Mediterr. J. Math. (2018) 15:19
https://doi.org/10.1007/s00009-017-1064-x
1660-5446/18/010001-12
*published online* January 12, 2018


**Mediterranean Journal of Mathematics**

# Existence of Positive Solutions for Higher Order $p$-Laplacian Boundary Value Problems

K. R. Prasad, N. Sreedhar and L. T. Wesen

**Abstract.** In this paper, we consider a higher order $p$-Laplacian boundary value problem

$$(-1)^n[\phi_p(u^{(2n-2)} + k^2u^{(2n-4)})]'' = f(t, u),\ 0 \leq t \leq 1,$$
$$u^{(2i)}(0) = 0 = u^{(2i)}(1),\ 0 \leq i \leq n-1,$$

where $n \geq 1$ and $k \in (0, \frac{\pi}{2})$ is a constant. By applying fixed point index theory, we derive sufficient conditions for the existence of positive solutions to the boundary value problem.

**Mathematics Subject Classification.** 34A40, 34B15, 34L30.

**Keywords.** Green's function, $p$-Laplacian, Boundary value problem, Positive solution, Cone, Fixed point index theory.

## 1. Introduction

In many branches of applied mathematics and physics, the main objective of studying the differential equations is to analyze the given situations of the real world problems by formulating suitable mathematical models. The theory of differential equations offers a broad mathematical basis to understand the problems of modern society, which are complex and interdisciplinary by nature. In this theory, one of the most applicable operator is the classical one dimensional $p$-Laplacian operator and is given by $\phi_p(s) = |s|^{p-2}s$, where $p > 1, \phi_p^{-1} = \phi_q$ and $\frac{1}{p} + \frac{1}{q} = 1$. These type of problems appear in mathematical modeling of viscoelastic flows, image processing, turbulent filtration in porous media, biophysics, plasma physics, rheology, glaciology, radiation of heat, plastic molding, etc. Some recent advances indicate that even the Brownian motion has its counter part and a mathematical game 'tug of war' leads to the case $p = \infty$. For more details on applications, we refer [9].

Due to wide mathematical and physical background, the existence of positive solutions for nonlinear boundary value problems with $p$-Laplacian operators have received great attention in recent years. To mention a few papers along these lines are Wang [30], Lian and Wong [21], Agarwal et al. [2], Li and Ge [18], Liu and Ge [22], Avery and Henderson [3], Li and Shen [20] and for further development in the topic, see [11,12,25,31,32,34,36,37].

Motivated by the papers mentioned above, in this paper, we establish the existence of positive solutions for higher order $p$-Laplacian boundary value problems of the form

$$(-1)^n[\phi_p(u^{(2n-2)}+k^2u^{(2n-4)})]''=f(t,u),\quad 0\le t\le 1, \tag{1.1}$$

$$u^{(2i)}(0)=0=u^{(2i)}(1),\quad 0\le i\le n-1, \tag{1.2}$$

where $n\ge 1, k\in(0,\frac{\pi}{2})$ is a constant and $f:[0,1]\times\mathbb{R}^+\to\mathbb{R}^+$ is a continuous function, by applying fixed point index theory. In the past few decades for $k=0$ and $p=2$, a lot of works has been done on the existence of positive solutions of the boundary value problems associated with differential equations using various methods, see [4,6,7,10,17,29,33,35] and for $k\neq 0$ and $p=2$, most of the authors focussed on the existence of positive solutions of second order differential equations satisfying Neumann and Sturm–Liouville boundary conditions, see [5,14,15,19,23,24,26–28]. However, as far as we know, this work is path breaking on study of higher order $p$-Laplacian boundary value problems with the parameter $k$.

The rest of the paper is organized as follows. In Sect. 2, we express the solution of the boundary value problem (1.1) and (1.2) in to an equivalent integral equation and estimate bounds for the Green functions. In Sect. 3, we establish criteria for the existence of at least one positive solution for the boundary value problem (1.1) and (1.2). Finally as an application, we give an example to illustrate our results.

## 2. Green's Function and Bounds

In this section, we express the solution of the boundary value problem (1.1) and (1.2) into an equivalent integral equation involving Green functions and estimate bounds for these Green functions.

First, we construct the Green's function $H(t,s)$ for the homogeneous problem,

$$-(u''+k^2u)=0,\quad 0\le t\le 1, \tag{2.1}$$

$$u(0)=0=u(1), \tag{2.2}$$

and then we derive the Green's function $G_1(t,s)$ for the homogeneous boundary value problem,

$$-y''=0,\quad 0\le t\le 1, \tag{2.3}$$

$$y(0)=0=y(1), \tag{2.4}$$

by taking $y(t) = (-1)^{n-2}[\phi_p(x)^{(2n-4)}]$ and $x(t) = -(u'' + k^2u)$. Using the Green's function $G_1(t,s)$, we derive Green's function $G_{n-2}(t,s), n \geq 3$, recursively for the homogeneous boundary value problem,

$$(-1)^{n-2}x^{(2n-4)} = 0, \quad 0 \leq t \leq 1, \tag{2.5}$$

$$x^{(2i)}(0) = 0 = x^{(2i)}(1), \quad 0 \leq i \leq n-3. \tag{2.6}$$

**Lemma 2.1.** *The Green's function $H(t,s)$ for the homogeneous boundary value problem* (2.1), (2.2) *is given by*

$$H(t,s) = \begin{cases} \dfrac{\sin(kt)\sin(k(1-s))}{k\sin(k)}, & t \leq s, \\ \\ \dfrac{\sin(ks)\sin(k(1-t))}{k\sin(k)}, & s \leq t. \end{cases} \tag{2.7}$$

*Proof.* By algebraic calculations, we can establish the result. □

**Lemma 2.2.** [1] *The Green's function $G_1(t,s)$ for the homogeneous boundary value problem* (2.3), (2.4) *is given by*

$$G_1(t,s) = \begin{cases} t(1-s), & t \leq s, \\ \\ s(1-t), & s \leq t. \end{cases} \tag{2.8}$$

**Lemma 2.3.** [1,33] *The Green's function for the homogeneous boundary value problem* (2.5), (2.6) *is $G_{n-2}(t,s)$, where $G_{n-2}(t,s)$ is defined recursively as*

$$G_j(t,s) = \int_0^1 G_{j-1}(t,\xi)G_1(\xi,s)d\xi, \quad \text{for } 2 \leq j \leq n-2, \tag{2.9}$$

*and $G_1(t,s)$ is defined as in* (2.8).

Now, the equivalent integral equation for the boundary value problem (1.1), (1.2) is given by

$$u(t) = \int_0^1 G(t,s)\phi_q\left[\int_0^1 G_1(s,r)f(r,u(r))\mathrm{d}r\right]\mathrm{d}s, \tag{2.10}$$

where

$$G(t,s) = \int_0^1 H(t,\tau)G_{n-2}(\tau,s)\mathrm{d}\tau. \tag{2.11}$$

**Lemma 2.4.** *The Green's function $H(t,s)$ in* (2.7) *satisfies the following inequalities:*

(i) $H(t,s) \geq 0$, *for all* $t,s \in [0,1]$,
(ii) $H(t,s) \leq H(s,s)$, *for all* $t,s \in [0,1]$,
(iii) $H(t,s) \geq \eta H(s,s)$, *for all* $t \in I$ *and* $s \in [0,1]$,

*where* $\eta = \frac{\sin(\frac{k}{4})}{\sin(k)}$ *and* $I = \left[\frac{1}{4}, \frac{3}{4}\right]$.

*Proof.* By algebraic calculations, we can establish the result. □

**Lemma 2.5.** [33] *The Green's function $G_1(t,s)$ in* (2.8) *satisfies the following inequalities:*

(i) $G_1(t,s) \geq 0, \quad \textit{for all} \quad t,s \in [0,1],$
(ii) $G_1(t,s) \leq G_1(s,s), \quad \textit{for all} \quad t,s \in [0,1],$
(iii) $G_1(t,s) \geq \dfrac{1}{4} G_1(s,s), \quad \textit{for all} \quad t \in I \textit{ and } s \in [0,1],$

*where* $I = \left[\frac{1}{4}, \frac{3}{4}\right]$.

**Lemma 2.6.** [33] *The Green's function* $G_{n-2}(t,s)$ *in* (2.9) *satisfies the following inequalities:*

(i) $G_{n-2}(t,s) \geq 0, \quad \textit{for all} \quad t,s \in [0,1],$
(ii) $G_{n-2}(t,s) \leq \dfrac{1}{6^{n-3}} G_1(s,s), \quad \textit{for all} \quad t,s \in [0,1],$
(iii) $G_{n-2}(t,s) \geq \dfrac{1}{4^{n-2}} \left(\dfrac{11}{96}\right)^{n-3} G_1(s,s), \quad \textit{for all} \quad t \in I \textit{ and } s \in [0,1],$

*where* $I = [\frac{1}{4}, \frac{3}{4}]$.

**Lemma 2.7.** *The Kernel* $G(t,s)$ *in* (2.10) *satisfies the following inequalities:*

(i) $G(t,s) \geq 0, \quad \textit{for all} \quad t,s \in [0,1],$
(ii) $G(t,s) \leq \dfrac{K}{6^{n-3}} G_1(s,s), \quad \textit{for all} \quad t,s \in [0,1],$
(iii) $G(t,s) \geq \dfrac{\eta L}{4^{n-2}} \left(\dfrac{11}{96}\right)^{n-3} G_1(s,s), \quad \textit{for all} \quad t \in I \textit{ and } s \in [0,1],$

*where* $K = \displaystyle\int_0^1 H(\tau,\tau)d\tau$ *and* $L = \displaystyle\int_{\tau \in I} H(\tau,\tau)d\tau$.

*Proof.* By algebraic calculations, we can establish the result. □

For the reader's convenience, we present some necessary definitions and theorems we may use through the entire paper.

**Definition 2.8.** Let $X$ be a Banach space over $\mathbb{R}$. A nonempty, closed set $P \subset X$ is a *cone*, provided

(i) $\alpha\mathbf{u} + \beta\mathbf{v} \in P$ for all $\mathbf{u}, \mathbf{v} \in P$ and all $\alpha, \beta \geq 0$, and
(ii) $\mathbf{u}, -\mathbf{u} \in P$ implies $\mathbf{u} = \mathbf{0}$.

**Definition 2.9.** An operator, $T$, is completely continuous if $T$ is continuous and compact, i.e., $T$ maps bounded sets into precompact sets.

Let $X$ be a Banach Space and $K$ be a cone in $X$. For $r > 0$, define

$$K_r = \{x \in K : \|x\| < r\} \quad \text{and} \quad \partial K_r = \{x \in K : \|x\| = r\}.$$

We use the following well-known fixed point index theorem will be the fundamental tool to prove our main results.

**Theorem 2.10.** [8,13,16] *Let* $X$ *be a Banach Space and* $K$ *be a cone in* $X$. *Assume that* $F : \overline{K_r} \to K$ *is a completely continuous such that* $Fx \neq x$ *for* $x \in \partial K_r$.

(i) *If* $\|x\| < \|Fx|$ *for* $x \in \partial K_r$, *then* $i(F, K_r, K) = 0$.
(ii) *If* $\|x\| > \|Fx\|$ *for* $x \in \partial K_r$, *then* $i(F, K_r, K) = 1$.

*Here* $i(F, K_r, K)$ *is called the fixed point index of* $F$ *on* $K_r$ *with respect to* $K$.

## 3. Existence Positive Solutions

In this section, we establish the existence of at least one positive solution for nonlinear $p$-Laplacian boundary value problem (1.1), (1.2) using fixed point index theory.

Let $X = \{u | u \in C[0,1]\}$ be a Banach space with the norm $\|u\| = \max_{t\in[0,1]} |u|$, and let

$$P = \{u \in X | u(t) > 0, t \in [0,1] \quad \text{and} \min_{t\in I} |u(t)| \geq \mathcal{M}\|u\|\},$$

where $\mathcal{M} = (\frac{\eta L}{K})(\frac{11^{n-3}}{2^{6n-16}})$. We note that $P$ is a cone in $X$. Let the operator $F : P \to X$ be defined as

$$Fu(t) = \int_0^1 G(t,s)\phi_q \left[\int_0^1 G_1(s,r) f(r, u(r))\mathrm{d}r\right] \mathrm{d}s. \tag{3.1}$$

To obtain a positive solution of (1.1), (1.2), we shall seek a fixed point of the operator $F$ in the cone $P$.

We assume the following conditions hold throughout this paper:

(A1) $0 < \int_0^1 G_1(t,s)\mathrm{d}s < \infty$,
(A2) $f(t,x)$ is a nondecreasing function with respect to $x$.

Define the nonnegative extended real numbers $f_0$, $f^0$, $f_\infty$ and $f^\infty$ by

$$f_0 = \lim_{u\to 0^+} \min_{t\in[0,1]} \frac{f(t,u)}{\phi_p(u)}, \; f^0 = \lim_{u\to 0^+} \max_{t\in[0,1]} \frac{f(t,u)}{\phi_p(u)},$$
$$f_\infty = \lim_{u\to \infty} \min_{t\in[0,1]} \frac{f(t,u)}{\phi_p(u)} \text{ and } f^\infty = \lim_{u\to \infty} \max_{t\in[0,1]} \frac{f(t,u)}{\phi_p(u)},$$

assuming that they will exist. When $f^0 = 0$ and $f_\infty = \infty$ is called super linear case and $f_0 = \infty$ and $f^\infty = 0$ is called the sub linear case.

**Lemma 3.1.** *The operator $F : P \to X$ defined by (3.1) is a self-map on $P$.*

*Proof.* From (A1) and the positivity of the Green's function $G(t,s)$ in Lemma 2.7 that for $u \in P$, $Fu(t) \geq 0$ on $t \in [0,1]$. Now, for $u \in P$ and by Lemma 2.7, we have

$$\begin{aligned} Fu(t) &= \int_0^1 G(t,s)\phi_q\bigg(\int_0^1 G_1(s,r) f\big(r, u(r)\big)\mathrm{d}r\bigg)\mathrm{d}s \\ &\leq \frac{K}{6^{n-3}} \int_0^1 G_1(s,s)\phi_q\bigg(\int_0^1 G_1(s,r) f\big(r, u(r)\big)\mathrm{d}r\bigg)\mathrm{d}s. \end{aligned}$$

So that

$$\|Fu\| \leq \frac{K}{6^{n-3}} \int_0^1 G_1(s,s)\phi_q\bigg(\int_0^1 G_1(s,r) f\big(r, u(r)\big)\mathrm{d}r\bigg)\mathrm{d}s. \tag{3.2}$$

Then, by Lemma 2.7, for $u \in P$ that

$$\begin{aligned}\min_{t\in I} Fu(t) &= \min_{t\in I}\int_0^1 G(t,s)\phi_q\left(\int_0^1 G_1(s,r)f\big(r,u(r)\big)\mathrm{d}r\right)\mathrm{d}s.\\ &\geq \left(\frac{\eta L}{4^{n-2}}\right)\left(\frac{11}{96}\right)^{n-3}\int_0^1 G_1(s,s)\phi_q\left(\int_0^1 G_1(s,r)f\big(r,u(r)\big)\mathrm{d}r\right)\mathrm{d}s\\ &= \left(\frac{\eta L}{K}\right)\left(\frac{11^{n-3}}{2^{6n-16}}\right)\frac{K}{6^{n-3}}\int_0^1 G_1(s,s)\phi_q\left(\int_0^1 G_1(s,r)f\big(r,u(r)\big)\mathrm{d}r\right)\mathrm{d}s\\ &\geq \left(\frac{\eta L}{K}\right)\left(\frac{11^{n-3}}{2^{6n-16}}\right)\|Fu\|\\ &= \mathcal{M}\|Fu\|.\end{aligned}$$

Therefore, $F : P \to P$, and hence the proof is complete. □

Furthermore, the operator $F$ is completely continuous by an application of the Arzela–Ascoli theorem.

**Theorem 3.2.** *Assume that the conditions* (A1), (A2) *are satisfied. If* $f^0 = 0$ *and* $f_\infty = \infty$, *then the boundary value problem* (1.1), (1.2) *has at least one positive solution that lies in* $P$.

*Proof.* Let $F$ be the cone preserving, completely continuous operator defined by (3.1). Since $f^0 = 0$, we may choose $\xi_1 > 0$ and $\mathcal{H}_1 > 0$ such that

$$f(t,u) \leq \xi_1\phi_p(u), \quad \text{for } \ 0 < u < \mathcal{H}_1,$$

where $\xi_1$ satisfies

$$(\xi_1)^{q-1}\left(\frac{K}{6^{n-3}}\right)\int_0^1 G_1(s,s)\phi_q\left(\int_0^1 G_1(r,r)\mathrm{d}r\right)\mathrm{d}s < 1. \tag{3.3}$$

Now, let $u \in P$ with $\|u\| = \mathcal{H}_1$. Then, by Lemma 2.7 and for $t \in [0,1]$, we have

$$\begin{aligned}Fu(t) &= \int_0^1 G(t,s)\phi_q\left(\int_0^1 G_1(s,r)f\big(r,u(r)\big)\mathrm{d}r\right)\mathrm{d}s\\ &\leq \frac{K}{6^{n-3}}\int_0^1 G_1(s,s)\phi_q\left(\int_0^1 G_1(r,r)\xi_1\phi_p(u)\big)\mathrm{d}r\right)\mathrm{d}s\\ &\leq (\xi_1)^{q-1}\left(\frac{K}{6^{n-3}}\right)\int_0^1 G_1(s,s)\phi_q\left(\int_0^1 G_1(r,r)\mathrm{d}r\right)\mathrm{d}s\|u\|\\ &\leq \|u\|.\end{aligned}$$

Therefore, $\|Fu\| \leq \|u\|$. If we set $\Omega_1 = \{u \in X : \|u\| < \mathcal{H}_1\}$, then

$$\|Fu\| \leq \|u\|, \quad \text{for}\, u \in P \cap \partial\Omega_1. \tag{3.4}$$

Hence by Theorem 2.10, we have $i(F, P \cap \Omega_1, P) = 1$.

Furthermore, since $f_\infty = \infty$, there exist $\xi_2 > 0$ and $\bar{\mathcal{H}}_2 > 0$ such that

$$f(t,u(t)) > \xi_2\phi_p(u), \quad \text{for } \ u \geq \bar{\mathcal{H}}_2,$$

where $\xi_2$ satisfies

$$(\xi_2)^{q-1}\left(\frac{\eta L\mathcal{M}}{4^{n-2}}\right)\left(\frac{11}{96}\right)^{n-3}\int_{s\in I} G_1(s,s)\phi_q\left(\frac{1}{4}\int_{r\in I} G_1(r,r)\mathrm{d}r\right)\mathrm{d}s \geq 1. \tag{3.5}$$

Let $\mathcal{H}_2 = \max\left\{2\mathcal{H}_1, \frac{\bar{\mathcal{H}}_2}{\mathcal{M}}\right\}$. Choose $u \in P$ and $\|u\| = \mathcal{H}_2$. Then,

$$\min_{t\in I} u(t) \geq \mathcal{M}\|u\| \geq \bar{\mathcal{H}}_2.$$

From Lemmas 2.5, 2.7, and for $t \in [0,1]$, we have

$$\begin{aligned} Fu(t) &= \int_0^1 G(t,s)\phi_q\bigg(\int_0^1 G_1(s,r)f(r,u(r))\mathrm{d}r\bigg)\mathrm{d}s \\ &\geq \min_{t\in I}\int_0^1 G(t,s)\phi_q\bigg(\int_0^1 G_1(s,r)f(r,u(r))\mathrm{d}r\bigg)\mathrm{d}s \\ &\geq \left(\frac{\eta L}{4^{n-2}}\right)\left(\frac{11}{96}\right)^{n-3}\int_0^1 G_1(s,s)\phi_q\bigg(\int_0^1 G_1(s,r)f(r,u(r))\mathrm{d}r\bigg)\mathrm{d}s \\ &\geq \left(\frac{\eta L}{4^{n-2}}\right)\left(\frac{11}{96}\right)^{n-3}\int_{s\in I} G_1(s,s)\phi_q\bigg(\frac{1}{4}\int_{r\in I} G_1(r,r)\xi_2\phi_p(u)\mathrm{d}r\bigg)\mathrm{d}s \\ &\geq \left(\frac{\eta L}{4^{n-2}}\right)\left(\frac{11}{96}\right)^{n-3}(\xi_2)^{q-1}\int_{s\in I} G_1(s,s)\phi_q\bigg(\frac{1}{4}\int_{r\in I} G_1(r,r)\mathrm{d}r\bigg)\mathcal{M}\|u\|\mathrm{d}s \\ &\geq \|u\|. \end{aligned}$$

Therefore, $\|Fu(t)\| \geq \|u\|$. So, if we set $\Omega_2 = \{u \in X : \|u\| < \mathcal{H}_2\}$, then

$$\|Fu\| \geq \|u\| \quad \text{for } u \in P \cap \partial\Omega_2. \tag{3.6}$$

By Theorem 2.10, we have $i(F, P\cap\Omega_2, P) = 0$.
If $\mathcal{H}_1 < \mathcal{H}_2$, then $i(F, P\cap(\Omega_2\backslash\bar{\Omega}_1), P) = i(F, P\cap\Omega_2, P) - i(F, P\cap\Omega_1, P) = 0 - 1 = -1$. It follows from Theorem 2.10 that $F$ has a fixed point $u \in P\cap(\Omega_2\backslash\bar{\Omega}_1)$ and that $u$ is the positive solution of the boundary value problem (1.1) and (1.2).
If $\mathcal{H}_1 > \mathcal{H}_2$, then $i(F, P\cap(\Omega_1\backslash\bar{\Omega}_2), P) = i(F, P\cap\Omega_1, P) - i(F, P\cap\Omega_2, P) = 1 - 0 = 1$. It follows from Theorem 2.10 that $F$ has a fixed point $u \in P\cap(\Omega_1\backslash\bar{\Omega}_2)$ and that $u$ is the positive solution of the boundary value problem (1.1) and (1.2). □

We now establish the existence of at least one positive solution of the boundary value problem (1.1), (1.2) for sublinear case.

**Theorem 3.3.** *Assume that the conditions* (A1), (A2) *are satisfied. If* $f_0 = \infty$ *and* $f^\infty = 0$*, then the boundary value problem* (1.1),(1.2) *has at least one positive solution that lies in* $P$.

*Proof.* Let $F$ be the cone preserving, completely continuous operator defined by (3.1). Since $f_0 = \infty$, there exist $\bar{\xi}_1 > 0$ and $J_1 > 0$ such that

$$f(t,u) \geq \bar{\xi}_1\phi_p(u), \qquad \text{for } 0 < u < J_1,$$

where $\bar{\xi}_1 \geq \xi_2$ and $\xi_2$ is given in (3.5). Then, for $u \in P$ and $\|u\| = J_1$, we have

$$
\begin{aligned}
Fu(t) &= \int_0^1 G(t,s)\phi_q\Big(\int_0^1 G_1(s,r)f\big(r,u(r)\big)\mathrm{d}r\Big)\mathrm{d}s.\\
&\geq \min_{t\in I}\int_0^1 G(t,s)\phi_q\Big(\int_0^1 G_1(s,r)f\big(r,u(r)\big)\mathrm{d}r\Big)\mathrm{d}s\\
&\geq \left(\frac{\eta L}{4^{n-2}}\right)\left(\frac{11}{96}\right)^{n-3}\int_0^1 G_1(s,s)\phi_q\Big(\int_0^1 G_1(r,r)f(r,u(r)))\mathrm{d}r\Big)\mathrm{d}s\\
&\geq \left(\frac{\eta L}{4^{n-2}}\right)\left(\frac{11}{96}\right)^{n-3}\int_{s\in I} G_1(s,s)\phi_q\Big(\frac{1}{4}\int_{r\in I} G_1(r,r)\bar{\xi}_1\phi_p(u))\mathrm{d}r\Big)\mathrm{d}s\\
&\geq \left(\frac{\eta L}{4^{n-2}}\right)\left(\frac{11}{96}\right)^{n-3}(\bar{\xi}_1)^{q-1}\int_{s\in I} G_1(s,s)\phi_q\Big(\frac{1}{4}\int_{r\in I} G_1(r,r))\mathrm{d}r\Big)\mathcal{M}\|u\|\mathrm{d}s\\
&\geq \|u\|.
\end{aligned}
$$

Therefore, $\|Fu\| \geq \|u\|$. Now, if we set $\Omega_3 = \{u \in X : \|u\| < J_1\}$, then,

$$\|Fu\| \geq \|u\|, \quad \text{for}\, u \in P \cap \partial\Omega_3. \tag{3.7}$$

By Theorem 2.10, we have $i(F, P\cap\Omega_3, P) = 0$.

Furthermore, since $f^\infty = 0$, then there exist $\bar{\xi}_2 > 0$ and $\bar{J}_2 > 0$ such that

$$f(t,u(t)) \leq \bar{\xi}_2\phi_p(u), \qquad \text{for } \; u \geq \bar{J}_2,$$

where $\bar{\xi}_2 \leq \xi_1$ and $\xi_1$ is given in (3.3).

Now, we consider two cases: $f$ is either bounded or unbounded.
Case (i): Suppose that $f$ is bounded. Then, there exists $N > 0$ such that

$$f(t,u(t)) \leq N, \qquad \text{for } \; 0 < u < \infty.$$

In this case, we may choose

$$J_2 = \max\left\{2J_1, \frac{KN^{q-1}}{6^{n-3}}\int_0^1 G_1(s,s)\phi_q\left(\int_0^1 G_1(r,r)\mathrm{d}r\right)\mathrm{d}s\right\},$$

then for $u \in P$ and $\|u\| = J_2$, we have

$$
\begin{aligned}
Fu(t) &= \int_0^1 G(t,s)\phi_q\Big(\int_0^1 G_1(s,r)f(r,u(r))\mathrm{d}r\Big)\mathrm{d}s\\
&\leq \frac{K}{6^{n-3}}\int_0^1 G_1(s,s)\phi_q\Big(\int_0^1 G_1(s,r)f(r,u(r))\mathrm{d}r\Big)\mathrm{d}s\\
&\leq \frac{K}{6^{n-3}}\int_0^1 G_1(s,s)\phi_q\Big(\int_0^1 G_1(r,r)N\mathrm{d}r\Big)\mathrm{d}s\\
&= \frac{KN^{q-1}}{6^{n-3}}\int_0^1 G_1(s,s)\phi_q\Big(\int_0^1 G_1(r,r)\mathrm{d}r\Big)\mathrm{d}s\\
&\leq J_2 = \|u\|.
\end{aligned}
$$

Case (ii): Suppose that $f$ is unbounded. Choose $J_2 > \max\{2J_1, \bar{J}_2\}$ such that

$$f(t,u) \leq f(t,J_2), \; \text{for } \; 0 < u < J_2.$$

Then, for $u \in P$ and $\|u\| = J_2$, we have

$$\begin{aligned} Fu(t) &= \int_0^1 G(t,s)\phi_q\left(\int_0^1 G_1(s,r)f(r,u(r))\mathrm{d}r\right)\mathrm{d}s \\ &\le \frac{K}{6^{n-3}}\int_0^1 G_1(s,s)\phi_q\left(\int_0^1 G_1(s,r)f(r,J_2)\mathrm{d}r\right)\mathrm{d}s \\ &\le \frac{K}{6^{n-3}}\int_0^1 G_1(s,s)\phi_q\left(\int_0^1 G_1(s,r)\bar{\eta}_2\phi_p(J_2)\mathrm{d}r\right)\mathrm{d}s \\ &\le \frac{K(\bar{\eta}_2)^{q-1}}{6^{n-3}}\int_0^1 G_1(s,s)\phi_q\left(\int_0^1 G_1(r,r)\mathrm{d}r\right)\mathrm{d}s J_2 \\ &\le J_2 = \|u\|. \end{aligned}$$

Therefore, in either case by setting $\Omega_4 = \{u \in P : \|u\| < J_2\}$, we have

$$\|Fu\| \le \|u\|, \qquad \text{for}\, u \in P \cap \partial\Omega_4. \tag{3.8}$$

Hence by Theorem 2.10, $i(F, P \cap \Omega_4, P) = 1$.
If $J_1 < J_2$, then $i(F, P\cap(\Omega_4\setminus\bar{\Omega}_3), P) = i(F, P\cap\Omega_4, P) - i(F, P\cap\Omega_3, P) = 1 - 0 = 1$. It follows from Theorem 2.10 that $F$ has a fixed point $u \in P\cap(\Omega_4\setminus\bar{\Omega}_3)$ and that $u$ is the positive solution of the boundary value problem (1.1) and (1.2).
If $J_1 > J_2$, then $i(F, P\cap(\Omega_3\setminus\bar{\Omega}_4), P) = i(F, P\cap\Omega_3, P) - i(F, P\cap\Omega_4, P) = 0 - 1 = -1$. It follows from Theorem 2.10 that $F$ has a fixed point $u \in P\cap(\Omega_3\setminus\bar{\Omega}_4)$ and that $u$ is the positive solution of the boundary value problem (1.1) and (1.2). □

## 4. Example

Let us consider an example to illustrate our established results.

*Example* 4.1. Consider the boundary value problem

$$(-1)^3[\phi_p(u^{(4)}(t) + k^2u''(t))]'' = f(t,u(t)), \quad 0 \le t \le 1, \tag{4.1}$$

$$\left.\begin{aligned} u(0) &= 0 = u(1), \\ u''(0) &= 0 = u''(1,) \\ u^{(4)}(0) &= 0 = u^{(4)}(1). \end{aligned}\right\} \tag{4.2}$$

For simplicity, we take $p = 2$ and $k = 1$. By algebraic computations, we get $\eta = 0.2474$, $\mathcal{M} = 0.05059$, $K = 0.1693$ and $L = 0.1268$.

(a) If $f(t,u(t)) = u^2e^{t(1-3t)}$, then $f^0 = 0$ and $f_\infty = \infty$. So, all the assumptions of Theorem 3.2 are satisfied and hence, the boundary value problem (4.1), (4.2) has at least one positive solution.
(b) If $f(t,u(t)) = \sqrt{t^2+1}\sqrt[3]{u}$, then $f_0 = \infty$ and $f^\infty = 0$. So, all the assumptions of Theorem 3.3 are satisfied and hence, the boundary value problem (4.1) and (4.2) has at least one positive solution.

**Acknowledgements**

The authors thank the referees for their valuable comments.

## References


[1] Agarwal, R.P.: Boundary Value Problems for Higher Order Differential Equations. World Scientific, Singapore (1986)

[2] Agarwal, R.P., Lu, H., O'Regan, D.: Eigenvalues and the one dimensional $p$-Laplacian. J. Math. Anal. Appl. **266**, 383–400 (2002)

[3] Avery, R.I., Henderson, J.: Existence of three positive pseudo-symmetric solutions for a one-dimensional $p$-Laplacian. J. Math. Anal. Appl. **277**, 395–404 (2003)

[4] Bai, Z., Ge, W.: Solutions of $2n$th Lidstone boundary value problems and dependence on higher order derivatives. J. Math. Anal. Appl. **279**, 442–450 (2003)

[5] Chu, J., Sun, Y., Chen, H.: Positive solutions of Neumann problems with singularities. J. Math. Anal. Appl. **337**, 1267–1272 (2008)

[6] Davis, J.M., Henderson, J.: Triple positive symmetric solutions for a Lidstone boundary value problem. Differ. Equ. Dyn. Syst. **7**, 321–330 (1999)

[7] Davis, J.M., Henderson, J., Wong, P.J.Y.: General Lidstone problems: multiplicity and symmetry of solutions. J. Math. Anal. Appl. **251**, 527–548 (2000)

[8] Deimling, K.: Nonlinear Functional Analysis. Springer, New York (1985)

[9] Diening, L., Lindqvist, P., Kawohl, B.: Mini-workshop: the $p$-Laplacian operator and applications. Oberwolfach Rep. **10**, 433–482 (2013)

[10] Ehme, J., Henderson, J.: Existence and local uniqueness for nonlinear Lidstone boundary value problems. J. Inequalities Pure Appl. Math. **1**(1), 1–9 (2000). (Article 8)

[11] Feng, H., Pang, H., Ge, W.: Multiplicity of symmetric positive solutions for a multi-point boundary value problem with a one-dimensional $p$-Laplacian. Nonlinear Anal. **69**, 3050–3059 (2008)

[12] Guo, Y., Ji, Y., Liu, X.: Multiple positive solutions for some multi-point boundary value problems with $p$-Laplacian. J. Comput. Appl. Math. **216**, 144–156 (2008)

[13] Guo, D., Lakshmikantham, V.: Nonlinear Problems in Abstract Cones. Acadamic Press, San Diego, CA (1988)

[14] Henderson, J., Kosmatov, N.: Positive solutions of the semipositone Neumann boundary value problem. Math. Modell. Anal. **20**, 578–584 (2015)

[15] Jiang, D., Liu, H.: Existence of positive solutions to second order Neumann boundary value problems. J. Math. Res. Expo. **20**, 360–364 (2000)

[16] Krasnoselskii, M.: Positive Solutions of Operator Equations. Noordhoff, Groningen (1964)

[17] Li, Y.: Existence of positive solutions for the cantilever beam equations with fully nonlinear terms. Nonlinear Anal. Real World Appl. **27**, 221–237 (2016)

[18] Li, C., Ge, W.: Existence of positive solutions for $p$-Laplacian singular boundary value problems. Indian J. Pure Appl. Math. **34**, 187–203 (2003)

[19] Li, X., Jiang, D.: Optimal existence theory for single and multiple positive solutions Neumann boundary value problems. Indian J. Pure Appl. Math. **35**, 573–586 (2004)

[20] Li, J., Shen, J.: Existence of three positive solutions for boundary value problems with $p$-Laplacian. J. Math. Anal. Appl. **311**, 457–465 (2005)

[21] Lian, W.C., Wong, F.H.: Existence of positive solutions for higher order generalized $p$-Laplacian BVPs. Appl. Math. Lett. **13**, 35–43 (2000)

[22] Liu, Y., Ge, W.: Multiple positive solutions to a three-point boundary value problem with $p$-Laplacian. J. Math. Anal. Appl. **273**, 293–302 (2003)

[23] Prasad, K.R., Sreedhar, N., Wesen, L.T.: Multiplicity of positive solutions for second order Sturm–Liouville boundary value problems. Creat. Math. Info. **25**, 215–222 (2016)

[24] Prasad, K.R., Wesen, L.T., Sreedhar, N.: Existence of positive solutions for second order undamped Sturm–Liouville boundary value problems. Asian Eur. J. Math. **9**, 1–9 (2016)

[25] Shi, G., Zhang, J.: Positive solutions for higher order singular $p$-Laplacian boundary value problems. Proc. Indian Acad. Sci. **118**, 295–305 (2008)

[26] Sun, Y., Cho, Y.J., O'Regan, D.: Positive solutions for singular second order Neumann boundary value problems via a cone fixed point theorem. Appl. Math. Comput. **210**, 80–86 (2009)

[27] Sun, J.P., Li, W.T.: Multiple positive solutions to second order Neumann boundary value problems. Appl. Math. Comput. **146**, 187–194 (2003)

[28] Sun, J.P., Li, W.T., Cheng, S.S.: Three positive solutions for second order Neumann boundary value problems. Appl. Math. Lett. **17**, 1079–1084 (2004)

[29] Wanjun, L.: Positive solutions for higher order singular nonlinear ordinary differential systems. Int. J. Nonlinear Sci. **9**, 28–40 (2010)

[30] Wang, J.: The existence of positive solutions for the one-dimensional $p$-Laplacian. Proc. Am. Math. Soc. **125**, 2275–2283 (1997)

[31] Wei, Z.L.: Existence of positive solutions for $n$th-order $p$-Laplacian singular sublinear boundary value problems. Appl. Math. Lett. **36**, 25–30 (2014)

[32] Wei, Z.L.: Existence of positive solutions for $n$th- order $p$-Laplacian singular super-linear boundary value problems. Appl. Math. Lett. **50**, 133–140 (2015)

[33] Wong, P.J.Y., Agarwal, R.P.: Eigenvalues of Lidstone boundary value problems. Appl. Math. Comput. **104**, 15–31 (1999)

[34] Yang, J., Wei, Z.: Existence of positive solutions for fourth-order $m$-point boundary value problems with a one dimensional $p$-Laplacian operator. Nonlinear Anal. **71**, 2985–2996 (2009)

[35] Zhang, B., Liu, X.: Existence of multiple symmetric positive solutions of higher order Lidstone problems. J. Math. Anal. Appl. **284**, 672–689 (2003)

[36] Zhang, X., Liu, L.: Positive solutions of fourth order four-point boundary value problems with $p$-Laplacian operators. J. Math. Anal. Appl. **336**, 1414–1423 (2007)

[37] Zhu, Y., Zhu, J.: Existence of multiple positive solutions for $n$-th order $p$-Laplacian $m$-point singular boundary value problems. J. Appl. Math. Comput. **34**, 393–405 (2010)

K. R. Prasad
Department of Applied Mathematics
Andhra University
Visakhapatnam 530 003
India
e-mail: rajendra92@rediffmail.com

N. Sreedhar
Department of Mathematics
GITAM (Deemed to be University)
Visakhapatnam 530 045
India
e-mail: sreedharnamburi13@gmail.com

L. T. Wesen
Department of Mathematics
Jimma University
378-Jimma
Oromia
Ethiopia
e-mail: wesen08@gmail.com

Received: August 24, 2017.
Accepted: December 26, 2017.