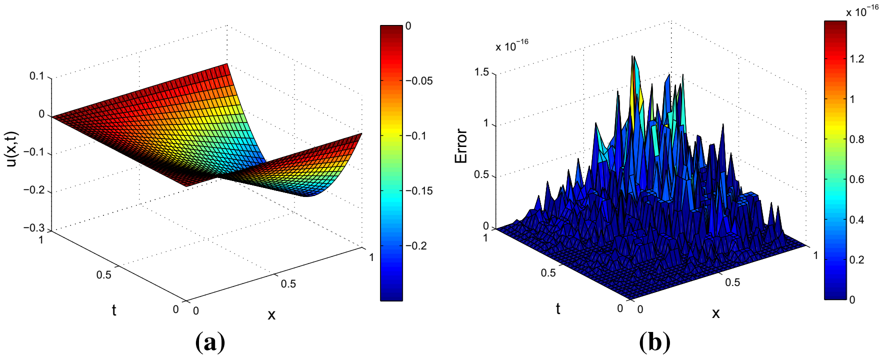


Figure 1. Approximate solution (**a**) and absolute error (**b**) for Example 1 with $\alpha = 1.95$, $k = 2$ and $M = 3$

and the boundary conditions

$$\mu(0, t) = \mu(1, t) = 0,\ 0 < t \leq 1,$$

where $f(x,t) = \left(\frac{\Gamma(2)}{\Gamma(3-\alpha)} t^{2-\alpha} + t\right)(x^2 - x) - 2t$. The exact solution of this problem is $\mu(x,t) = (x^2 - x)t$ ([16]). The problem is solved by the proposed method for $k = 2, M = 3$. Figure 1 shows the approximate solution and the absolute error of the problem for $\alpha = 1.95$. Table 1 gives the absolute errors for different values of α at different points. To make a comparison, the absolute error obtained by the present method has been compared with Sinc–Legendre method ([16]) in the case of $\alpha = 1.95$ and $t = 1$ in Table 2. It is apparent that the numerical solution is in good agreement with the exact solution and the results obtained by the proposed method are more accurate than the those given in [16].

Example 2. Consider the time-fractional telegraph equations of order α ($1 < \alpha \leq 2$)

$$\begin{aligned} &\frac{\partial^\alpha \mu(x,t)}{\partial t^\alpha} + \frac{\partial^{\alpha-1} \mu(x,t)}{\partial t^{\alpha-1}} + \mu(x,t) \\ &\quad = \pi \frac{\partial^2 \mu(x,t)}{\partial x^2} + f(x,t), \quad 0 < x < 1,\ 0 < t \leq 1, \end{aligned}$$

with the initial conditions

$$\mu(x, 0) = 0,\ \mu_t(x, 0) = 0,\ 0 \leq x \leq 1,$$

and the boundary conditions

$$\mu(0, t) = 0, \mu(1, t) = t^3 sin(1),\ 0 < t \leq 1,$$

where $f(x,t) = \left(\frac{\Gamma(4)}{\Gamma(4-\alpha)} t^{3-\alpha} + \frac{\Gamma(4)}{\Gamma(5-\alpha)} t^{4-\alpha} + t^3\right) \sin^2(x) - 2\pi t^3 \cos(2x)$. The exact solution of this problem is given by $\mu(x,t) = t^3 \sin^2 x$ ([16,31]). The problem is solved by the proposed method for $k = 2, M = 6$. Figure 2 shows the approximate solution and the absolute error of this problem for $\alpha = 1.95$. Table 3 gives the absolute errors for different values of α at some different

Table 1. The absolute errors for Example 1 for some different values $1 < \alpha \leq 2$ at some different points

(x, t)	$\alpha = 1.1$	$\alpha = 1.3$	$\alpha = 1.5$	$\alpha = 1.7$	$\alpha = 1.9$
(0.1,0.1)	1.73472e−18	1.73472e−18	1.73472e−18	1.73472e−18	1.73472e−18
(0.2,0.2)	6.93889e−18	6.93889e−18	0	6.93889e−18	6.93889e−18
(0.3,0.3)	0	0	6.93889e−18	0	0
(0.4,0.4)	1.38778e−17	1.38778e−17	1.38778e−17	1.38778e−17	1.38778e−17
(0.5,0.5)	2.77556e−17	0	0	0	2.77556e−17
(0.6,0.6)	0	2.77556e−17	2.77556e−17	0	2.77556e−17
(0.7,0.7)	5.55112e−17	5.55112e−17	2.77556e−17	0	2.77556e−17
(0.8,0.8)	2.77556e−17	1.11022e−16	8.32667e−17	5.55112e−17	2.77556e−17
(0.9,0.9)	6.93889e−17	5.55112e−17	4.16334e−17	1.38778e−17	1.38778e−17

Table 2. Comparison of absolute errors for Example 1 at $t = 1$

Method	Legendre wavelet($k = 2$)		Sinc–Legendre $n = 3$			
x	$M = 3$	$M = 4$	$m = 5$	$m = 7$	$m = 10$	$m = 15$
0.1	0	1.38778e−17	1.6276e−3	1.6932e−4	2.9057e−4	6.8187e−5
0.2	0	2.77556e−17	2.4790e−3	1.0916e−3	3.7898e−4	8.7378e−5
0.3	0	5.55112e−17	2.3211e−3	1.0749e−3	3.8165e−4	8.8787e−5
0.4	0	2.77556e−17	2.1722e−3	1.0102e−3	3.6413e−4	8.4142e−5
0.5	2.77556e−17	2.77556e−17	2.1507e−3	9.9270e−4	3.5476e−4	8.2186e−5
0.6	2.77556e−17	8.32667e−17	2.1772e−3	1.0102e−3	3.6413e−4	8.4142e−5
0.7	2.77556e−17	8.32667e−17	2.3211e−3	1.0749e−3	3.8165e−4	8.8787e−5
0.8	2.77556e−17	5.55112e−17	2.4790e−3	1.0916e−3	3.7898e−4	8.7378e−5
0.9	4.16333e−17	5.55112e−17	1.6276e−3	7.6032e−4	2.9057e−4	6.8187e−5

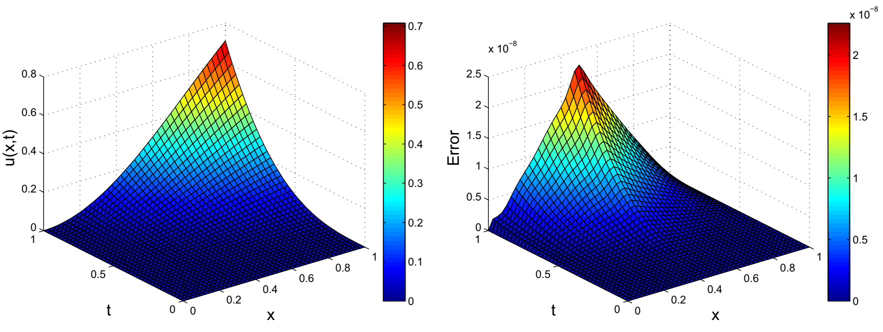


Figure 2. Approximate solution (**a**) and absolute error (**b**) for Example 2 with $\alpha = 1.75$, $k = 2$ and $M = 6$

points with $k = 2, M = 6$. In Table 4, we give the maximum absolute errors obtained by the proposed method for different choices of M and α at the points (x_i, t_j), where $x_i = i/40, t_j = j/40, i, j = 0, 1, 2, \ldots, 40$. By observing the graphs of the absolute errors given in Fig. 2b and graphs in [16,32], it is obvious that the results provided by the proposed method are more accurate than those given in [16,32].

Example 3. Consider the time-fractional telegraph equations of order α $(1 < \alpha \leq 2)$

$$\frac{\partial^\alpha \mu(x,t)}{\partial t^\alpha} + \frac{\partial^{\alpha-1}\mu(x,t)}{\partial t^{\alpha-1}} + \mu(x,t) = \frac{\partial^2 \mu(x,t)}{\partial x^2} + f(x,t), \quad 0 < x < 1,\ 0 < t \leq 1,$$

with the initial conditions

$$\mu(x,0) = x\mathrm{e}^{-x^2},\ \mu_t(x,0) = \mathrm{e}^{-x^2},\ 0 \leq x \leq 1,$$

and the boundary conditions

$$\mu(0,t) = t, \mu(1,t) = (t+1)\mathrm{e}^{-1},\ 0 < t \leq 1,$$

where $f(x,t) = \left(\frac{\Gamma(2)}{\Gamma(3-\alpha)}t^{2-\alpha} + t + x + 4x^3 - 4tx^2 + 7x + 3t\right)\mathrm{e}^{-x^2}$. The exact solution of this problem is $\mu(x,t) = (t+x)\mathrm{e}^{-x^2}$ ([12,27]). The problem is solved by the proposed method for $k = 2$ and $M = 3$. Figure 3 shows the approximate solution and the absolute error of this problem in the case of $\alpha = 1.75, k = 2$ and $M = 3$. Table 5 gives the absolute errors of the approximate solutions for different values of α at different points with $k = 2, M = 3$. In Table 6, there is a comparison between our method and the method in [16]. Table 6 and the graph for the absolute error given in Fig. 3 show that the results obtained by the proposed method are more accurate than the results achieved by Sinc–Legendre method in [16].

Table 3. The absolute errors for Example 2 for some different values $1 < \alpha \leq 2$ at some different points

(x,t)	$\alpha = 1.1$	$\alpha = 1.3$	$\alpha = 1.5$	$\alpha = 1.7$	$\alpha = 1.9$
(0.1,0.1)	5.36194e−13	2.86189e−13	5.9049e−13	5.50959e−13	9.85549e−13
(0.2,0.2)	3.83265e−11	2.36148e−11	1.06104e−11	2.52390e−12	2.23833e−13
(0.3,0.3)	2.63138e−10	2.0649e−10	1.41799e−10	8.21434e−11	3.84587e−11
(0.4,0.4)	9.54295e−10	8.31688e−10	6.75917e−10	5.0091e−10	3.40671e−10
(0.5,0.5)	2.71767e−9	2.52402e−9	2.27019e−9	1.95013e−9	1.60355e−9
(0.6,0.6)	3.70085e−9	3.45439e−9	3.13799e−9	2.71614e−9	2.19494e−9
(0.7,0.7)	4.41086e−9	4.15043e−9	3.82576e−9	3.39384e−9	2.80503e−9
(0.8,0.8)	4.40271e−9	4.17711e−9	3.90480e−9	3.55754e−9	3.05944e−9
(0.9,0.9)	3.14315e−9	3.00556e−9	2.84449e−9	2.65070e−9	2.37776e−9

Table 4. Maximum absolute error for Example 2 with various choices of M and α

M	$\alpha = 1.15$	$\alpha = 1.35$	$\alpha = 1.55$	$\alpha = 1.75$	$\alpha = 1.95$
3	4.6439e−5	4.5474e−5	4.4293e−5	4.2909e−5	4.1016e−5
4	7.5593e−6	7.3935e−6	7.1980e−6	6.9749e−6	6.6688e−6
5	2.0881e−7	2.0391e−7	1.9830e−7	1.9191e−7	1.8310e−7
6	2.3589e−8	2.3039e−8	2.2413e−8	2.1700e−8	2.0719e−8
7	5.0824e−10	4.9575e−10	4.8283e−10	4.6684e−10	4.4514e−10
8	4.2920e−11	3.9333e−11	4.1304e−11	4.0374e−11	3.4691e−11

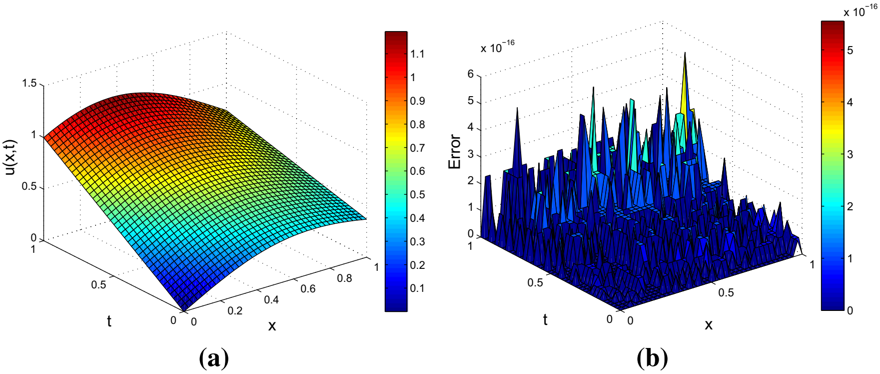


Figure 3. Approximate solution (**a**) and absolute error (**b**) for Example 3 with $\alpha = 1.75$, $k = 2$ and $M = 3$

Example 4. Consider the time-fractional telegraph equations of order α $(1 < \alpha \leq 2)$

$$\frac{\partial^\alpha \mu(x,t)}{\partial t^\alpha} + \frac{\partial^{\alpha-1}\mu(x,t)}{\partial t^{\alpha-1}} + \mu(x,t) = \frac{\partial^2 \mu(x,t)}{\partial x^2} + f(x,t), \ \ 0 < x < 1, \ \ 0 < t \leq 1,$$

with the initial conditions

$$\mu(x,0) = 0, \ \ \mu_t(x,0) = 0, \ \ 0 \leq x \leq 1,$$

and the boundary conditions

$$\mu(0,t) = t^{2\alpha}, \mu(1,t) = cos(7)t^{2\alpha}, \ \ 0 < t \leq 1,$$

where $f(x,t) = \left(\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)}t^\alpha + \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+2)}t^{\alpha+1} + 50t^{2\alpha}\right)\cos(7x)$. The exact solution of this problem is $\mu(x,t) = \cos(7x)t^{2\alpha}$. The space–time graph of the approximate solution and the absolute error for $\alpha = 1.85, k = 2$ and $M = 6$ is presented in Fig. 4. The absolute errors for different values of α at different points with $k = 2$ and $M = 6$ are tabulated in Table 7.

Table 5. The absolute errors for Example 3 for some different values $1 < \alpha \leq 2$ at some different points

(x,t)	$\alpha = 1.1$	$\alpha = 1.3$	$\alpha = 1.5$	$\alpha = 1.7$	$\alpha = 1.9$
(0.1,0.1)	0	0	0	0	0
(0.2,0.2)	5.55112e−17	5.55112e−17	5.55112e−17	5.55112e−17	5.55112e−17
(0.3,0.3)	0	0	0	0	0
(0.4,0.4)	0	0	0	0	0
(0.5,0.5)	1.11022e−16	1.11022e−16	1.11022e−16	1.11022e−16	0
(0.6,0.6)	1.11022e−16	0	2.22045e−16	0	1.11022e−16
(0.7,0.7)	0	1.11022e−16	0	1.11022e−16	1.11022e−16
(0.8,0.8)	0	0	0	0	1.11022e−16
(0.9,0.9)	2.22045e−16	1.11022e−16	1.11022e−16	2.22045e−16	1.11022e−16

Table 6. Comparison of absolute errors for Example 3 at $t = 1$

Method	Legendre wavelet($k = 2, M = 3$)				Sinc–Legendre($m = 20, n = 3$)			
x	$\alpha = 1.05$	$\alpha = 1.25$	$\alpha = 1.75$	$\alpha = 1.95$	$\alpha = 1.05$	$\alpha = 1.25$	$\alpha = 1.75$	$\alpha = 1.95$
0.1	0	0	0	0	6.0614e−5	4.7741e−5	2.0185e−5	1.1972e−5
0.2	0	0	0	0	2.4765e−5	6.5947e−6	2.6225e−5	3.4438e−5
0.3	5.5511e−17	2.7755e−17	2.7755e−17	2.7755e−17	3.5064e−6	1.6315e−5	4.6190e−5	5.1757e−5
0.4	2.7755e−17	0	2.7755e−17	0	7.5471e−6	2.7626e−5	5.3615e−5	5.6784e−5
0.5	2.7755e−17	0	2.7755e−17	0	1.0063e−5	3.0144e−5	5.4628e−5	5.6943e−5
0.6	5.5511e−17	5.5511e−17	0	2.7755e−17	4.2526e−6	2.4434e−5	5.0771e−5	5.4138e−5
0.7	5.5511e−17	5.5511e−17	0	2.7755e−17	9.9226e−6	1.0078e−5	4.0542e−5	4.6434e−5
0.8	0	0	5.5511e−17	2.7755e−17	3.5107e−5	1.6737e−5	1.6704e−5	2.5237e−5
0.9	2.7755e−17	0	5.5511e−17	0	7.5204e−5	6.2187e−5	3.4230e−5	2.5841e−5

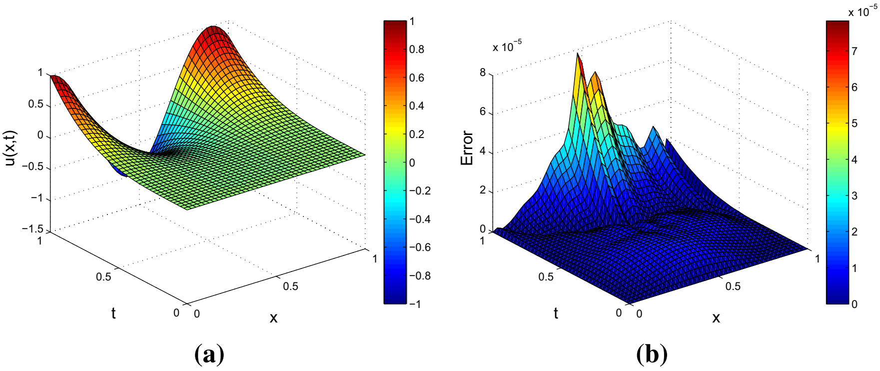


Figure 4. Approximate solution (**a**) and absolute error (**b**) for Example 4 with $\alpha = 1.85$, $k = 2$ and $M = 6$

Example 5. Consider the time-fractional telegraph equations of order α $(1 < \alpha \leq 2)$

$$\frac{\partial^{\alpha}\mu(x,t)}{\partial t^{\alpha}} + \frac{\partial^{\alpha-1}\mu(x,t)}{\partial t^{\alpha-1}} + \mu(x,t) = \frac{\partial^2\mu(x,t)}{\partial x^2} + f(x,t), \quad 0 < x < 1, \ 0 < t \leq 1,$$

with the initial conditions

$$\mu(x,0) = 0, \ \mu_t(x,0) = \mathrm{e}^x, \ 0 \leq x \leq 1,$$

and the boundary conditions

$$\mu(0,t) = t^{\alpha+3} + t, \mu(1,t) = e(t^{\alpha+3} + t), \ 0 < t \leq 1,$$

where $f(x,t) = \left(\frac{\Gamma(\alpha+4)}{\Gamma(4)}t^3 + \frac{\Gamma(\alpha+4)}{\Gamma(5)}t^4 + \frac{\Gamma(2)}{\Gamma(3-\alpha)}t^{2-\alpha}\right)\mathrm{e}^x$. The exact solution of this problem is $\mu(x,t) = (t^{\alpha+3} + t)\mathrm{e}^x$. The space–time graph of the approximate solution and the absolute error for $\alpha = 2, k = 2$ and $M = 6$ are presented in Fig. 5. Table 8 gives the absolute errors for different values of α at different points with $k = 2, M = 6$. In Table 9, we list the maximum absolute errors obtained by the proposed method for different choices of M and α.

Example 6. Consider the following time-fractional telegraph equations:

$$\frac{\partial^{1.5}\mu(x,t)}{\partial t^{1.5}} + \frac{\partial^{0.5}\mu(x,t)}{\partial t^{0.5}} + 2\mu(x,t) = \frac{\partial^2\mu(x,t)}{\partial x^2} + f(x,t), \ 0 < x < 1, \ 0 < t \leq 1,$$

with the initial conditions

$$\mu(x,0) = \mathrm{e}^{x^2}, \ \mu_t(x,0) = \mathrm{e}^{x^2}, \ 0 \leq x \leq 1,$$

and the boundary conditions

$$\mu(0,t) = \mathrm{e}^t, \ \mu(1,t) = \mathrm{e}^{1+t}, \ 0 < t \leq 1,$$

where $f(x,t) = \left(\frac{\sqrt{\pi}}{\Gamma(1/2)}erf(\sqrt{t}) - 4x^2\right)\mathrm{e}^{x^2+t}$ and erf is error function defined as $erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x \mathrm{e}^{-t^2}\mathrm{d}t$. The exact solution of this problem is $\mu(x,t) = \mathrm{e}^{x^2+t}$. We solved this problem with $k = 2$, $M = 3$ to 8 and compared our

Table 7. The absolute errors for Example 4 for some different values $1 < \alpha \leq 2$ at some different points

(x,t)	$\alpha = 1.25$	$\alpha = 1.45$	$\alpha = 1.65$	$\alpha = 1.85$	$\alpha = 1.95$
(0.1,0.1)	6.56143e−6	1.44302e−6	2.31695e−6	1.38000e−6	3.83479e−7
(0.2,0.2)	4.19945e−6	1.88774e−6	6.14709e−6	6.31448e−6	2.38872e−6
(0.3,0.3)	1.16117e−6	1.60510e−6	6.38326e−6	9.73774e−6	4.42841e−6
(0.4,0.4)	6.98697e−6	2.94178e−6	5.23872e−6	9.42029e−6	4.76428e−6
(0.5,0.5)	9.03251e−5	1.18209e−5	8.88721e−7	5.50334e−6	1.98029e−6
(0.6,0.6)	8.37918e−5	2.20565e−5	1.94768e−6	3.79861e−6	2.02016e−7
(0.7,0.7)	6.21527e−5	2.22687e−5	5.05434e−6	5.52133e−7	1.82243e−7
(0.8,0.8)	4.45163e−5	2.23114e−5	9.00314e−6	4.17944e−6	2.82841e−6
(0.9,0.9)	5.52851e−5	2.92289e−5	2.20003e−5	1.73389e−5	1.58564e−5

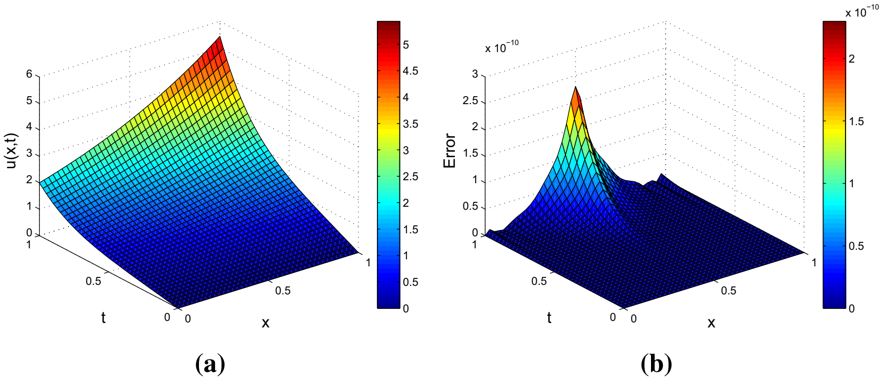


Figure 5. Approximate solution (**a**) and absolute error (**b**) for Example 5 with $\alpha = 2$, $k = 2$ and $M = 6$

results with the exact solutions in Table 10. Figure 6 shows the approximate solution and the absolute error of this problem in the case of $k = 2$ and $M = 7$.

Example 7. To further demonstrate the superiority of the proposed algorithm, we consider the time-fractional telegraph equations of order α $(1 < \alpha \leq 2)$, which is also studied by the author in [17] using two-dimensional Legendre wavelets together with their operational matrix of fractional integration.

$$\frac{\partial^\alpha \mu(x,t)}{\partial t^\alpha} + \frac{\partial^{\alpha-1}\mu(x,t)}{\partial t^{\alpha-1}} + \mu(x,t) = \frac{\partial^2 \mu(x,t)}{\partial x^2} + f(x,t), \quad 0 < x < 1,\ 0 < t \leq 1,$$

with the initial conditions

$$\mu(x,0) = x^2,\ \mu(x,1) = 1 + x^2,\ 0 \leq x \leq 1,$$

and the boundary conditions

$$\mu(0,t) = t, \mu(1,t) = 1 + t,\ 0 < t \leq 1,$$

where $f(x,t) = \frac{\Gamma(2)}{\Gamma(3-\alpha)} + x^2 + t - 2$. The exact solution of this problem is $\mu(x,t) = x^2 + t$. To compare our numerical solutions with the results obtained in [17], the root mean square error L_2 and maximum error L_∞ for some (x,t) in $(0,1)$ in case $\alpha = 2$ are presented in Table 11. The space–time graph of the approximate solution and the absolute error for $\alpha = 2, k = 2$ and $M = 3$ is shown in Fig. 7. Table 12 gives the absolute errors for different values of α at different points with $k = 2, M = 3$.

From Figs. 1, 2, 3, 4, 5, 6 and 7 and Tables 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 and 12, it can be seen that the proposed method is very efficient and accurate in solving this problem and the obtained approximate solutions are very close to the exact ones for all chosen α with $1 < \alpha \leq 2$.

Table 8. The absolute errors for Example 5 for some different values $1 < \alpha \leq 2$ at some different points

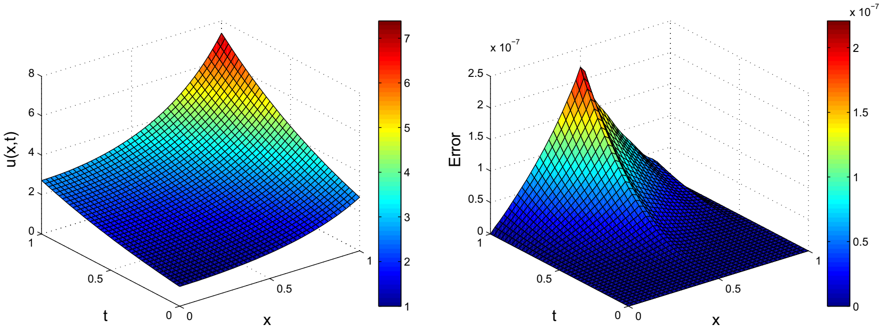
(x, t)	$\alpha = 1.1$	$\alpha = 1.3$	$\alpha = 1.5$	$\alpha = 1.7$	$\alpha = 1.9$
(0.1,0.1)	1.35319e−9	4.07841e−9	1.68836e−8	2.68133e−8	1.79555e−8
(0.2,0.2)	1.32832e−9	5.26396e−9	2.61160e−8	5.82506e−8	5.12206e−8
(0.3,0.3)	3.20547e−9	3.48706e−9	2.12329e−8	7.83247e−8	8.52865e−8
(0.4,0.4)	2.39446e−9	7.16483e−9	2.23624e−8	8.12507e−8	1.06713e−7
(0.5,0.5)	9.56645e−8	7.92492e−8	3.34593e−8	5.64275e−8	9.87810e−8
(0.6,0.6)	1.10489e−7	2.63172e−7	2.32402e−7	1.49424e−7	1.02135e−7
(0.7,0.7)	4.11335e−8	1.40763e−7	1.90783e−7	1.47798e−7	8.42311e−8
(0.8,0.8)	2.39614e−9	4.75584e−8	9.99702e−8	9.31169e−8	5.14595e−8
(0.9,0.9)	2.12963e−8	5.32955e−10	3.13807e−8	3.62929e−8	1.87748e−8

Table 9. Maximum absolute error for Example 5 with various choices of M and α

M	$\alpha = 1.1$	$\alpha = 1.3$	$\alpha = 1.5$	$\alpha = 1.7$	$\alpha = 1.9$
3	5.51177e−5	6.86346e−5	8.60388e−5	1.27440e−4	1.17578e−4
4	7.35883e−6	7.38658e−6	5.31993e−6	4.57049e−6	2.76544e−6
5	1.88294e−6	1.61842e−6	9.35373e−7	7.30801e−7	3.70593e−7
6	6.89600e−7	5.40155e−7	2.45587e−7	1.75043e−7	1.13799e−7
7	3.00209e−7	2.16288e−7	8.88981e−8	6.17699e−7	4.40735e−8
8	1.52542e−7	9.87635e−8	3.81930e−8	2.35588e−8	2.01534e−8

Table 10. The absolute errors for Example 6 at some different points with $k = 2$, $M = 3$–8.

(x, t)	$M = 3$	$M = 4$	$M = 5$	$M = 6$	$M = 7$	$M = 8$
(0.1,0.1)	5.72299e−8	9.68516e−9	6.45132e−10	6.25242e−11	2.20490e−12	8.59313e−14
(0.2,0.2)	1.39180e−7	8.33936e−8	1.01266e−8	1.46717e−9	1.31050e−10	1.71356e−11
(0.3,0.3)	8.51538e−6	1.54290e−6	1.53090e−7	2.47532e−8	2.20417e−9	2.85300e−10
(0.4,0.4)	3.79062e−5	7.55160e−6	7.75779e−7	1.21327e−7	1.09169e−8	1.41185e−9
(0.5,0.5)	1.09183e−4	2.40384e−5	2.33133e−6	3.78638e−7	3.34379e−8	4.36894e−9
(0.6,0.6)	9.40495e−5	2.11198e−5	2.19361e−6	3.79021e−7	3.52032e−8	4.68838e−9
(0.7,0.7)	6.95004e−5	2.20188e−5	2.76178e−6	4.01214e−7	3.76155e−8	4.81099e−9
(0.8,0.8)	2.15857e−4	1.67122e−5	2.42911e−6	3.51572e−7	3.64944e−8	4.31197e−9
(0.9,0.9)	1.37827e−4	1.24114e−6	2.88286e−6	1.53645e−7	2.95434e−8	2.72259e−9

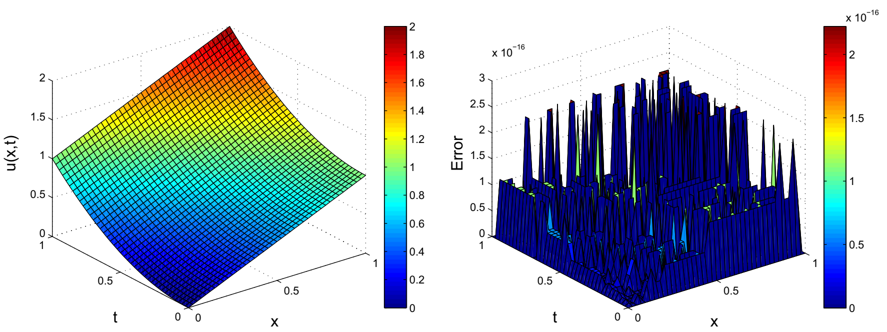


(a) **(b)**

Figure 6. Approximate solution (**a**) and absolute error (**b**) for Example 6 with $k = 2$ and $M = 7$

Table 11. Comparison of L_∞ and L_2 errors for Example 7 for some different values of t in case $\alpha = 2$

Method	Present method ($k = 3, M = 3$)		Method in [17]($k = 3, M = 3$)	
t	L_∞	L_2	L_∞	L_2
0.1	2.22044e−16	7.53756e−17	4.90e−3	8.64e−4
0.3	1.11022e−16	5.55111e−17	1.47e−3	8.06e−4
0.5	1.11022e−16	7.85046e−17	2.28e−3	1.38e−3
0.7	2.22044e−16	1.24126e−16	1.17e−3	7.50e−4
0.9	4.44089e−16	2.00148e−16	6.45e−3	8.60e−5
1.0	2.22044e−16	2.02698e−16	0.00	0.00



(a) **(b)**

Figure 7. Approximate solution (**a**) and absolute error (**b**) for Example 7 with $\alpha = 2$, $k = 2$ and $M = 3$

Table 12. The absolute errors for Example 7 for some different values $1 < \alpha \leq 2$ at some different points

(x, t)	$\alpha = 1.1$	$\alpha = 1.3$	$\alpha = 1.5$	$\alpha = 1.7$	$\alpha = 1.9$	$\alpha = 2.0$
(0.1,0.1)	0	0	0	4.16334e−17	1.94289e−16	0
(0.2,0.2)	0	0	0	5.55112e−17	2.49800e−16	0
(0.3,0.3)	0	0	0	5.55112e−17	3.33067e−16	0
(0.4,0.4)	0	0	0	2.22045e−16	5.55112e−16	0
(0.5,0.5)	0	0	0	3.33067e−16	5.55112e−16	0
(0.6,0.6)	1.11022e−16	0	1.11022e−16	1.11022e−16	3.33067e−16	0
(0.7,0.7)	0	0	2.22045e−16	2.22045e−16	4.44089e−16	0
(0.8,0.8)	0	0	0	2.22045e−16	8.88178e−16	0
(0.9,0.9)	0	0	2.22045e−16	2.22045e−16	8.88178e−16	0

7. Conclusion

In this paper, the fractional integral formula of a single Legendre wavelet in the Riemann–Liouville sense is derived. Legendre wavelet collocation method has been successfully used to obtain the approximate solution of the time-fractional order telegraph equations. The proposed method is very convenient for solving the time-fractional order telegraph equations, since the initial and boundary conditions are all considered during the process of constructing the approximate solution. The numerical results obtained by the proposed method are compared with those obtained by Sinc–Legendre method and radial basis functions to illustrate validity and applicability of the proposed technique. Moreover, the convergence analysis and error analysis of the proposed method are studied. The illustrative examples show that the method is an effective tool to solve time-fractional order telegraph equations and is expected to solve other fractional order partial differential equations numerically.

References

[1] Weston, V.H., He, S.: Wave splitting of the telegraph equation in R^3 and its application to inverse scattering. Inverse Probl. **9**, 789–812 (1993)

[2] Banasiak, J., Mika, J.R.: Singular perturbed telegraph equations with applications in the random walk theory. J. Appl. Math. Stoch. Anal. **11**, 9–28 (1998)

[3] Jordan, P.M., Puri, A.: Digital signal propagation in dispersive media. J. Appl. Phys. **85**, 1273–1282 (1999)

[4] Saadatmandi, A., Dehghan, M.: Numerical solution of hyperbolic telegraph equation using the Chebyshev Tau method. Numer. Methods Partial Differ. Equ. **26**, 239–252 (2010)

[5] Chen, J., Liu, F., Anh, V.: Analytical solution for the time-fractional telegraph equation by the method of separating variables. J. Math. Anal. Appl. **338**(2), 1364–1377 (2008)

[6] Dehghan, M., Shokri, A.: A numerical method for solving the hyperbolic telegraph equation. Numer. Methods Partial Differ. Equ. **24**(4), 1080–1093 (2008)

[7] Dehghan, M., Lakestani, M.: The use of Chebyshev cardinal functions for solution of the second-order one-dimensional telegraph equation. Numer. Methods Partial Differ. Equ. **25**(4), 931–938 (2009)

[8] Saadatmandi, A., Dehghan, M.: Numerical solution of hyperbolic telegraph equation using the Chebyshev tau method. Numer. Methods Partial Differ. Equ. **26**(1), 239–252 (2010)

[9] Das, S., Gupta, P.K.: Homotopy analysis method for solving fractional hyperbolic partial differential equations. Int. J. Comput. Math. **88**(3), 578–588 (2011)

[10] Mollahasani, N., Moghadam, M.M., Afrooz, K.: A new treatment based on hybrid functions to the solution of telegraph equations of fractional order. Appl. Math. Model. **40**(4), 2804–2814 (2016)

[11] Biazar, J., Eslami, M.: Analytic solution for telegraph equation by differential transform method. Phys. Lett. A **374**(29), 2904–2906 (2010)

[12] Biazar, J., Ebrahimi, H., Ayati, Z.: An approximation to the solution of telegraph equation by variational iteration method. Numer. Methods Partial Differ. Equ. **25**(4), 797–801 (2009)

[13] Biazar, J., Ebrahimi, H.: An approximation to the solution of telegraph equation by Adomian decomposition method. Int. Math. Forum **2**(45), 2231–2236 (2007)

[14] Yildrim, A.: He's homotopy perturbation method for solving the space-and time-fractional telegraph equations. Int. J. Comput. Math. **87**(13), 2998–3006 (2010)

[15] Wei, L., Dai, H., Zhang, D., et al.: Fully discrete local discontinuous Galerkin method for solving the fractional telegraph equation. Calcolo **51**(1), 175–192 (2014)

[16] Sweilam, N.H., Nagy, A.M., El-Sayed, A.A.: Solving time-fractional order telegraph equation via Sinc–Legendre collocation method. Mediterr. J. Math. **13**(6), 5119–5133 (2016)

[17] Heydari, M.H., Hooshmandasl, M.R., Mohammadi, F.: Two-dimensional Legendre wavelets for solving time-fractional telegraph equation. Adv. Appl. Math. Mech. **6**(2), 247–260 (2014)

[18] Shiralashetti, S.C., Deshi, A.B.: An efficient Haar wavelet collocation method for the numerical solution of multi-term fractional differential equations. Nonliner Dyn. **83**(1–2), 293–303 (2016)

[19] Zhang, Q., Feng, Z., Tang, Q., Zhang, Y.: An adaptive wavelet collocation method for solving optimal control problem. Proc. Inst. Mech. Eng. G J. Aerosp. **229**(9), 1640–1649 (2015)

[20] Sahu, P.K., Ray, S.S.: Legendre spectral collocation method for Fredholm integro-differential-difference equation with variable coefficients and mixed conditions. Appl. Math. Comput. **268**, 575–580 (2015)

[21] Heydari, M.H., Hooshmandasl, M.R., Ghaini, F.M.M.: A new approach of the Chebyshev wavelets method for partial differential equations with boundary conditions of the telegraph type. Appl. Math. Model. **38**(5), 1597–1606 (2014)

[22] Rahimkhani, P., Ordokhani, Y., Babolian, E.: A new operational matrix based on Bernoulli wavelets for solving fractional delay differential equations. Numer. Algorithm **74**(1), 223–245 (2017)

[23] Rehman, M.U., Khan, R.A.: The Legendre wavelet method for solving fractional differential equations. Commun. Nonlinear Sci. **16**(11), 4163–4173 (2011)

[24] Heydari, M.H., Hooshmandasl, M.R., Mohammadi, F.: Legendre wavelets method for solving fractional partial differential equations with Dirichlet boundary conditions. Appl. Math. Comput. **234**, 267–276 (2014)

[25] Meng, Z., Wang, L., Li, H., Zhang, W.: Legendre wavelets method for solving fractional integro-differential equations. Int. J. Comput. Math. **92**(6), 1–17 (2015)

[26] Li, Y.: Solving a nonlinear fractional differential equation using Chebyshev wavelets. Commun. Non-linear. Sci. **15**(9), 2284–2292 (2010)

[27] Celik, I.: Chebyshev Wavelet collocation method for solving generalized Burgers–Huxley equation. Math. Method Appl. Sci. **39**(3), 366–377 (2016)

[28] Miller, K.S., Ross, B.: An Introduction to the Fractional Calculus and Fractional Differential Equations. Wiley, New York (1993)

[29] Liu, N., Lin, E.B.: Legendre wavelet method for numerical solutions of partial differential equations. Numer. Methods Partial Differ. Equ. **26**(1), 81–94 (2010)

[30] Parsian, H.: Two dimension Legendre wavelets and operational matrices of integration. Acta. Math. Acad. Paedagog. Nyiregyhziens **21**, 101–106 (2005)

[31] Das, S., Vishal, K., Gupta, P.K., Yildirim, : An approximate analytical solution of time-fractional telegraph equation. Appl. Math. Comput. **217**(18), 7405–7411 (2011)

[32] Hosseini, V.R., Chen, W., Avazzadeh, Z.: Numerical solution of fractional telegraph equation by using radial basis functions. Eng. Anal. Bound. Elem. **38**, 31–39 (2014)

Xiaoyong Xu and Da Xu
College of Mathematics and Computer Science
Hunan Normal University
Changsha 410006
People's Republic of China
e-mail: daxu@hunnu.edu.cn

Xiaoyong Xu
School of Science
East China University of Technology
Nanchang 330013
People's Republic of China
e-mail: xxy@ecit.cn

Received: November 20, 2016.
Revised: September 13, 2017.
Accepted: January 19, 2018.