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On a Degenerate p-Fractional Kirchhoff Equations Involving Critical Sobolev–Hardy Nonlinearities

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Abstract. In this paper, we study a class of degenerate p-fractional Kirchhoff equations with critical Hardy–Sobolev nonlinearities. By means of the Kajikiya's new version of the symmetric mountain pass lemma, we obtain the existence of infinitely many solutions which tend to zero under a suitable value of λ. The main feature and difficulty of our equations is the fact that the Kirchhoff term M could be zero at zero, that is the equation is *degenerate*. To our best knowledge, our results are new even in the Laplacian and p-Laplacian cases.

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1. Introduction and Main Result

In this paper, we consider a class of degenerate fractional p-Laplacian equation of Schrödinger–Kirchhoff with critical Hardy–Sobolev nonlinearities:

$$M\left(\|u\|^p\right)\left[(-\Delta)_p^s u+V(x)|u|^{p-2}u\right]=\lambda f(x,u)+\frac{|u|^{p_s^*(\alpha)-2}u}{|x|^\alpha},\quad \text{in } \mathbb{R}^N, \tag{1.1}$$

where

$$\|u\|=\left([u]_s^p+\int_{\mathbb{R}^N}V(x)|u|^p\mathrm{d}x\right)^{1/p},\quad [u]_{s,p}=\left(\iint_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y\right)^{1/p}, \tag{1.2}$$

$M:\mathbb{R}_0^+\to\mathbb{R}_0^+$ is a Kirchhoff function, $V:\mathbb{R}^N\to\mathbb{R}^+$ is a scalar potential, $N>ps$, $p\in(1,\infty)$ and $s\in(0,1)$, $p_s^*(\alpha)=p(N-\alpha)/(N-ps)$ is the critical exponent of the fractional Hardy–Sobolev exponent with $\alpha\in[0,ps)$, λ is a positive parameter, and $f:\mathbb{R}^N\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function.

Here, $(-\Delta)_p^s$ is the fractional p–Laplace operator which, up to normalization factors, may be defined by the Riesz potential as follows:

$$(-\Delta)_p^s u(x) = 2\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^N\setminus B_\varepsilon(x)} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+ps}}\mathrm{d}y, \quad x\in\mathbb{R}^N,$$

along any $u\in C_0^\infty(\mathbb{R}^N)$, where $B_\varepsilon(x)=\{y\in\mathbb{R}^N : |x-y|<\varepsilon\}$. For more details about the fractional p-Laplacian, we refer to [15,34] and the references therein.

The main framework for (1.1) is the space E, defined as the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm $\|\cdot\|$, introduced in (1.2). Denote by $D^{s,p}(\mathbb{R}^N)$ the p-fractional Beppo–Levi space, that is the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to Gagliardo semi-norm $[u]_{s,p}$. From Theorem 6.5 in [8] and condition (V) that $E\hookrightarrow D^{s,p}(\mathbb{R}^N)$. While Lemma 2.1 of [12] gives

$$\|u\|_{H_\alpha}\le C[u]_{s,p}, \quad \|u\|_{H_\alpha}^{p_s^*(\alpha)} = \int_{\mathbb{R}^N}|u(x)|^{p_s^*(\alpha)}\frac{\mathrm{d}x}{|x|^\alpha},$$

for all $u\in D^{s,p}(\mathbb{R}^N)$, where C is a positive constant depending only on N,p,s, and α. Thus, the fractional Sobolev embedding $D^{s,p}(\mathbb{R}^N)\hookrightarrow L^{p_s^*}(\mathbb{R}^N)$ and the fractional Hardy–Sobolev embedding $D^{s,p}(\mathbb{R}^N)\hookrightarrow L^{p_s^*(\alpha)}(\mathbb{R}^N,|x|^{-\alpha})$ are continuous, but not compact. However, we can introduce the best fractional critical Hardy–Sobolev constant $H_\alpha=H(N,p,s,\alpha)$, given by the following:

$$H_\alpha = \inf_{u\in D^{s,p}(\mathbb{R}^N)\setminus\{0\}}\frac{[u]_{s,p}^p}{\|u\|_{p_s^*(\alpha)}^p}. \tag{1.3}$$

Of course, the number H_α is strictly positive and it coincides with the best fractional Sobolev constant when $\alpha=0$.

Throughout the paper, without explicit mention, we assume $(\mathcal{V})$ and $(\mathcal{M})$.

$(\mathcal{V})$ *$V:\mathbb{R}^N\to\mathbb{R}^+$ is a continuous function and there exists $V_0>0$, such that* $\inf_{\mathbb{R}^N}V\ge V_0$.

$(\mathcal{M})$ *$M:\mathbb{R}_0^+\to\mathbb{R}_0^+$ is assumed to be continuous and to verify*

- (M_1) *there exists $\theta\in[1,p_s^*(\alpha)/p)$, such that $tM(t)\le\theta\mathscr{M}(t)$ for all $t\in\mathbb{R}_0^+$, where $\mathscr{M}(t)=\int_0^t M(\tau)\mathrm{d}\tau$;*
- (M_2) *for any $\tau>0$ there exists $m=m(\tau)>0$, such that $M(t)\ge m$ for all $t\ge\tau$;*
- (M_3) *there exists $m_0>0$, such that $M(t)\ge m_0t^{\theta-1}$ for all $t\in[0,1]$.*

A prototype for M, due to Kirchhoff, is given by

$$M(t)=a+b\theta t^{\theta-1}, \quad a,\, b\ge 0,\ a+b>0,\ \theta\ge 1. \tag{1.4}$$

When $M(t)\ge c>0$ for all $t\in\mathbb{R}_0^+$, Kirchhoff equations like (1.1) are said to be *nondegenerate* and this happens for example if $a>0$ in the model case (1.4). While, if $M(0)=0$ but $M(t)>0$ for all $t\in\mathbb{R}^+$, Kirchhoff equations as (1.1) are called *degenerate*. Of course, for (1.4), this occurs when $m_0=0$.

Concerning the function $f:\mathbb{R}^N\times\mathbb{R}\to\mathbb{R}$, we suppose that

(f_1) *There exist* $q\in(1,\theta p)$ *and a nonnegative function* $w\in L^{\vartheta}(\mathbb{R}^N)$, *such that* $|f(x,t)|\le w(x)t^{q-1}$ *for all* $(x,t)\in\mathbb{R}^N\times\mathbb{R}^+$, *where* $\vartheta=p_s^*(\alpha)/(p_s^*(\alpha)-q)$.

(f_2) *There exist* $\xi\in(1,p)$, $\delta>0$, $a_0>0$, *and a nonempty open subset* Ω *of* $\mathbb{R}^N$, *such that*

$$F(x,t)\ge a_0t^{\xi}\quad \textit{for all } (x,t)\in\Omega\times(0,\delta).$$

A simple example of f, verifying (f_1)–(f_2), is $f(x,t)=(1+|x|^2)^{(l-2)/2}(t^+)^{l-1}$ with $1<l<p$, where $t^+=\max\{t,0\}$.

In the appendix of the recent paper [13], the authors provide a detailed discussion about the physical meaning underlying the fractional Kirchhoff problems and their applications. Indeed, they propose in [13] a stationary Kirchhoff variational problem, which models, as a special significant case, the nonlocal aspect of the tension arising from nonlocal measurements of the fractional length of the string. In this case, M measures the change of the tension on the string caused by the change of its length during the vibration. For this, the fact that $M(0)=0$ means that the base tension of the string is zero, a very realistic model.

Several recent papers are focused both on theoretical aspects and applications related to nonlocal fractional models. Always in [13], the following critical fractional problem on Ω bounded was studied, for the first time, in the literature:

$$\begin{cases} M\left(\displaystyle\iint_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y\right)(-\Delta)^su=\lambda f(x,u)+|u|^{2_s^*-2}u & \text{in } \Omega\\ u=0 & \text{in } \mathbb{R}^N\backslash\Omega. \end{cases}\tag{1.5}$$

The authors prove the existence of a nontrivial nonnegative solution for (1.5) on a nondegenerate setting, combining a truncation argument with a concentration compactness principle. The degenerate case of equation (1.5) is studied in [4], by introducing a new technical approach based on the asymptotic property of the critical mountain pass level. Furthermore, the existence of a solution for different critical fractional Kirchhoff problems set on the whole space $\mathbb{R}^N$ is given in [2,7,11,12,24,29] adapting the variational technique developed in [4]. For multiplicity results, we refer to [27], where they consider a nonhomogeneus fractional Schrödinger–Kirchhoff equation. By combining the mountain pass theorem with Ekeland's variational principle, [27] establish the existence of two solutions on a nondegenerate situation. With a similar approach of [27,32] prove the existence of two solutions for a degenerate Kirchhoff equation in $\mathbb{R}^N$ with a concave–convex nonlinearity. In [33], by the Fountain theorem and the dual Fountain theorem, the authors get the existence of infinitely many solutions for a symmetric subcritical Kirchhoff problem on Ω, with suitable nondegenerate assumptions for M. The existence of infinitely many solutions is still proved in [26,28] using Krasnoselskii's genus theory, under degenerate frameworks. Moreover, to get infinitely many solutions, Krasnoselskii's genus theory is used in [10] for a critical problem similar to (1.5) and Kajikiya's new version of the symmetric mountain pass lemma is applied in [31] for a critical problem similar to (1.1), but just on the nondegenerate case. Finally, in [34], a subcritical degenerate Kirchhoff system on Ω is studied using the symmetric mountain pass theorem given by Ambrosetti and Rabinowitz in [30].

Motivated by the above works, in the present paper, we provide the existence of infinitely many solutions for (1.1) on a degenerate setting. As far as we know, our multiplicity result for degenerate Kirchhoff equations similar to (1.1) is new even in the Laplacian and p-Laplacian cases. Indeed, while there is a wide literature concerning the study of multiplicity results for critical Kirchhoff problems under a nondegenerate setting; see, for example, [9,14,17–20,23,35–37], very few attempts have been made to cover also the degenerate case. We refer to [21,22], where the authors just consider M like the prototype in (1.4) with $a = 0$. Furthermore, they are able to give multiplicity results either when $N = 4$ or by considering a small perturbation. Here, using a different approach, we allow M to be more general in (1.1).

Denoting with $\mathcal{J}_\lambda : E \to E$ the Euler–Lagrange functional related to variational equation (1.1), we are ready to state the main result of our paper as follows.

Theorem 1.1. *Let $M(0) = 0$, $N \in (ps, \infty)$, $q \in (1, p)$, with $s \in (0, 1)$ and $p \in (1, \infty)$. Assume that M and V satisfy assumptions (M_1)–(M_3) and $(\mathcal{V})$. Then, there exists $\overline{\lambda} > 0$, such that for any $\lambda \in (0, \overline{\lambda})$, Eq. (1.1) admits a sequence of solutions $\{u_n\}_n$ in E with $\mathcal{J}_\lambda(u_n) \le 0$, $\mathcal{J}_\lambda(u_n) \to 0$ and $\{u_n\}_n$ converges to zero as $n \to \infty$.*

The proof of Theorem 1.1 is mainly based on the application of the symmetric mountain pass lemma, introduced by Kajikiya in [16]. For this, we need a truncation argument which allow us to control from below functional $\mathcal{J}_\lambda$. Furthermore, as usual in elliptic problems involving critical nonlinearities, we must pay attention to the lack of compactness at critical level $L^{p_s^*(\alpha)}(\mathbb{R}^N)$. To overcome this difficulty, we fix parameter λ under a suitable threshold strongly depending on assumptions (M_2) and (M_3).

The paper is organized as follows. In Sect. 2, we discuss the variational formulation of the Eq. (1.1) and introduce some topological notions. In Sect. 3, we prove the Palais–Smale condition for the functional $\mathcal{J}_\lambda$. In Sect. 4, we introduce a truncation argument for our functional. In Sect. 5, we prove Theorem 1.1.

2. Preliminaries

In this section, we first give the variational formulation of Eq. (1.1) and then provide some useful technical results, which will be used in the sequel.

Let $L^q(\mathbb{R}^N, w)$ be the weighted Lebesgue space, endowed with the norm:

$$\|u\|_{q,w}^q = \int_{\mathbb{R}^N} w(x)|u(x)|^q \mathrm{d}.$$

By Proposition A.6 of [3], the Banach space $L^q(\mathbb{R}^N, w) = (L^q(\mathbb{R}^N, w), \|\cdot\|_{q,w})$ is uniformly convex. Furthermore, by Lemma 2.1 of [7], the embedding $D^{s,p}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N, w)$ is compact, with

$$\|u\|_{q,w} \leq C_w [u]_{s,p} \quad \text{for all } u \in D^{s,p}(\mathbb{R}^N), \tag{2.1}$$

and $C_w = H_0^{-1/p} \|w\|_r^{1/q} > 0$, where $S = S(N,p,s)$ is the best fractional critical Sobolev constant, given by the following:

$$S = \inf_{u \in D^{s,p}(\mathbb{R}^N) \setminus \{0\}} \frac{[u]_{s,p}^p}{\|u\|_{p_s^*}^p}. \tag{2.2}$$

Of course, number S is positive, since Theorem 1 of [25].

We say that $u \in D^{s,p}(\mathbb{R}^N)$ is a (weak) solution of Eq. (1.1), if u satisfies

$$M(\|u\|^p)\langle u, \varphi\rangle = \lambda \int_{\mathbb{R}^N} f(x,u)\varphi \mathrm{d}x + \langle u, \varphi\rangle_{H_\alpha},$$

for all $\varphi \in E$, where

$$\begin{aligned}
\langle u, \varphi\rangle &= \langle u, \varphi\rangle_{s,p} + \langle u, \varphi\rangle_{p,V}, \\
\langle u, \varphi\rangle_{s,p} &= \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}[u(x)-u(y)]\cdot[\varphi(x)-\varphi(y)]}{|x-y|^{N+sp}} \mathrm{d}x\mathrm{d}y, \\
\langle u, \varphi\rangle_{p,V} &= \int_{\mathbb{R}^N} V(x)|u(x)|^{q-2}u(x)\varphi(x)\mathrm{d}x, \\
\langle u, \varphi\rangle_{H_\alpha} &= \int_{\mathbb{R}^N} |u(x)|^{p_s^*(\alpha)-2}u(x)\varphi(x)\frac{\mathrm{d}x}{|x|^\alpha}.
\end{aligned}$$

Equation (1.1) has a variational structure and $\mathcal{J}_\lambda : E \to \mathbb{R}$, defined by the following:

$$\mathcal{J}_\lambda(u) = \frac{1}{p}\mathscr{M}(\|u\|^p) - \lambda \int_{\mathbb{R}^N} F(x,u)\mathrm{d}x - \frac{1}{p_s^*(\alpha)} \int_{\mathbb{R}^N} \frac{|u|^{p_s^*(\alpha)}}{|x|^\alpha}\mathrm{d}x,$$

is the underlying functional associated with (1.1). Essentially, as shown in Lemma 4.2 of [7], the functional $\mathcal{J}_\lambda$ is of class $C^1(E, \mathbb{R})$.

To handle the degenerate Kirchhoff coefficient, we need appropriate lower and upper bounds for M, given by (M_1) and (M_2). Indeed, condition (M_2) implies that $M(t) > 0$ for any $t > 0$, and consequently, by (M_1) for all $t \in (0,1]$, we have $M(t)/\mathscr{M}(t) \leq \theta/t$. Thus, integrating on $[t,1]$, with $0 < t < 1$, we get

$$\mathscr{M}(t) \geq \mathscr{M}(1)t^\theta, \tag{2.3}$$

and (2.3) holds for all $t \in [0,1]$ by continuity. Hence, (M_3) is a stronger request. Furthermore (2.3) is compatible with (M_3), since integrating (M_3), we have $\mathscr{M}(t) \geq m_0 t^\theta/\theta$ for any $t \in [0,1]$, from which $\mathscr{M}(1) \geq m_0/\theta$.

Similarly, for any $\varepsilon > 0$, there exists $\delta_\varepsilon = \mathscr{M}(\varepsilon)/\varepsilon^\theta > 0$, such that

$$\mathscr{M}(t) \leq \delta_\varepsilon t^\theta \quad \text{for any } t \geq \varepsilon. \tag{2.4}$$

We conclude this section recalling the symmetric mountain pass lemma introduced by Kajikiya in [16]. The proof of Theorem 1.1 is based on the application of the following result.

Lemma 2.1. *Let E be an infinite-dimensional space and $J \in C^1(E, \mathbb{R})$ and suppose the following conditions hold.*

(J_1) *$J(u)$ is even, bounded from below, $J(0) = 0$ and $J(u)$ satisfies the local Palais–Smale condition, i.e., for some $\bar{c} > 0$, in the case when every sequence $\{u_n\}_n$ in E satisfying $\lim_{n\to\infty} J(u_n) = c < \bar{c}$ and $\lim_{n\to\infty} \|J'(u_n)\|_{E'} = 0$ has a convergent subsequence;*

(J_2) *For each $n \in \mathbb{N}$, there exists an $A_n \in \Sigma_n$, such that $\sup_{u\in A_n} J(u) < 0$.*

Then, either (i) or (ii) below holds.

(*i*) *There exists a sequence $\{u_n\}_n$, such that $J'(u_n) = 0$, $J(u_n) < 0$, and $\{u_n\}$ converges to zero.*

(*ii*) *There exist two sequences $\{u_n\}_n$ and $\{v_n\}_n$, such that $J'(u_n) = 0$, $J(u_n) < 0$, $u_n \neq 0$, $\lim_{n\to\infty} u_n = 0$, $J'(v_n) = 0$, $J(v_n) < 0$, $\lim_{n\to\infty} J(v_n) = 0$, and $\{v_n\}_n$ converges to a nonzero limit.*

3. The Palais–Smale Condition

Throughout this paper, we consider $N > ps$ with $s \in (0, 1)$ and $p \in (1,\infty)$, $M(0) = 0$ and we assume that M and V satisfy (M_1)–(M_3) and $(\mathcal{V})$, without further mentioning.

To apply Lemma 2.1, we discuss now the compactness property for the functional $\mathcal{J}_\lambda$, given by the Palais–Smale condition. We recall that $\{u_n\}_n \subset E$ is a Palais–Smale sequence for $\mathcal{J}_\lambda$ at level $c \in \mathbb{R}$ if

$$\mathcal{J}_\lambda(u_n) \to c \quad \text{and} \quad \mathcal{J}'_\lambda(u_n) \to 0 \quad \text{in } E' \text{ as } n \to \infty. \tag{3.1}$$

We say that $\mathcal{J}_\lambda$ satisfies the Palais–Smale condition at level c if any Palais–Smale sequence $\{u_n\}_n$ at level c admits a convergent subsequence in E.

Before going to prove Theorem 1.1, we first give some auxiliary lemmas.

Lemma 3.1. *If (f_3) holds, then there exist $\rho \in (0, 1]$ and $\lambda_0 = \lambda_0(\rho) > 0$, $\ell = \ell(\rho)$, such that $\mathcal{J}_\lambda(u) \geq \ell > 0$ for any $u \in E$, with $\|u\| = \rho$, and for all $\lambda \leq \lambda_0$.*

Proof. For all $u \in E$, with $\|u\| \leq 1$, by (f_3), (2.1), and (2.3), there exists a positive constant K_w, such that

$$\begin{aligned}
\mathcal{J}_\lambda(u) &\geq \frac{\mathscr{M}(1)}{p}\|u\|^{\theta p} - \lambda \int_\Omega w(x)|u|^q \mathrm{d}x - \frac{1}{p_s^*(\alpha)}\|u\|_{H_\alpha}^{p_s^*(\alpha)} \\
&\geq \frac{\mathscr{M}(1)}{p}\|u\|^{\theta p} - \lambda \|w\|_{L^{\frac{p_s^*}{p_s^*-q}}(\mathbb{R}^N)} \|u\|^p_{L^{p_s^*}(\mathbb{R}^N)} \\
&\quad - \frac{1}{H_\alpha^{p_s^*(\alpha)/p} p_s^*(\alpha)}\|u\|^{p_s^*(\alpha)} \\
&\geq \frac{\mathscr{M}(1)}{p}\|u\|^{\theta p} - \lambda K_w \|u\|^q - \frac{1}{H_\alpha^{p_s^*(\alpha)/p} p_s^*(\alpha)}\|u\|^{p_s^*(\alpha)}.
\end{aligned} \tag{3.2}$$

By the Young inequality, for any $\varepsilon > 0$, we have

$$\lambda K_w \|u\|^q \leq \varepsilon \|u\|^{\theta p} + \varepsilon^{-q/(\theta p - q)} (\lambda K_w)^{\theta p/(\theta p - q)},$$

since $q < \theta p$. Thus, for $\varepsilon = \mathscr{M}(1)/2p$,

$$\mathcal{J}_\lambda(u) \geq \frac{\mathscr{M}(1)}{2p}\|u\|^{\theta p} - \left(\frac{2p}{\mathscr{M}(1)}\right)^{-q/(\theta p - q)} (\lambda K_w)^{\theta p/(\theta p - q)} - \frac{1}{H_\alpha^{p_s^*(\alpha)/p} p_s^*(\alpha)} \|u\|^{p_s^*(\alpha)}.$$

Since $\theta p < p_s^*(\alpha)$, the function

$$\eta(t) = \frac{\mathscr{M}(1)}{2p} t^{\theta p} - \frac{1}{H_\alpha^{p_s^*(\alpha)/p} p_s^*(\alpha)} t^{p_s^*(\alpha)}, \quad t \in [0,1]$$

admits a maximum at some $\rho \in (0,1]$ small enough, that is $\eta(\rho) = \max_{t\in[0,1]} \eta(t) > 0$. Thus, let

$$\lambda_0 = \frac{1}{2K_w}\eta(\rho)^{(\theta p - q)/(\theta p)} \left(\frac{\mathscr{M}(1)}{p}\right)^{q/\theta p}.$$

Then, for all $u \in E$, with $\|u\| = \rho \leq 1$, and for all $\lambda \leq \lambda_0$, we get

$$\mathcal{J}_\lambda(u) \geq \eta(\rho) - 2^{\frac{\theta p}{\theta p - q}} \left(\frac{\mathscr{M}(1)}{p}\right)^{q/(q-\theta p)} (K_w \lambda_0)^{\theta p/(\theta p - q)} = \ell > 0,$$

being $q < \theta p$. The lemma is so proved. □

Set

$$c_\lambda = \inf\{\mathcal{J}_\lambda(u) : u \in \overline{B_\rho}\},$$

where $B_\rho = \{u \in E : \|u\| < \rho\}$ and $\rho \in (0,1]$ is the number determined in Lemma 3.1.

Lemma 3.2. *If (f_1) and (f_2) hold, then $c_\lambda < 0$ for each $\lambda \in (0, \lambda_0]$.*

Proof. Let $x_0 \in \Omega$ and $R \in (0,1)$ be so small that $B(x_0, 2R) \subset \Omega$, where Ω is given in (f_2). Choose a function $\varphi \in C_0^\infty(B(x_0, 2R))$, such that $0 \leq \varphi \leq 1$, $0 < \|\varphi\| \leq \rho$ and $\bar{\varphi} = \int_{B(x_0,2R)} |\varphi|^\xi \mathrm{d}x > 0$. Fix $\lambda \in (0, \lambda_0]$. Then, by (f_2) and continuity of M, for all t, with $0 < t < \min\{1, \delta\}$, we have

$$\begin{aligned}
\mathcal{J}_\lambda(t\varphi) &= \frac{1}{p}\mathscr{M}(\|t\varphi\|^p) - \lambda \int_\Omega F(x, t\varphi)\mathrm{d}x - \frac{1}{p_s^*(\alpha)}\|tu\|_{H_\alpha}^{p_s^*(\alpha)} \\
&\leq \frac{1}{p}\left(\sup_{0\leq\xi\leq\rho^p} M(\xi)\right)\|\varphi\|^p t^p - \lambda\int_\Omega F(x, t\varphi)\mathrm{d}x \\
&\leq \frac{\rho^p}{p}\left(\sup_{0\leq\xi\leq\rho^p} M(\xi)\right) t^p - \lambda\left(a_0 \int_{B(x_0,2R)} |\varphi|^\xi \mathrm{d}x\right) t^\xi \\
&= \frac{\rho^p}{p} M_1 t^p - \lambda a_0 \bar{\varphi}\, t^\xi.
\end{aligned}$$

Since $1 < \xi < p$, fixing $t > 0$ even smaller, so that

$$t < \min\{1, \delta, (\lambda p a_0 \bar{\varphi} \rho^{-p}/M_1)^{1/(p-\xi)}\},$$

we have that $t\varphi \in B_\rho$ and $\mathcal{J}_\lambda(t\varphi) < 0$. This proves that $c_\lambda < 0$ for all $\lambda \in (0, \lambda_0]$, as stated. □

Lemma 3.3. *If* (f_3) *and* (f_4) *hold, then there exists* $\lambda_1 > 0$*, such that, up to a subsequence,* $(u_n)_n$ *strongly converges to some* u_λ *in* E *for all* $\lambda \in (0, \lambda_1]$.

Proof. Due to the degenerate nature of (1.1), two situations must be considered: either $\inf_{n\in\mathbb{N}} \|u_n\| = d_\lambda > 0$ or $\inf_{n\in\mathbb{N}} \|u_n\| = 0$. For this, we divide the proof in two cases.

• *Case* $\inf_{n\in\mathbb{N}}[u_n]_{s,p} = d_\lambda > 0$. By Lemmas 3.1 and 3.2 and the Ekeland variational principle, applied in $\overline{B}_\rho$, there exists a sequence $\{u_n\}_n \subset B_\rho$, such that

$$c_\lambda \le \mathcal{J}_\lambda(u_n) \le c_\lambda + 1/n \quad \text{and} \quad \mathcal{J}_\lambda(v) \ge \mathcal{J}_\lambda(u_n) - \|u_n - v\|/n \tag{3.3}$$

for all $n \in \mathbb{N}$ and for any $v \in \overline{B}_\rho$. Fixed $n \in \mathbb{N}$, for all $\upsilon \in S_E$, where $S_E = \{u \in E : \|u\| = 1\}$, and for all $\sigma > 0$ so small that $u_n + \sigma\upsilon \in \overline{B}_\rho$, we have

$$\mathcal{J}_\lambda(u_n + \sigma\upsilon) - \mathcal{J}_\lambda(u_n) \ge -\frac{\sigma}{n}$$

by (3.3). Since $\mathcal{J}_\lambda$ is Gâteaux differentiable in E, we get

$$\langle \mathcal{J}'_\lambda(u_n), \upsilon\rangle_{E',E} = \lim_{\sigma\to 0} \frac{\mathcal{J}_\lambda(u_n + \sigma\upsilon) - \mathcal{J}_\lambda(u_n)}{\sigma} \ge -\frac{1}{n}$$

for all $\upsilon \in S_E$. Hence

$$|\langle \mathcal{J}'_\lambda(u_n), \upsilon\rangle_{E',E}| \le \frac{1}{n},$$

since $\upsilon \in S_E$ is arbitrary. Consequently, $\mathcal{J}'_\lambda(u_n) \to 0$ in E' as $n \to \infty$, and clearly, up to a subsequence, the bounded sequence $\{u_n\}_n$ weakly converges to some $u_\lambda \in \overline{B}_\rho$ and has the following properties:

$$\begin{aligned} &u_n \rightharpoonup u_\lambda \text{ in } E, && \|u_n\| \to \mu_\lambda,\\ &u_n \rightharpoonup u_\lambda \text{ in } L^{p_s^*(\alpha)}(\mathbb{R}^N, |x|^{-\alpha}), && \|u_n - u_\lambda\|_{H_\alpha} \to \kappa_\lambda,\\ &u_n \to u_\lambda \text{ in } L^q(\mathbb{R}^N, w), && u_n \to u_\lambda \text{ a.e. in } \mathbb{R}^N \end{aligned} \tag{3.4}$$

as $n \to \infty$, by Lemma 2.1 of [7]. Clearly, $\mu_\lambda > 0$, since we are in the case in which $d_\lambda > 0$. Therefore, $M(\|u_n\|^p) \to M(\mu_\lambda^p) > 0$ as $n \to \infty$, by continuity of M and the fact that 0 is the unique zero of M.

For any subset $U \subset \mathbb{R}^N$, by (f_3), we have

$$\int_U |f(x, u_n)(u_n - u_\lambda)|\mathrm{d}x \le \|w\|_{L^{\frac{p_s^*(\alpha)}{p_s^*(\alpha)-q}}(U)} \|u_n - u_\lambda\|_{p_s^*(\alpha)} \le C\|w\|_{L^{\frac{p_s^*(\alpha)}{p_s^*(\alpha)-q}}(U)}.$$

It follows from $w \in L^{\frac{p_s^*(\alpha)}{p_s^*(\alpha)-q}}(\mathbb{R}^N)$ that sequence $\{f(x, u_n)(u_n - u_\lambda)\}_n$ is equi-integrable in $L^1(\mathbb{R}^N)$. Clearly, $f(x, u_n)(u_n - u_\lambda) \to 0$ a.e. in $\mathbb{R}^N$ as $n \to \infty$. Hence, the Vitali convergence theorem implies that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} f(x, u_n)(u_n - u_\lambda)\mathrm{d}x = 0. \tag{3.5}$$

A similar argument shows that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} f(x, u_\lambda)(u_n - u_\lambda)\mathrm{d}x = 0 \tag{3.6}$$

and

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} f(x,u_n)u_\lambda \mathrm{d}x = \lim_{n\to\infty}\int_{\mathbb{R}^N} f(x,u_\lambda)u_\lambda \mathrm{d}x. \tag{3.7}$$

Furthermore, by (f_2) for equi-integrability, we get

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} F(x,u_n)\mathrm{d}x = \lim_{n\to\infty}\int_{\mathbb{R}^N} F(x,u_\lambda)\mathrm{d}x. \tag{3.8}$$

As shown in the proof of Lemma 2.4 of [7], the sequence $\{\mathcal{U}_n\}_n$, defined in $\mathbb{R}^{2N}\backslash\mathrm{Diag}\,\mathbb{R}^{2N}$ by

$$(x,y)\mapsto \mathcal{U}_n(x,y) = \frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))}{|x-y|^{(N+ps)/p'}},$$

is bounded in $L^{p'}(\mathbb{R}^{2N})$ as well as $\mathcal{U}_n \to \mathcal{U}_\lambda$ a.e. in $\mathbb{R}^{2N}$, where

$$\mathcal{U}_\lambda(x,y) = \frac{|u_\lambda(x)-u_\lambda(y)|^{p-2}(u_\lambda(x)-u_\lambda(y))}{|x-y|^{(N+ps)/p'}}.$$

Thus, up to a subsequence, we get $\mathcal{U}_n \rightharpoonup \mathcal{U}_\lambda$ in $L^{p'}(\mathbb{R}^{2N})$, and so as $n\to\infty$. Furthermore, $|u_n|^{p-2}u_n \rightharpoonup |u_\lambda|^{p-2}u_\lambda$ in $L^{p'}(\mathbb{R}^N, V)$ by Proposition A.8 of [3]. Hence

$$\langle u_n,\varphi\rangle \to \langle u,\varphi\rangle \tag{3.9}$$

for any $\varphi\in E$, since $|\varphi(x)-\varphi(y)|\cdot|x-y|^{-(N+ps)/p}\in L^p(\mathbb{R}^{2N})$ and $\varphi\in L^p(\mathbb{R}^N,V)$. Similarly, (3.4) and Proposition A.8 of [3] imply that $|u_n|^{p_s^*(\alpha)-2}u_n \rightharpoonup |u_\lambda|^{p_s^*(\alpha)-2}u_\lambda$ in $L^{p_s^*(\alpha)'}(\mathbb{R}^N,|x|^{-\alpha})$, from which as $n\to\infty$

$$\langle u_n,\varphi\rangle_{H_\alpha} \to \langle u_\lambda,\varphi\rangle_{H_\alpha}, \tag{3.10}$$

for any $\varphi\in E$, Then, (3.4), (3.9), and (3.10) give

$$M(\mu_\lambda^p)\langle u_\lambda,\varphi\rangle = \lambda\int_{\mathbb{R}^N} f(x,u_\lambda)\varphi(x)\mathrm{d}x + \langle u,\varphi\rangle_{H_\alpha},$$

for any $\varphi\in E$. Hence, u_λ is a critical point of the $C^1(E)$ functional

$$\widetilde{\mathcal{J}}_\lambda(u) = \frac{1}{p}M(\mu_\lambda^p)\|u\|^p - \lambda\int_{\mathbb{R}^N} F(x,u)\mathrm{d}x - \frac{1}{p_s^*(\alpha)}\|u\|_{H_\alpha}^{p_s^*(\alpha)}. \tag{3.11}$$

By the Hölder inequality, we have

$$|\langle u,v\rangle| \le \|u\|^{p-1}\|v\| \quad \text{for all } u,v\in E$$

and so, for any $u\in E$, the functional $\langle u,\cdot\rangle$ is linear and continuous on E. Furthermore, using again (3.4) and the celebrated Brézis and Lieb lemma of [6]

$$\begin{aligned}\|u_n\|^p &= \|u_n-u\|^p + \|u\|^p + o(1),\\ \|u_n\|_{H_\alpha}^{p_s^*(\alpha)} &= \|u_n-u\|_{H_\alpha}^{p_s^*(\alpha)} + \|u\|_{H_\alpha}^{p_s^*(\alpha)} + o(1)\end{aligned} \tag{3.12}$$

as $n\to\infty$. Consequently, we deduce from (2.1), (3.4), (3.10), (3.11), and (3.12) that as $n\to\infty$

$$\begin{aligned} o(1) &= \langle \mathcal{J}_\lambda'(u_n) - \widetilde{\mathcal{J}}_\lambda'(u_\lambda), u_n - u_\lambda\rangle \\ &= M(\|u_n\|^p)\|u_n\|^p + M(\mu_\lambda^p)\|u_\lambda\|^p - M(\|u_n\|^p)\langle u_n, u_\lambda\rangle - M(\mu_\lambda^p)\langle u_\lambda, u_n\rangle \\ &\quad - \int_{\mathbb{R}^N} (|u_n(x)|^{p_s^*(\alpha)-2}u_n(x) - |u_\lambda(x)|^{p_s^*(\alpha)-2}u_\lambda(x)) \\ &\qquad (u_n(x)-u_\lambda(x))\frac{\mathrm{d}x}{|x|^\alpha} \\ &= M(\mu_\lambda^p)(\mu_\lambda^p - \|u_\lambda\|^p) - \|u_n\|_{H_\alpha}^{p_s^*(\alpha)} + \|u_\lambda\|_{H_\alpha}^{p_s^*(\alpha)} + o(1) \\ &= M(\mu_\lambda^p)\|u_n - u\|^p - \|u_n - u_\lambda\|_{H_\alpha}^{p_s^*(\alpha)} + o(1). \end{aligned} \tag{3.13}$$

Therefore, we have proved the crucial formula

$$M(\mu_\lambda^p)\lim_{n\to\infty}\|u_n - u\|^p = \lim_{n\to\infty}\|u_n - u_\lambda\|_{H_\alpha}^{p_s^*(\alpha)}. \tag{3.14}$$

Hence, using the notation in (3.4), we have

$$M(\mu_\lambda^p)\lim_{n\to\infty}\|u_n - u\|^p = M(\mu_\lambda^p)(\mu_\lambda^p - \|u_\lambda\|^p) = \lim_{n\to\infty}\|u_n - u_\lambda\|_{H_\alpha}^{p_s^*(\alpha)} = \kappa_\lambda^{p_s^*(\alpha)}. \tag{3.15}$$

By (1.3), for all $\lambda>0$, we have

$$\kappa_\lambda^{p_s^*(\alpha)} \ge H_\alpha M(\mu_\lambda^p)\kappa_\lambda^p. \tag{3.16}$$

By sending $n\to\infty$ in (3.4), and by (M_1) and (3.12), we have

$$\begin{aligned} c_\lambda &= \frac{1}{p}\mathscr{M}(\|u_n\|^p) - \lambda\int_{\mathbb{R}^N} F(x,u_n)\mathrm{d}x - \frac{1}{p_s^*(\alpha)}\|u_n\|_{H_\alpha}^{p_s^*(\alpha)} + o(1) \\ &\ge \frac{1}{\theta p}M(\|u_n\|^p)\|u_n\|^p - \lambda\int_{\mathbb{R}^N} F(x,u_\lambda)\mathrm{d}x - \frac{1}{p_s^*(\alpha)}\|u_n\|_{H_\alpha}^{p_s^*(\alpha)} + o(1) \\ &= \frac{1}{\theta p}M(\mu_\lambda^p)\|u_n - u_\lambda\|^p + \frac{1}{\theta p}M(\mu_\lambda^p)\|u_\lambda\|^p - \lambda\int_{\mathbb{R}^N} F(x,u_\lambda)\mathrm{d}x \\ &\quad - \frac{1}{p_s^*(\alpha)}\|u_n - u_\lambda\|_{H_\alpha}^{p_s^*(\alpha)} - \frac{1}{p_s^*(\alpha)}\|u_\lambda\|_{H_\alpha}^{p_s^*(\alpha)} + o(1) \\ &\ge \frac{1}{\theta p}M(\mu_\lambda^p)\|u_\lambda\|^p + \left(\frac{1}{\theta p} - \frac{1}{p_s^*(\alpha)}\right)\|u_n - u_\lambda\|_{H_\alpha}^{p_s^*(\alpha)} - \frac{1}{p_s^*(\alpha)}\|u_\lambda\|_{H_\alpha}^{p_s^*(\alpha)} \\ &\quad - \lambda\int_{\mathbb{R}^N} F(x,u_\lambda)\mathrm{d}x + o(1). \end{aligned} \tag{3.17}$$

By (f_2), clearly, $|f(x,u_\lambda)|\le w(x)|u_\lambda|^q$ and $|F(x,u_\lambda)|\le w(x)|u_\lambda|^q$ for all $x\in\mathbb{R}^N$ and for all $\lambda\in(0,\lambda_0]$. In view of the choice of ρ in Lemma 3.1, we know that ρ is independent of λ. Thus, $\{u_\lambda\}_{\lambda\in(0,\lambda_0]}$ is uniformly bounded in E. Furthermore, there exists $C>0$, which does not depend on λ, such that

$$\int_{\mathbb{R}^N} F(x,u_\lambda)\mathrm{d}x \le C \quad \text{and} \quad \Big|\int_{\mathbb{R}^N} f(x,u_\lambda)u_\lambda\mathrm{d}x\Big| \le C. \tag{3.18}$$

Hence, by (3.15), we deduce

$$\left(\frac{1}{\theta p} - \frac{1}{p_s^*(\alpha)}\right) \|u_n - u_\lambda\|_{H_\alpha}^{p_s^*(\alpha)} \le c_\lambda + 2C\lambda + o(1).$$

This, together with Lemma 3.1, implies that

$$\lim_{\lambda\to 0} \kappa_\lambda = \lim_{\lambda\to 0}\lim_{n\to\infty} \|u_n - u_\lambda\|_{H_\alpha} = 0. \tag{3.19}$$

Here, we claim there exists a $\lambda_* \in (0, \lambda_0]$, such that $\kappa_\lambda = 0$ for all $\lambda \in (0, \lambda_1]$. Otherwise, there exists a sequence $\lambda_n \to 0$, such that $\kappa_{\lambda_n} > 0$. By (3.16), we have

$$\kappa_{\lambda_n}^{p_s^*(\alpha)-p} \ge H_\alpha M(\mu_{\lambda_n}^p),$$

from which, sending $n \to \infty$, considering (3.17), the continuity of M and the fact that 0 is the unique zero of M, we deduce $\mu_{\lambda_n} \to 0$ as $n \to \infty$. Using (3.16), it follows that:

$$\kappa_{\lambda_n}^{p_s^*(\alpha)-p} = \left(\kappa_{\lambda_n}^{p_s^*(\alpha)}\right)^{\frac{ps-\alpha}{N-\alpha}} = M(\mu_{\lambda_n}^p)^{\frac{ps-\alpha}{N-\alpha}} \left(\mu_{\lambda_n}^p - \|u_{\lambda_n}\|^p\right)^{\frac{ps-\alpha}{N-\alpha}} \ge H_\alpha\, M(\mu_{\lambda_n}^p). \tag{3.20}$$

Hence, we obtain for all n sufficiently large by (M_3) and (3.19)

$$\mu_{\lambda_n}^{\frac{p(ps-\alpha)}{N-\alpha}} \ge \left(\mu_{\lambda_n}^p - \|u_{\lambda_n}\|^p\right)^{\frac{ps-\alpha}{N-\alpha}} \ge H_\alpha\, M(\mu_{\lambda_n}^p)^{\frac{N-ps}{N-\alpha}} \ge m_0^{\frac{N-ps}{N-\alpha}}\, H_\alpha\, \mu_{\lambda_n}^{\frac{p(\theta-1)(N-ps)}{N-\alpha}}.$$

The restriction $ps\theta + N > \alpha + N\theta$ follows directly from the fact that $1 < \theta < p_s^*(\alpha)/p = (N-\alpha)/(N-ps)$. It follows that:

$$\mu_{\lambda_n}^p \ge \left(m_0^{\frac{N-ps}{N-\alpha}}\, H_\alpha\right)^{\frac{N-\alpha}{ps\theta+N-\alpha-N\theta}}.$$

This is impossible, since $\mu_{\lambda_n} \to 0$ as $n \to \infty$.

Hence, for any $\lambda \in (0, \lambda_1]$

$$\lim_{n\to\infty} \|u_n - u_\lambda\|_{H_\alpha}^{p_s^*(\alpha)} = 0.$$

This, together with (3.14), gives that $u_n \to u_\lambda$ strongly in E as $n \to \infty$.
• *Case* $\inf_{n\in\mathbb{N}} \|u_n\| = 0$. If 0 is an isolated point for the real sequence $\{\|u_n\|\}_n$, then there is a subsequence $\{u_{n_k}\}_k$, such that

$$\inf_{k\in\mathbb{N}} \|u_{n_k}\| = d > 0,$$

and we can proceed as before. Otherwise, 0 is an accumulation point of the sequence $\{\|u_n\|\}_n$ and so there exists a subsequence $\{u_{n_k}\}_k$ of $\{u_n\}_n$, such that $u_{n_k} \to 0$ strongly in E as $n \to \infty$.

In conclusion, $\mathcal{J}_\lambda$ satisfies the (PS) condition in E at the level c_λ in all the possible cases. □

4. A Truncation Argument

We note that our functional $\mathcal{J}_\lambda$ is not bounded from below in E. Indeed, by fixing $\varepsilon > 0$ in (2.4), we see that for any $u \in E$

$$\begin{aligned} \mathcal{J}_\lambda(tu) &\le t^{p\theta}\frac{\delta_\varepsilon}{p}\|u\|^{p\theta} - \lambda a_0 \int_\Omega |u|^\xi \mathrm{d}x - t^{p_s^*(\alpha)}\frac{1}{p_s^*(\alpha)}\|u\|_{H_\alpha}^{p_s^*(\alpha)} \\ &\to -\infty \quad \text{as } t \to \infty, \end{aligned}$$

since $q < p \le p\theta < p_s^*(\alpha)$.

For this, in the sequel, we introduce a truncation like in [3], to get a special lower bound which will be worth to construct critical values for $\mathcal{J}_\lambda$. Let us denote

$$\mathcal{G}_\lambda(t) = \frac{\mathscr{M}(1)}{p}t^{p\theta} - \lambda C_w^q\, t^q - \frac{1}{p_s^*(\alpha)\, H_\alpha}t^{p_s^*(\alpha)}$$

where C_w comes from (2.1), while H_α is defined in (1.3). Denoting $m = m(1)$ the constant given by (M_2) with $\tau = 1$, we can take $R_1 \in (0, 1)$ sufficiently small, such that

$$\frac{m}{p\theta}R_1^p \ge \frac{\mathscr{M}(1)}{p}R_1^{p\theta} > \frac{1}{p_s^*(\alpha)\, S}R_1^{p_s^*(\alpha)}, \tag{4.1}$$

since $p \le p\theta < p_s^*(\alpha)$, and we define

$$\lambda^* = \frac{1}{2\, C_w^q\, R_1^q}\left(\frac{\mathscr{M}(1)}{p}R_1^{p\theta} - \frac{1}{p_s^*(\alpha)\, H_\alpha}R_1^{p_s^*(\alpha)}\right), \tag{4.2}$$

so that $\mathcal{G}_{\lambda^*}(R_1) > 0$. From this, we consider

$$R_0 = \max\left\{t \in (0, R_1) :\ \mathcal{G}_{\lambda^*}(t) \le 0\right\}.$$

Since, by $q < p$, we have $\mathcal{G}_\lambda(t) \le 0$ for t near to 0 and since also $\mathcal{G}_{\lambda^*}(R_1) > 0$, it easily follows that $\mathcal{G}_{\lambda^*}(R_0) = 0$.

We can choose $\psi \in C_0^\infty([0, \infty), [0, 1])$, such that $\psi(t) = 1$ if $t \in [0, R_0]$ and $\psi(t) = 0$ if $t \in [R_1, \infty)$. Thus, we consider the truncated functional

$$\mathcal{I}_\lambda(u) = \frac{1}{p}\mathscr{M}(\|u\|^p) - \lambda \int_{\mathbb{R}^N} F(x, u)\mathrm{d}x - \psi(\|u\|)\frac{1}{p_s^*(\alpha)}\|u\|_{H_\alpha}^{p_s^*(\alpha)}.$$

It immediately follows that $\mathcal{I}_\lambda(u) \to \infty$ as $\|u\| \to \infty$, by (M_1) and (M_2). Hence, $\mathcal{I}_\lambda$ is coercive and bounded from below.

Now, we prove a local Palais–Smale and a topological result for the truncated functional $\mathcal{I}_\lambda$.

Lemma 4.1. *There exists $\overline{\lambda} > 0$, such that, for any $\lambda \in (0, \overline{\lambda})$*

(i) if $\mathcal{I}_\lambda(u) \le 0$, then $\|u\| < R_0$ and also $\mathcal{J}_\lambda(v) = \mathcal{I}_\lambda(v)$ for any v in a sufficiently small neighborhood of u;

(ii) $\mathcal{I}_\lambda$ satisfies the $(PS)_{c_\lambda}$ condition on E.

Proof. Considering λ_0 and λ_1 given, respectively, by Lemma 3.1 and 3.3, we choose $\overline{\lambda}$ sufficiently small, such that $\overline{\lambda} \le \min\{\lambda_0, \lambda_1\}$. Let $\lambda < \overline{\lambda}$.

For proving (i), we assume that $\mathcal{I}_\lambda(u) \le 0$. When $\|u\| \ge 1$, using (M_1) and (M_2) with $\tau = 1$ and $\lambda < \lambda_1$, we see that

$$\mathcal{I}_\lambda(u) \ge \frac{m}{p\theta}\|u\|^p - \frac{\lambda^*}{q}C_w^q\|u\|^q > 0,$$

where the last inequality follows by $q < p$, and because by $\mathcal{G}_{\lambda^*}(R_1) > 0$ and (4.1), we have

$$\frac{m}{p\theta}R_1^p - \frac{\lambda^*}{q}C_w^q R_1^q > 0.$$

Thus, we get the contradiction $0 \ge \mathcal{I}_\lambda(u) > 0$. Similarly, when $R_1 \le \|u\| < 1$, using (2.3), (4.1), and $\lambda < \lambda^*$, we obtain

$$\mathcal{I}_\lambda(u) \ge \frac{\mathscr{M}(1)}{p}\|u\|^{p\theta} - \frac{\lambda^*}{q}C_w^q\|u\|^q > 0,$$

where the last inequality follows by $q < p \le p\theta$, and because by $\mathcal{G}_{\lambda^*}(R_1) > 0$, we have

$$\frac{\mathscr{M}(1)}{p}R_1^{p\theta} - \frac{\lambda^*}{q}C_w^q R_1^q > 0.$$

We get again the contradiction $0 \ge \mathcal{I}_\lambda(u) > 0$. When $\|u\| < R_1$, since $\phi(t) \le 1$ for any $t \in [0, \infty)$ and $\lambda < \lambda^*$, we have

$$0 \ge \mathcal{I}_\lambda(u) \ge \mathcal{G}_\lambda(\|u\|) \ge \mathcal{G}_{\lambda^*}(\|u\|),$$

and this yields $\|u\| \le R_0$, by definition of R_0. Furthermore, for any $u \in B(0, R_0/2)$, we have $\mathcal{I}_\lambda(u) = \mathcal{J}_\lambda(u)$.

Arguing exactly as Lemma 3.3, we know that $\mathcal{I}_\lambda$ satisfies the $(PS)_{c_\lambda}$ condition on E for $\lambda < \lambda_1$. This completes the proof of Lemma 3.1. □

Here, to get the next technical result, we need a finite-dimensional subspace of E. For this, since E is a separable and reflexive Banach space, see, for example, [1], there exists $\{\varphi_n\}_n \subset E$. Then, for any $n \in \mathbb{N}$, we can set $X_n = span\,\{\varphi_n\}$ and $Y_n = \oplus_{i=1}^n X_i$.

Lemma 4.2. *For any $\lambda > 0$ and $n \in \mathbb{N}$, there exists $\varepsilon = \varepsilon(\lambda, n) > 0$, such that*

$$\gamma(\mathcal{I}_\lambda^{-\varepsilon}) \ge n,$$

where $\mathcal{I}_\lambda^{-\varepsilon} = \{u \in E : \mathcal{I}_\lambda(u) \le -\varepsilon\}$.

Proof. Fix $\lambda > 0$, $n \in \mathbb{N}$. Let Y_n be a n-dimensional subspace of E. For any $u \in Y_n$, $u \ne 0$ write $u = r_n\phi$ with $\phi \in Y_n$, $\|\phi\| = 1$ and $\bar{\phi} = \int_\Omega |\phi|^\xi \mathrm{d}x > 0$. Then, by (f_2) and continuity of M, for all r_n, with $0 < r_n < \min\{1, \delta\}$, we have

$$\begin{aligned}\mathcal{I}_\lambda(u) &= \frac{1}{p}\mathscr{M}(\|u\|^p) - \lambda\int_\Omega F(x,u)\mathrm{d}x - \frac{1}{p_s^*(\alpha)}\|u\|_{H_\alpha}^{p_s^*(\alpha)} \\ &\le \frac{1}{p}\left(\sup_{0\le s\le r_n^p} M(s)\right)\|\phi\|^p r_n^p - \lambda\left(a_0\int_{B(x_0,2R)}|\varphi|^\xi \mathrm{d}x\right) r_n^\xi - \frac{c}{p_s^*(\alpha)}r_n^{p_s^*(\alpha)} \\ &= \frac{1}{p}M_1 r_n^p - \lambda a_0\bar{\phi}\, r_n^\xi - \frac{c}{p_s^*(\alpha)}r_n^{p_s^*(\alpha)} = \varepsilon_n,\end{aligned}$$

where $M^* = \max_{\tau\in[0,R_0]} M(\tau) < \infty$; here, we use all the norms are equivalent for finite-dimensional Y_n. Hence, we can choose r_n so small that $\mathcal{I}_\lambda(u) < \varepsilon_n < 0$. Let

$$S_{r_n} = \{u \in E; \|u\| = r_n\}.$$

Then, $S_{r_n} \cap Y_n \subset \mathcal{I}_\lambda^{\varepsilon_m}$. Then, we have $\gamma(\mathcal{I}_\lambda^{\varepsilon_m}) \geq \gamma(S_{r_n} \cap Y_n) \geq n$. Therefore, we can denote $\Gamma_n = \{A \in \Sigma; \gamma(A) \geq n\}$ and let

$$c_m := \inf_{A\in\Gamma_m} \sup_{u\in A} \widetilde{\mathcal{I}_\lambda}(u), \tag{4.3}$$

then

$$-\infty < c_n \leq \varepsilon_n < 0, \quad n \in \mathbb{N}, \tag{4.4}$$

because $\widetilde{\mathcal{I}_\lambda}^{\varepsilon_n} \in \Gamma_n$ and $\widetilde{\mathcal{I}_\lambda}$ is bounded from below. □

5. Main Results

Here, we define for any $n \in \mathbb{N}$ the sets

$$\Sigma_n = \{A \subset E : \ A \text{ is closed}, A = -A \text{ and } \gamma(A) \geq n\},$$
$$K_c = \{u \in E : \ \mathcal{I}'_\lambda(u) = 0 \text{ and } \mathcal{I}_\lambda(u) = c\},$$

and the number

$$c_n = \inf_{A\in\Sigma_n} \sup_{u\in A} \mathcal{I}_\lambda(u). \tag{5.1}$$

Before proving our main results, we state some crucial properties of the family of numbers $\{c_n\}_{n\in\mathbb{N}}$.

Lemma 5.1. *For any $\lambda > 0$ and $n \in \mathbb{N}$, the number c_n is negative.*

Proof. Let $\lambda > 0$ and $n \in \mathbb{N}$. By Lemma 4.2, there exists $\varepsilon > 0$, such that $\gamma(\mathcal{I}_\lambda^{-\varepsilon}) \geq n$. Since also $\mathcal{I}_\lambda$ is continuous and even, $\mathcal{I}_\lambda^{-\varepsilon} \in \Sigma_n$. From $\mathcal{I}_\lambda(0) = 0$, we have $0 \notin \mathcal{I}_\lambda^{-\varepsilon}$. Furthermore, $\sup_{u\in\mathcal{I}_\lambda^{-\varepsilon}} \mathcal{I}_\lambda(u) \leq -\varepsilon$. In conclusion, remembering also that $\mathcal{I}_\lambda$ is bounded from below, we get

$$-\infty < c_n = \inf_{A\in\Sigma_n} \sup_{u\in A} \mathcal{I}_\lambda(u) \leq \sup_{u\in\mathcal{I}_\lambda^{-\varepsilon}} \mathcal{I}_\lambda(u) \leq -\varepsilon < 0.$$

□

Lemma 5.2. *Let $\lambda \in (0, \overline{\lambda})$, with $\overline{\lambda}$ given in Lemma 4.1. Then, all c_n given by (3.3) are critical values for $\mathcal{I}_\lambda$ and $c_n \to 0$ as $n \to \infty$.*

Proof. It is clear that $c_n \leq c_{n+1}$. By Lemma 5.1, we have $c_n < 0$. Hence, $c_n \to \overline{c} \leq 0$. Moreover, by Lemma 4.1, the functional $\mathcal{I}_\lambda$ satisfies the Palais–Smale condition at c_n. Thus, it follows from standard arguments as in [30] that all c_n are critical values of $\mathcal{I}_\lambda$. We claim that $\overline{c} = 0$. If $\overline{c} < 0$, then still by Lemma 4.1, we have $K_{\overline{c}}$ is compact. It follows that $\gamma(K_{\overline{c}}) = n_0 < \infty$ and there exists $\delta > 0$, such that $\gamma(K_{\overline{c}}) = \gamma(N_\delta(K_{\overline{c}})) = n_0$.

By Theorem 3.4 of [5], there exist $\varepsilon \in (0, \overline{c})$ and an odd homeomorphism $\eta : E \to E$, such that

$$\eta\left(\mathcal{I}_\lambda^{\overline{c}+\varepsilon} \backslash N_\delta(K_{\overline{c}})\right) \subset \mathcal{I}_\lambda^{\overline{c}-\epsilon}. \tag{5.2}$$

Since c_n is increasing and converges to $\overline{c}$, there exists $n \in \mathbb{N}$, such that $c_n > \overline{c} - \varepsilon$ and $c_{n+n_0} \le \overline{c}$. There exists $A \in \Gamma_{n+n_0}$, such that $\sup_{u\in A} \mathcal{I}_\lambda(u) < \overline{c} + \varepsilon$. Therefore, we have

$$\gamma(\overline{A\backslash N_\delta(K_{\overline{c}})}) \ge \gamma(A) - \gamma(N_\delta(K_{\overline{c}})), \quad \gamma(\eta(\overline{A\backslash N_\delta(K_{\overline{c}})})) \ge n. \tag{5.3}$$

Therefore, we have

$$\eta(\overline{A\backslash N_\delta(K_{\overline{c}})}) \in \Gamma_n.$$

Consequently

$$\sup_{u\in\eta(\overline{A\backslash N_\delta(K_{\overline{c}})})} \mathcal{I}_\lambda(u) \ge c_n > \overline{c} - \epsilon. \tag{5.4}$$

On the other hand, by (5.2) and (5.3), we have

$$\eta(\overline{A\backslash N_\delta(K_{\overline{c}})}) \subset \eta(\mathcal{I}_\lambda^{\overline{c}+\epsilon}\backslash N_\delta(K_{\overline{c}})) \subset \mathcal{I}_\lambda^{\overline{c}}, \tag{5.5}$$

which contradicts (5.4). Hence, $c_n \to 0$ as $n \to \infty$.

Now, we are ready to give the proof of Theorem 1.1, as follows.

Proof of Theorem 1.1. By Lemma 4.1, $\mathcal{I}_\lambda(u) = \mathcal{J}_\lambda(u)$ if $\mathcal{I}_\lambda(u) < 0$. Then, by Lemmas 4.1, 4.2, 5.1, and 5.2, one can see that all the assumptions of the new version of symmetric mountain pass lemma due to Kajikiya in [16] are satisfied. Hence, the proof is finished. □

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