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Some Existence and Stability Results for Hilfer-fractional Implicit Differential Equations with Nonlocal Conditions

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Abstract. The aim of this paper is to establish the existence and uniqueness results for implicit differential equations of Hilfer-type fractional order via Schaefer's fixed point theorem and Banach contraction principle. Next, we establish the equivalent mixed-type integral for nonlocal condition. Further we prove the Ulam stability results. The Gronwall's lemma for singular kernels plays an important role to prove our results. We verify our results by providing examples.

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1. Introduction

The topic of fractional calculus is as old as the differential calculus and it has been developed up to nowadays (see Kilbas et al. [19], Hilfer [15]). Fractional differential and integral equations have recently been applied in different areas of engineering, mathematics, physics and bio-engineering and so on. There has been considerable development in ordinary and partial fractional differential and integral equations in recent years; see the monographs of Kilbas et al. [19], Hilfer [15] and Podlubny [21]. There are several definitions of fractional integrals and derivatives in the literature, but the most popular definitions are in the sense of the Riemann–Liouville and Caputo. Recently, Hilfer has introduced a generalized form of the Riemann–Liouville fractional derivative. In short, Hilfer fractional derivative is an interpolation between the Riemann–Liouville and Caputo fractional derivatives. This set of parameters gives an extra degree of freedom on the initial conditions and produces more types of stationary states. For some recent results and applications of Hilfer fractional derivative, we refer the reader to a series of papers [1,2,12–16,28] and the references cited therein.

Birkhäuser

The aim of this paper is to study the implicit differential equation with nonlocal condition involving Hilfer fractional derivative of the form

$$\begin{cases} D_{0^+}^{\alpha,\beta}x(t) = f(t, x(t), D_{0^+}^{\alpha,\beta}x(t)), \quad t \in J := [0, T], \\ I_{0^+}^{1-\gamma}x(0) = \sum_{i=1}^{m} c_i x(\tau_i), \quad \alpha \le \gamma = \alpha + \beta - \alpha\beta < 1, \ \tau_i \in [0, T], \end{cases} \tag{1}$$

where $D_{0^+}^{\alpha,\beta}$ is the Hilfer fractional derivative, $0 < \alpha < 1$, $0 \le \beta \le 1$ and let $\mathbb{R}$ be a Banach space, $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given continuous function. $I_{0^+}^{1-\gamma}$ is the left-sided Riemann–Liouville fractional integral of order $1-\gamma$, c_i are real numbers and $\tau_i = 1, 2, \ldots, m$ are prefixed points satisfying $0 < \tau_1 \le \tau_2 \le \cdots \le \tau_m < T$.

In passing, we remark that the application of nonlocal condition $I_{0^+}^{1-\gamma}$ $x(0) = \sum_{i=1}^{m} c_i x(\tau_i)$ in physical problems yields better effect than the initial condition $I_{0^+}^{1-\gamma}x(0) = x_0$. In recent research work, several researchers of mathematics community studied implicit differential equations with fractional order due to their applications in different fields of science and engineering. The papers [3,4,6,7,9,11,23,24] treated fractional implicit differential equations (FIDEs). Very recently, Tidke and Mahajan [24] studied an initial value problem for nonlinear FIDE with Riemann–Liouville fractional derivative and the uniqueness result is based on the application of Bihari and Medved inequalities. In [11], sufficient conditions for existence and stability of solutions for system of nonlinear FIDE are established via method of successive approximations. Next, Sousa et al. [10] investigated the Ulam–Hyers–Rassias stability for FIDEs using the ψ-Hilfer operator. Weak solutions for a class of functional FIDEs of Hilfer–Hadamard type are studied by Abbas et al. [3].

Since 1940, Ulam-type stability problems [5,18,20,22] have been studied by a large number of mathematicians. This stability analysis is very useful in many applications, such as numerical analysis, optimization, etc., where finding the exact solution is quite difficult. For detailed study of Ulam-type stability with different approaches, we recommend papers such as [17,29,30]. Recently, Vivek et al. [25] investigated the existence and Ulam–Hyers stability results for pantograph differential equations with Hilfer fractional derivative. They also studied the dynamics and stability results for Hilfer-type thermistor problem in [27]. Some existence and stability analysis of nonlinear neutral pantograph equations are studied via Hilfer fractional derivative in [26].

This paper is presented as follows. Section 2 contains some fundamental concepts of Hilfer fractional derivative. In Sect. 3, we present our main result by using Schaefer's fixed point theorem. In Sect. 4, we discuss stability analysis. Section 5 contains some examples.

2. Prerequisites

In this section, we gather some basic facts, definitions and lemmas regarding fractional differential equations, which we utilized throughout this paper, to obtain our main results. The following observations are taken from [12,13, 15,16,28].

Definition 2.1. (*see* [19]) The left-sided Riemann–Liouville fractional integral of order $\alpha \in \mathbb{R}^+$ of function $f(t)$ is defined by

$$(I_{0+}^{\alpha} f)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\mathrm{d}s, \quad (t > 0),$$

where $\Gamma(\cdot)$ is the Gamma function.

Definition 2.2. (*see* [19]) The left-sided Riemann–Liouville fractional derivative of order $\alpha \in [n-1, n),\ n \in \mathbb{Z}^+$ of function $f(t)$ is defined by

$$(D_{0+}^{\alpha} f)(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_0^t (t-s)^{n-\alpha-1} f(s)\mathrm{d}s,\ (t > 0).$$

Based on differentiating fractional integrals, a generalized definition called Hilfer fractional derivative can be introduced.

Definition 2.3. (*see* [15]) The left-sided Hilfer fractional derivative of order $0 < \alpha < 1$ and $0 \leq \beta \leq 1$ of function $f(t)$ is defined by

$$D_{0+}^{\alpha,\beta} f(t) = \left(I_{0+}^{\beta(1-\alpha)} D \left(I_{0+}^{(1-\beta)(1-\alpha)} f\right)\right)(t),$$

where $D := \frac{\mathrm{d}}{\mathrm{d}t}$.

The Hilfer fractional derivative is considered as an interpolator between the Riemann–Liouville and Caputo derivative, then the following remarks can be presented to show the relation with Caputo and Riemann–Liouville operators.

Remark 2.4. (see [15])

1. The operator $D_{0+}^{\alpha,\beta}$ also can be written as
$$D_{0+}^{\alpha,\beta} = I_{0+}^{\beta(1-\alpha)} D I_{0+}^{(1-\beta)(1-\alpha)} = I_{0+}^{\beta(1-\alpha)} D_{0+}^{\gamma},\ \gamma = \alpha + \beta - \alpha\beta.$$
2. Let $\beta = 0$, the left-sided Riemann–Liouville fractional derivative can be presented as $D_{0+}^{\alpha} := D_{0+}^{\alpha,0}$.
3. Let $\beta = 1$, the left-sided Caputo fractional derivative can be presented as ${}^cD_{0+}^{\alpha} := I_{0+}^{1-\alpha} D$.

Throughout this paper, let $C[J, \mathbb{R}]$ be the Banach space of all continuous functions from J into $\mathbb{R}$ with the norm $\|x\|_C = \max\{|x(t)| : t \in [0, T]\}$ and $L^1(J)$ is the space of Lebesgue-integrable functions $x : J \to \mathbb{R}$ with the norm

$$\|x\|_1 = \int_0^T |x(s)|\,\mathrm{d}s.$$

For $0 \leq \gamma < 1$, we denote the space $C_\gamma[J, \mathbb{R}]$ as

$$C_\gamma[J, \mathbb{R}] := \{f(t) : (0, T] \to \mathbb{R} | t^\gamma f(t) \in C[J, \mathbb{R}]\},$$

where $C_\gamma[J, \mathbb{R}]$ is the weighted space of the continuous functions f on the finite interval $[0, T]$.

Obviously, $C_\gamma[J, \mathbb{R}]$ is the Banach space with the norm

$$\|f\|_{C_\gamma} = \|t^\gamma f(t)\|_C.$$

Meanwhile, $C_\gamma^n[J,\mathbb{R}] := \left\{f \in C^{n-1}[J,\mathbb{R}] : f^{(n)} \in C_\gamma[J,\mathbb{R}]\right\}$ is the Banach space with the norm

$$\|f\|_{C_\gamma^n} = \sum_{i=0}^{n-1} \left\|f^{(k)}\right\|_C + \left\|f^{(n)}\right\|_{C_\gamma}, \; n \in \mathbb{N}.$$

Moreover, $C_\gamma^0[J,\mathbb{R}] := C_\gamma[J,\mathbb{R}]$.

Lemma 2.5. (see [19], p. 74, Property 2.1) *If $\alpha > 0$ and $\beta > 0$, there exist*

$$\left[I_{0^+}^\alpha s^{\beta-1}\right](t) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)} t^{\beta+\alpha-1},$$

and

$$\left[D_{0^+}^\alpha s^{\alpha-1}\right](t) = 0, \quad 0 < \alpha < 1.$$

Lemma 2.6. (see [19], Lemmas 2.3, 2.4, 2.9) *If $\alpha > 0$, $\beta > 0$, and $f \in L^1(J)$, for $t \in [0,T]$ there exist the following properties*

$$\left(I_{0^+}^\alpha I_{0^+}^\beta f\right)(t) = \left(I_{0^+}^{\alpha+\beta} f\right)(t)$$

and

$$\left(D_{0^+}^\alpha I_{0^+}^\alpha f\right)(t) = f(t).$$

In particular, if $f \in C_\gamma[J,\mathbb{R}]$ or $f \in C[J,\mathbb{R}]$, then these equalities hold at each $t \in (0,T]$ or $t \in [0,T]$, respectively.

Lemma 2.7. (see [19], Lemmas 2.5, 2.9) *Let $0 < \alpha < 1$, $0 \le \gamma < 1$. If $f \in C_\gamma[J,\mathbb{R}]$ and $I_{0^+}^{1-\alpha} f \in C_\gamma^1[J,\mathbb{R}]$, then*

$$I_{0^+}^\alpha D_{0^+}^\alpha f(t) = f(t) - \frac{(I_{0^+}^{1-\alpha} f)(0)}{\Gamma(\alpha)} t^{\alpha-1}, \quad \forall\, t \in J.$$

Lemma 2.8. (see [16], Lemma 13) *For $0 \le \gamma < 1$ and $f \in C_\gamma[J,\mathbb{R}]$, then*

$$(I_{0^+}^\alpha f)(0) := \lim_{t\to 0^+} I_{0^+}^\alpha f(t) = 0, \quad 0 \le \gamma < \alpha.$$

The following spaces will be used later:

$$C_{1-\gamma}^{\alpha,\beta} = \left\{f \in C_{1-\gamma}[J,\mathbb{R}], D_{0^+}^{\alpha,\beta} f \in C_{1-\gamma}[J,\mathbb{R}]\right\}$$

and

$$C_{1-\gamma}^{\gamma} = \left\{f \in C_{1-\gamma}[J,\mathbb{R}], D_{0^+}^{\gamma} f \in C_{1-\gamma}[J,\mathbb{R}]\right\}.$$

It is obvious that

$$C_{1-\gamma}^{\gamma}[J,\mathbb{R}] \subset C_{1-\gamma}^{\alpha,\beta}[J,\mathbb{R}].$$

Lemma 2.9. (see [16], Lemma 20) *Let $\alpha > 0$, $\beta > 0$, and $\gamma = \alpha + \beta - \alpha\beta$. If $f \in C_{1-\gamma}^\gamma[J,\mathbb{R}]$, then*

$$I_{0^+}^\gamma D_{0^+}^\gamma f = I_{0^+}^\alpha D_{0^+}^{\alpha,\beta} f, \;\; D_{0^+}^\gamma I_{0^+}^\alpha f = D_{0^+}^{\beta(1-\alpha)} f(t).$$

Lemma 2.10. (see [16], Lemma 21) *Let* $f \in L^1(J)$ *and* $D_{0^+}^{\beta(1-\alpha)} f \in L^1(J)$ *existed, then*

$$D_{0^+}^{\alpha,\beta} I_{0^+}^{\alpha} f = I_{0^+}^{\beta(1-\alpha)} D_{0^+}^{\beta(1-\alpha)} f.$$

Lemma 2.11. (see [16], Theorem 23) *Let* $f : J \times \mathbb{R} \to \mathbb{R}$ *be a function such that* $f \in C_{1-\gamma}[J, \mathbb{R}]$ *for any* $x \in C_{1-\gamma}[J, \mathbb{R}]$. *A function* $x \in C_{1-\gamma}^{\gamma}[J, \mathbb{R}]$ *is a solution of fractional initial value problem:*

$$\begin{cases} D_{0^+}^{\alpha,\beta} x(t) = f(t, x(t)), \ 0 < \alpha < 1, \ 0 \le \beta \le 1, \\ I_{0^+}^{1-\gamma} x(0) = x_0, \quad \gamma = \alpha + \beta - \alpha\beta, \end{cases}$$

if and only if x *satisfies the following Volterra integral equation:*

$$x(t) = \frac{x_0 t^{\gamma-1}}{\Gamma(\gamma)} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s)) \mathrm{d}s.$$

For complete study on initial value problem for Hilfer-type FIDEs, one can refer to [1,3,11].

According to Lemma 2.11, a new and important equivalent mixed-type integral equation for our problem (1) can be established. We adopt some ideas from (see [28], Lemma 2.12) to establish an equivalent mixed-type integral equation:

$$x(t) = \frac{Z t^{\gamma-1}}{\Gamma(\alpha)} \sum_{i=1}^{m} c_i \int_0^{\tau_i} (\tau_i - s)^{\alpha-1} K_x(s) \mathrm{d}s + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} K_x(s) \mathrm{d}s, \tag{2}$$

where

$$Z := \frac{1}{\Gamma(\gamma) - \sum_{i=1}^{m} c_i (\tau_i)^{\gamma-1}}, \quad \text{if} \quad \Gamma(\gamma) \ne \sum_{i=1}^{m} c_i (\tau_i)^{\gamma-1}. \tag{3}$$

For brevity, we shall take

$$K_x(t) := D_{0^+}^{\alpha,\beta} x(t) = f(t, x(t), K_x(t)). \tag{4}$$

Lemma 2.12. *Let* $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *be a function such that* $f \in C_{1-\gamma}[J, \mathbb{R}]$ *for any* $x \in C_{1-\gamma}[J, \mathbb{R}]$. *A function* $x \in C_{1-\gamma}^{\gamma}[J, \mathbb{R}]$ *is a solution of the problem* (1) *if and only if* x *satisfies the mixed-type integral* (2).

Proof. According to Lemma 2.11, a solution of problem (1) can be expressed by

$$x(t) = \frac{I_{0^+}^{1-\gamma} x(0)}{\Gamma(\gamma)} t^{\gamma-1} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} K_x(s) \mathrm{d}s. \tag{5}$$

Next, we substitute $t = \tau_i$ into the above equation,

$$x(\tau_i) = \frac{I_{0^+}^{1-\gamma} x(0)}{\Gamma(\gamma)} (\tau_i)^{\gamma-1} + \frac{1}{\Gamma(\alpha)} \int_0^{\tau_i} (\tau_i - s)^{\alpha-1} K_x(s) \mathrm{d}s, \tag{6}$$

by multiplying c_i to both sides of (6), we can write

$$c_i x(\tau_i) = \frac{I_{0^+}^{1-\gamma} x(0)}{\Gamma(\gamma)} c_i (\tau_i)^{\gamma-1} + \frac{1}{\Gamma(\alpha)} c_i \int_0^{\tau_i} (\tau_i - s)^{\alpha-1} K_x(s) \mathrm{d}s.$$

Thus, we have

$$
\begin{aligned}
I_{0^+}^{1-\gamma}x(0) &= \sum_{i=1}^{m} c_i x(\tau_i) \\
&= \frac{I_{0^+}^{1-\gamma}x(0)}{\Gamma(\gamma)} \sum_{i=1}^{m} c_i(\tau_i)^{\gamma-1} + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{m} c_i \int_0^{\tau_i} (\tau_i - s)^{\alpha-1} K_x(s)\mathrm{d}s
\end{aligned}
$$

which implies

$$
I_{0^+}^{1-\gamma}x(0) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)} Z \sum_{i=1}^{m} c_i \int_0^{\tau_i} (\tau_i - s)^{\alpha-1} K_x(s)\mathrm{d}s. \tag{7}
$$

Submitting (7) to (5), we derive that (2). It is probative that x is also a solution of the integral equation (2), when x is a solution of (1).

The necessity has been already proved. Next, we are ready to prove its sufficiency. Applying $I_{0^+}^{1-\gamma}$ to both sides of (2), we have

$$
\begin{aligned}
I_{0^+}^{1-\gamma}x(t) = I_{0^+}^{1-\gamma} t^{\gamma-1} \frac{Z}{\Gamma(\alpha)} \sum_{i=1}^{m} c_i \int_0^{\tau_i} (\tau_i - s)^{\alpha-1} K_x(s)\mathrm{d}s \\
+ I_{0^+}^{1-\gamma} I_{0^+}^{\alpha} K_x(t),
\end{aligned}
$$

using the Lemmas 2.5 and 2.6,

$$
I_{0^+}^{1-\gamma}x(t) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)} Z \sum_{i=1}^{m} c_i \int_0^{\tau_i} (\tau_i - s)^{\alpha-1} K_x(s)\mathrm{d}s + I_{0^+}^{1-\beta(1-\alpha)} K_x(t).
$$

Since $1-\gamma < 1-\beta(1-\alpha)$, Lemma 2.8 can be used when taking the limit as $t \to 0$,

$$
I_{0^+}^{1-\gamma}x(0) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)} Z \sum_{i=1}^{m} c_i \int_0^{\tau_i} (\tau_i - s)^{\alpha-1} K_x(s)\mathrm{d}s. \tag{8}
$$

Substituting $t = \tau_i$ into (2), we have

$$
\begin{aligned}
x(\tau_i) = \frac{Z}{\Gamma(\alpha)} (\tau_i)^{\gamma-1} \sum_{i=1}^{m} c_i \int_0^{\tau_i} (\tau_i - s)^{\alpha-1} K_x(s)\mathrm{d}s \\
+ \frac{1}{\Gamma(\alpha)} \int_0^{\tau_i} (\tau_i - s)^{\alpha-1} K_x(s)\mathrm{d}s.
\end{aligned}
$$

Then, we derive

$$
\begin{aligned}
\sum_{i=1}^{m} c_i x(\tau_i) = \frac{Z}{\Gamma(\alpha)} \sum_{i=1}^{m} c_i \int_0^{\tau_i} (\tau_i - s)^{\alpha-1} K_x(s)\mathrm{d}s \sum_{i=1}^{m} c_i(\tau_i)^{\gamma-1} \\
+ \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{m} c_i \int_0^{\tau_i} (\tau_i - s)^{\alpha-1} K_x(s)\mathrm{d}s
\end{aligned}
$$

$$= \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{m} c_i \int_0^{\tau_i} (\tau_i - s)^{\alpha-1} K_x(s) \mathrm{d}s \left(1 + Z \sum_{i=1}^{m} c_i (\tau_i)^{\gamma-1} \right)$$

$$= \frac{\Gamma(\gamma)}{\Gamma(\alpha)} Z \sum_{i=1}^{m} c_i \int_0^{\tau_i} (\tau_i - s)^{\alpha-1} K_x(s) \mathrm{d}s,$$

that is

$$\sum_{i=1}^{m} c_i x(\tau_i) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)} Z \sum_{i=1}^{m} c_i \int_0^{\tau_i} (\tau_i - s)^{\alpha-1} K_x(s) \mathrm{d}s. \tag{9}$$

It follows (8) and (9) that

$$I_{0^+}^{1-\gamma} x(0) = \sum_{i=1}^{m} c_i x(\tau_i).$$

Now by applying D_{0+}^{γ} to both sides of (2), it follows from Lemmas 2.5 and 2.9 that

$$D_{0+}^{\gamma} x(t) = D_{0+}^{\beta(1-\alpha)} K_x(t) = D_{0+}^{\beta(1-\alpha)} f(t, x(t), D_{0+}^{\alpha,\beta} x(t)). \tag{10}$$

Since $x \in C_{1-\gamma}^{\gamma}[J, \mathbb{R}]$ and by the definition of $C_{1-\gamma}^{\gamma}[J, \mathbb{R}]$, we have $D_{0+}^{\gamma} x \in C_{1-\gamma}[J, \mathbb{R}]$, then, $D_{0+}^{\beta(1-\alpha)} f = D I_{0+}^{1-\beta(1-\alpha)} f \in C_{1-\gamma}[J, \mathbb{R}]$. For $f \in C_{1-\gamma}[J, \mathbb{R}]$, it is obvious that $I_{0+}^{1-\beta(1-\alpha)} f \in C_{1-\gamma}[J, \mathbb{R}]$, then $I_{0+}^{1-\beta(1-\alpha)} f \in C_{1-\gamma}^{1}[J, \mathbb{R}]$. Thus f and $I_{0+}^{1-\beta(1-\alpha)} f$ satisfy the conditions of Lemma 2.7.

Next, by applying $I_{0+}^{\beta(1-\alpha)}$ to both sides of (10) and using Lemma 2.7, we can obtain

$$D_{0+}^{\alpha,\beta} x(t) = K_x(t) - \frac{(I_{0+}^{1-\beta(1-\alpha)} K_x)(0)}{\Gamma(\beta(1-\alpha))} (t)^{\beta(1-\alpha)-1},$$

where $I_{0+}^{\beta(1-\alpha)} K_x(0) = 0$ is implied by Lemma 2.8.
Hence, it reduces to $D_{0+}^{\alpha,\beta} x(t) = K_x(t) = f(t, x(t), D_{0+}^{\alpha,\beta} x(t))$. The results are proved completely. □

3. Existence Theory

In this section, we are concerned with the existence of solutions for the problem (1).

Theorem 3.1. *Assume that*

(H1) *There exist* $l, p, q \in C_{1-\gamma}[J, \mathbb{R}]$ *with* $l^* = \sup_{t \in J} l(t) < 1$ *such that*

$$|f(t, u, v)| \leq l(t) + p(t)\,|u| + q(t)\,|v|$$

for $t \in J,\ u, v \in \mathbb{R}$.

Then, the problem (1) *has at least one solution in* $C_{1-\gamma}^{\gamma}[J, \mathbb{R}] \subset C_{1-\gamma}^{\alpha,\beta}[J, \mathbb{R}]$.

Proof. The proof will be given in several steps.

Consider the operator $N : C_{1-\gamma}[J,\mathbb{R}] \to C_{1-\gamma}[J,\mathbb{R}]$.

$$(Nx)(t) = \frac{Z}{\Gamma(\alpha)} t^{\gamma-1} \sum_{i=1}^{m} c_i \int_0^{\tau_i} (\tau_i - s)^{\alpha-1} K_x(s)\mathrm{d}s + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} K_x(s)\mathrm{d}s. \tag{11}$$

It is obvious that the operator N is well defined.

Claim 1 The operator N is continuous.

Let x_n be a sequence such that $x_n \to x$ in $C_{1-\gamma}[J,\mathbb{R}]$. Then for each $t \in J$,

$$\begin{aligned}
&\left|((Nx_n)(t) - (Nx)(t))t^{1-\gamma}\right| \\
&\quad\le \frac{|Z|}{\Gamma(\alpha)} \sum_{i=1}^{m} c_i \int_0^{\tau_i} (\tau_i - s)^{\alpha-1} \left|K_{x_n}(s) - K_x(s)\right| \mathrm{d}s \\
&\qquad + \frac{t^{1-\gamma}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left|K_{x_n}(s) - K_x(s)\right| \mathrm{d}s \\
&\quad\le \frac{|Z|\, B(\gamma,\alpha) \sum_{i=1}^{m} c_i (\tau_i)^{\alpha+\gamma-1}}{\Gamma(\alpha)} \left\|K_{x_n}(\cdot) - K_{x_n}(\cdot)\right\|_{C_{1-\gamma}} \\
&\qquad + \frac{T^{\alpha} B(\gamma,\alpha)}{\Gamma(\alpha)} \left\|K_{x_n}(\cdot) - K_x(\cdot)\right\|_{C_{1-\gamma}} \\
&\quad\le \frac{B(\gamma,\alpha)}{\Gamma(\alpha)} \left(|Z| \sum_{i=1}^{m} c_i (\tau_i)^{\alpha+\gamma-1} + T^{\alpha} \right) \left\|K_{x_n}(\cdot) - K_x(\cdot)\right\|_{C_{1-\gamma}},
\end{aligned}$$

where we use the formula

$$\begin{aligned}
\int_a^t (t-s)^{\alpha-1} |x(s)|\, \mathrm{d}s &\le \left(\int_a^t (t-s)^{\alpha-1} (s-a)^{\gamma-1} \mathrm{d}s \right) \|x\|_{C_{1-\gamma}} \\
&= (t-a)^{\alpha+\gamma-1} B(\gamma,\alpha) \|x\|_{C_{1-\gamma}}.
\end{aligned}$$

Since K_x is continuous (i.e., f is continuous), then we have

$$\|Nx_n - Nx\|_{C_{1-\gamma}} \to 0 \quad \text{as} \quad n \to \infty.$$

Claim 2 N maps bounded sets into bounded sets in $C_{1-\gamma}[J,\mathbb{R}]$.

Indeed, it is enough to show that for $\eta > 0$, there exists a positive constant l such that $x \in B_\eta = \{x \in C_{1-\gamma}[J,\mathbb{R}] : \|x\| \le \eta\}$, we have $\|N(x)\|_{C_{1-\gamma}} \le l$.

$$\begin{aligned}
&\left|(Nx)(t)t^{1-\gamma}\right| \\
&\quad\le \frac{|Z|}{\Gamma(\alpha)} \sum_{i=1}^{m} c_i \int_0^{\tau_i} (\tau_i - s)^{\alpha-1} |K_x(s)|\, \mathrm{d}s + \frac{t^{1-\gamma}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |K_x(s)|\, \mathrm{d}s \\
&\quad:= \mathscr{A}_1 + \mathscr{A}_2.
\end{aligned} \tag{12}$$

For computation work, we set

$$\begin{aligned}\mathscr{A}_1 &= \frac{|Z|}{\Gamma(\alpha)}\sum_{i=1}^{m} c_i \int_0^{\tau_i} (\tau_i - s)^{\alpha-1} |K_x(s)|\, \mathrm{d}s,\\ \mathscr{A}_2 &= \frac{t^{1-\gamma}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} |K_x(s)|\, \mathrm{d}s,\end{aligned}$$

and by (H1)

$$\begin{aligned}|K_x(t)| &= |f(t, x(t), K_x(t))|\\ &\le l(t) + p(t)|x(t)| + q(t)|K_x(t)|\\ &\le l^* + p^*|x(t)| + q^*|K_x(t)|\\ &\le \frac{l^* + p^*|x(t)|}{1-q^*}. \end{aligned} \tag{13}$$

We estimate $\mathscr{A}_1, \mathscr{A}_2$ terms separately, we have

$$\mathscr{A}_1 = \frac{|Z|}{(1-q^*)}\sum_{i=1}^{m} c_i \left(\frac{l^*(\tau_i)^{\alpha}}{\Gamma(\alpha+1)} + p^*\frac{(\tau_i)^{\alpha+\gamma-1}}{\Gamma(\alpha)}B(\gamma,\alpha)\|x\|_{C_{1-\gamma}}\right), \tag{14}$$

$$\mathscr{A}_2 = \frac{1}{1-q^*}\left(\frac{l^* T^{\alpha-\gamma+1}}{\Gamma(\alpha+1)} + p^*\frac{T^{\alpha}}{\Gamma(\alpha)}B(\gamma,\alpha)\|x\|_{C_{1-\gamma}}\right). \tag{15}$$

Bringing inequalities (14) and (15) into (12), we get

$$\begin{aligned}\left|(Nx)(t)t^{1-\gamma}\right| \le\ & \frac{l^*}{(1-q^*)\Gamma(\alpha+1)}\left(|Z|\sum_{i=1}^{m} c_i(\tau_i)^{\alpha} + T^{\alpha+\gamma-1}\right)\\ & + \frac{p^*}{(1-q^*)\Gamma(\alpha)}\left(|Z|\sum_{i=1}^{m} c_i(\tau_i)^{\alpha+\gamma-1} + T^{\alpha}\right)B(\gamma,\alpha)\|x\|_{C_{1-\gamma}}\\ :=\ & l.\end{aligned}$$

Claim 3 N maps bounded sets into equicontinuous set of $C_{1-\gamma}[J,\mathbb{R}]$.

Let $t_1, t_2 \in J, t_2 \le t_1$, B_η be a bounded set of $C_{1-\gamma}[J,\mathbb{R}]$ as in Claim 2, and let $x \in B_\eta$. Then

$$\begin{aligned}&\left|t_1^{1-\gamma}(Nx)(t_1) - t_2^{1-\gamma}(Nx)(t_2)\right|\\ &\le \left|\frac{t_1^{1-\gamma}}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}K_x(s)ds - \frac{t_2^{1-\gamma}}{\Gamma(\alpha)}\int_0^{t_2}(t_2-s)^{\alpha-1}K_x(s)\mathrm{d}s\right|\\ &\le \left|\frac{1}{\Gamma(\alpha)}\int_0^{t_1}\left[t_1^{1-\gamma}(t_1-s)^{\alpha-1} - t_2^{1-\gamma}(t_2-s)^{\alpha-1}\right]K_x(s)\mathrm{d}s\right|\\ &\quad + \left|\frac{t_2^{1-\gamma}}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}K_x(s)\mathrm{d}s\right|.\end{aligned}$$

As $t_1 \to t_2$, the right-hand side of the above inequality tends to zero. As a consequence of Claim 1–3, together with Arzela–Ascoli theorem, we can conclude that $N : C_{1-\gamma}[J,\mathbb{R}] \to C_{1-\gamma}[J,\mathbb{R}]$ is completely continuous.

Claim 4 A priori bounds.

Now it remains to show that the set

$$\omega = \{x \in C_{1-\gamma}[J,\mathbb{R}] : x = \delta(Nx), \quad 0 < \delta < 1\}$$

is bounded set.

Let $x \in \omega$, $x = \delta(Nx)$ for some $0 < \delta < 1$. Thus for each $t \in J$, we have

$$x(t) = \delta\Bigg[\frac{|Z|}{\Gamma(\alpha)} t^{\gamma-1} \sum_{i=1}^{m} c_i \int_0^{\tau_i} (\tau_i - s)^{\alpha-1} K_x(s) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} K_x(s)\mathrm{d}s\Bigg].$$

This implies by (H2)that for each $t \in J$, we have

$$\begin{aligned} \left|x(t)t^{1-\gamma}\right| &\le \left|(Nx)(t)t^{1-\gamma}\right| \\ &\le \frac{l^*}{(1-q^*)\Gamma(\alpha+1)}\left(|Z|\sum_{i=1}^{m} c_i(\tau_i)^{\alpha} + T^{\alpha+\gamma-1}\right) \\ &\quad + \frac{p^*}{(1-q^*)\Gamma(\alpha)}\left(|Z|\sum_{i=1}^{m} c_i(\tau_i)^{\alpha+\gamma-1} + T^{\alpha}\right) B(\gamma,\alpha)\,\|x\|_{C_{1-\gamma}} \\ &:= R. \end{aligned}$$

This shows that the set ω is bounded. As a consequence of Schaefer's fixed point theorem, we deduce that N has a fixed point which is a solution of problem (1). The proof is completed. □

The following arguments are based on the Banach contraction principle.

Theorem 3.2. *Assume that hypothesis*

(H2) *Let $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a function such that $f \in C_{1-\gamma}^{\beta(1-\alpha)}[J,\mathbb{R}]$ for any x in $C_{1-\gamma}^{\gamma}[J,\mathbb{R}]$ and there exist positive constants $K > 0$ and $L > 0$ such that*

$$|f(t,u,v) - f(t,\overline{u},\overline{v})| \le K\,|u-\overline{u}| + L\,|v-\overline{v}|$$

for any $u, v, \overline{u}, \overline{v} \in \mathbb{R}$ and $t \in J$.

If

$$\left(\frac{K}{1-L}\right)\frac{B(\gamma,\alpha)}{\Gamma(\alpha)}\left(|Z|\sum_{i=1}^{m} c_i(\tau_i)^{\alpha+\gamma-1} + T^{\alpha}\right) < 1, \tag{16}$$

then the system (1) has a unique solution.

Proof. Let the operator $N : C_{1-\gamma}[J,\mathbb{R}] \to C_{1-\gamma}[J,\mathbb{R}]$.

$$(Nx)(t) = \frac{|Z|}{\Gamma(\alpha)} t^{\gamma-1} \sum_{i=1}^{m} c_i \int_0^{\tau_i} (\tau_i - s)^{\alpha-1} K_x(s)\mathrm{d}s + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} K_x(s)\mathrm{d}s.$$

By Lemma 2.12, it is clear that the fixed points of N are solutions of system (1).
Let $x_1, x_2 \in C_{1-\gamma}[J,\mathbb{R}]$ and $t \in J$, then we have

$$\begin{aligned}&\left|((Nx_1)(t) - (Nx_2)(t))\, t^{1-\gamma}\right| \\ &\quad \le \frac{|Z|}{\Gamma(\alpha)} \sum_{i=1}^{m} c_i \int_0^{\tau_i} (\tau_i - s)^{\alpha-1} \left|K_{x_1}(s) - K_{x_2}(s)\right| \mathrm{d}s \\ &\quad\quad + \frac{t^{1-\gamma}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left|K_{x_1}(s) - K_{x_2}(s)\right| \mathrm{d}s, \end{aligned} \tag{17}$$

and

$$\begin{aligned}\left|K_{x_1}(t) - K_{x_2}(t)\right| &= \left|f(t, x_1(t), K_{x_1}(t)) - f(t, x_2(t), K_{x_2}(t))\right| \\ &\le K\left|x_1(t) - x_2(t)\right| + L\left|K_{x_1}(t) - K_{x_2}(t)\right| \\ &\le \frac{K}{1-L}\left|x_1(t) - x_2(t)\right|. \end{aligned} \tag{18}$$

By replacing (18) in the inequality (17), we get

$$\begin{aligned}&\left|((Nx_1)(t) - (Nx_2)(t))\, t^{1-\gamma}\right| \\ &\le \frac{|Z|}{\Gamma(\alpha)} \sum_{i=1}^{m} c_i \left(\frac{K}{(1-L)} B(\gamma,\alpha)(\tau)^{\alpha+\gamma-1} \left\|x_1 - x_2\right\|_{C_{1-\gamma}}\right) \\ &\quad + \frac{t^{1-\gamma}}{\Gamma(\alpha)} t^{\alpha+\gamma-1} \left(\frac{K}{1-L}\right) B(\gamma,\alpha) \left\|x_1 - x_2\right\|_{C_{1-\gamma}} \\ &\le \left(\frac{K}{1-L}\right) \frac{1}{\Gamma(\alpha)} B(\gamma,\alpha) \left(|Z| \sum_{i=1}^{m} c_i (\tau_i)^{\alpha+\gamma-1} + T^{\alpha}\right) \left\|x_1 - x_2\right\|_{C_{1-\gamma}}. \end{aligned}$$

Hence,

$$\begin{aligned}&\left\|(Nx_1) - (Nx_2)\right\|_{C_{1-\gamma}} \\ &\quad \le \left(\frac{K}{1-L}\right) \frac{1}{\Gamma(\alpha)} B(\gamma,\alpha) \left(|Z| \sum_{i=1}^{m} c_i (\tau_i)^{\alpha+\gamma-1} + T^{\alpha}\right) \left\|x_1 - x_2\right\|_{C_{1-\gamma}}. \end{aligned}$$

From (16), it follows that N has a unique fixed point which is solution of problem (1). □

4. Stability Analysis

In this section, we prove four different types of Ulam stability results for problem (1).

Now we consider the Ulam–Hyers stability for the problem

$$D_{0^+}^{\alpha,\beta} x(t) = f(t, x(t), D_{0^+}^{\alpha,\beta} x(t)), \quad t \in J := [0, T]. \tag{19}$$

Let $\epsilon > 0$ and $\varphi : J \to [0,\infty)$ be a continuous function. Akbar Zada et al. [31] used the Ulam stability concepts for the problem (19) that will be used in Section 4. We consider the following inequalities

$$\left| D_{0^+}^{\alpha,\beta} z(t) - f(t, z(t), D_{0^+}^{\alpha,\beta} z(t)) \right| \leq \epsilon, \quad t \in J, \tag{20}$$

$$\left| D_{0^+}^{\alpha,\beta} z(t) - f(t, z(t), D_{0^+}^{\alpha,\beta} z(t)) \right| \leq \epsilon\varphi(t), \quad t \in J, \tag{21}$$

$$\left| D_{0^+}^{\alpha,\beta} z(t) - f(t, z(t), D_{0^+}^{\alpha,\beta} z(t)) \right| \leq \varphi(t), \quad t \in J. \tag{22}$$

Definition 4.1. Equation (1) is Ulam–Hyers stable if there exists a real number $C_f > 0$ such that for each $\epsilon > 0$ and for each solution $z \in C_{1-\gamma}^{\gamma}[J, \mathbb{R}]$ of the inequality (20) there exists a solution $x \in C_{1-\gamma}^{\gamma}[J, \mathbb{R}]$ of Eq. (1) with

$$|z(t) - x(t)| \leq C_f \epsilon, \quad t \in J.$$

Definition 4.2. Equation (1) is generalized Ulam–Hyers stable if there exists $\psi_f \in C\left([0, \infty), [0, \infty)\right)$, $\psi_f(0) = 0$ such that for each solution $z \in C_{1-\gamma}^{\gamma}[J, \mathbb{R}]$ of the inequality (20) there exists a solution $x \in C_{1-\gamma}^{\gamma}[J, \mathbb{R}]$ of Eq. (1) with

$$|z(t) - x(t)| \leq \psi_f \epsilon, \quad t \in J.$$

Definition 4.3. Equation (1) is Ulam–Hyers–Rassias stable with respect to $\varphi \in C_{1-\gamma}[J, \mathbb{R}]$ if there exists a real number $C_f > 0$ such that for each $\epsilon > 0$ and for each solution $z \in C_{1-\gamma}^{\gamma}[J, \mathbb{R}]$ of the inequality (21) there exists a solution $x \in C_{1-\gamma}^{\gamma}[J, \mathbb{R}]$ of Eq. (1) with

$$|z(t) - x(t)| \leq C_f \epsilon\varphi(t), \quad t \in J.$$

Definition 4.4. Equation (1) is generalized Ulam–Hyers–Rassias stable with respect to $\varphi \in C_{1-\gamma}[J, \mathbb{R}]$ if there exists a real number $C_{f,\varphi} > 0$ such that for each solution $z \in C_{1-\gamma}^{\gamma}[J, \mathbb{R}]$ of the inequality (22) there exists a solution $x \in C_{1-\gamma}^{\gamma}[J, \mathbb{R}]$ of Eq. (1) with

$$|z(t) - x(t)| \leq C_{f,\varphi}\varphi(t), \quad t \in J.$$

Remark 4.5. It is clear that

1. Definition 4.1⇒ Definition 4.2.
2. Definition 4.3⇒ Definition 4.4.
3. Definition 4.3 for $\varphi(t) = 1$ ⇒ Definition 4.1.

Remark 4.6. A function $z \in C_{1-\gamma}^{\gamma}[J, \mathbb{R}]$ is a solution of the inequality

$$\left| D_{0^+}^{\alpha,\beta} z(t) - f(t, z(t), D_{0^+}^{\alpha,\beta} z(t)) \right| \leq \epsilon, \quad t \in J,$$

if and only if there exist a function $g \in C_{1-\gamma}^{\gamma}[J, \mathbb{R}]$ such that

(i) $\mid g(t) \mid \leq \epsilon, t \in J,$

(ii) $D_{0^+}^{\alpha,\beta} z(t) = f(t, z(t), D_{0^+}^{\alpha,\beta} z(t)) + g(t), t \in J.$

Lemma 4.7. *Let $0 < \alpha < 1$, $0 \leq \beta \leq 1$, if a function $z \in C_{1-\gamma}^{\gamma}[J, \mathbb{R}]$ is a solution of the inequality (20), then z is a solution of the following integral inequality*

$$\left| z(t) - A_z - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} K_z(s) \mathrm{d}s \right| \leq \left(\frac{|Z|\,(mc)T^{\gamma+\alpha-1}}{\Gamma(\alpha+1)} + \frac{T^{\alpha}}{\Gamma(\alpha+1)} \right) \epsilon, \tag{23}$$

where

$$A_z = \frac{Zt^{\gamma-1}}{\Gamma(\alpha)} \sum_{i=1}^{m} c_i \int_1^{\tau_i} (\tau_i - \alpha)^{\alpha-1} K_z(s) \mathrm{d}s.$$

Proof. Indeed by Remark 4.6, we have that

$$\begin{aligned} D_{0^+}^{\alpha,\beta} z(t) &= f(t, x(t), D_{0^+}^{\alpha,\beta} z(t)) + g(t) \\ &= K_z(t) + g(t). \end{aligned}$$

Then

$$\begin{aligned} z(t) = \frac{Zt^{\gamma-1}}{\Gamma(\alpha)} \sum_{i=1}^{m} c_i \left(\int_0^{\tau_i} (\tau_i - s)^{\alpha-1} K_z(s) \mathrm{d}s + \int_0^{\tau_i} (\tau_i - s)^{\alpha-1} g(s) \mathrm{d}s \right) \\ + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} K_z(s) \mathrm{d}s + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s) \mathrm{d}s. \end{aligned}$$

From this it follows that

$$\begin{aligned} &\left| z(t) - A_z - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} K_z(s) \mathrm{d}s \right| \\ &= \left| \frac{Zt^{\gamma-1}}{\Gamma(\alpha)} \sum_{i=1}^{m} c_i \int_0^{\tau_i} (\tau_i - s)^{\alpha-1} g(s) \mathrm{d}s + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s) \mathrm{d}s \right| \\ &\leq \frac{|Z| \, t^{\gamma-1}}{\Gamma(\alpha)} \sum_{i=1}^{m} c_i \int_0^{\tau_i} (\tau_i - s)^{\alpha-1} |g(s)| \, \mathrm{d}s + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |g(s)| \, \mathrm{d}s \\ &\leq \left(\frac{|Z| \, (mc) T^{\gamma+\alpha-1}}{\Gamma(\alpha+1)} + \frac{T^{\alpha}}{\Gamma(\alpha+1)} \right) \epsilon. \end{aligned}$$

Lemma 4.8. *Let $0 < \alpha < 1$, $0 \leq \beta \leq 1$, if a function $z \in C_{1-\gamma}^{\gamma}[J, \mathbb{R}]$ is a solution of the inequality (20), then z is a solution of the following integral inequality*

$$\left| z(t) - A_z - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} K_z(s) \mathrm{d}s \right| \leq \left(Zt^{\gamma-1}(mc) + 1 \right) \epsilon \lambda_{\varphi} \varphi(t), \tag{24}$$

where

$$A_z = \frac{Zt^{\gamma-1}}{\Gamma(\alpha)} \sum_{i=1}^{m} c_i \int_1^{\tau_i} (\tau_i - \alpha)^{\alpha-1} K_z(s) \mathrm{d}s.$$

Proof. The proof of the theorem directly follows from Remark 4.6 and Lemma 4.7. □

We have similar remark for the solution of inequalities (22).

We state the following generalization of Gronwall's lemma for singular kernels.

Lemma 4.9. [see [8], Lemma 3.4] *Let* $v : [0,T] \to [0,\infty)$ *be a real function and* $w(\cdot)$ *is a nonnegative, locally integrable function on* $[0,T]$ *and there are constants* $a > 0$ *and* $0 < \alpha < 1$ *such that*

$$v(t) \leq w(t) + a\int_0^t \frac{v(s)}{(t-s)^{\alpha}}\mathrm{d}s.$$

Then there exists a constant $K = K(\alpha)$ *such that*

$$v(t) \leq w(t) + Ka\int_0^t \frac{w(s)}{(t-s)^{\alpha}}\mathrm{d}s,$$

for every $t \in [0,T]$.

Now, we are ready to prove our stability results for problem (1).

Theorem 4.10. *If the hypothesis* (H2) *and* (16) *are satisfied, then the problem* (1) *is Ulam–Hyers stable.*

Proof. Let $\epsilon > 0$ and let $z \in C^{\gamma}_{1-\gamma}[J,\mathbb{R}]$ be a function which satisfies the inequality:

$$\left|D^{\alpha,\beta}_{0^+}z(t) - f(t,z(t),D^{\alpha,\beta}_{0^+}z(t))\right| \leq \epsilon \quad \text{for any } t \in J, \tag{25}$$

and let $x \in C^{\gamma}_{1-\gamma}[J,\mathbb{R}]$ the unique solution of the following implicit differential equation

$$D^{\alpha,\beta}_{0^+}x(t) = f(t,x(t),D^{\alpha,\beta}_{0^+}x(t)), \quad t \in J := [0,T],$$
$$I^{1-\gamma}_{0^+}x(0) = I^{1-\gamma}_{0^+}z(0) = \sum_{i=1}^{m} c_i x(\tau_i), \quad \tau_i \in [0,T],\ \gamma = \alpha+\beta-\alpha\beta,$$

where $0 < \alpha < 1, \quad 0 \leq \beta \leq 1$.

Using Lemma 2.12, we obtain

$$x(t) = A_x + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}K_x(s)\mathrm{d}s,$$

where

$$A_x = \frac{|Z|}{\Gamma(\alpha)}t^{\gamma-1}\sum_{i=1}^{m} c_i\int_0^{\tau_i}(\tau_i-s)^{\alpha-1}K_x(s)\mathrm{d}s.$$

On the other hand, if $x(\tau_i) = z(\tau_i)$, and $I^{1-\gamma}_{0^+}x(0) = I^{1-\gamma}_{0^+}z(0)$, then $A_x = A_z$. Indeed,

$$\begin{aligned}|A_x - A_z| &\leq \frac{|Z|}{\Gamma(\alpha)}t^{\gamma-1}\sum_{i=1}^{m} c_i\int_0^{\tau_i}(\tau_i-s)^{\alpha-1}\left|K_x(s)-K_z(s))\right|\mathrm{d}s\\ &\leq \frac{|Z|}{\Gamma(\alpha)}t^{\gamma-1}\sum_{i=1}^{m} c_i\int_0^{\tau_i}(\tau_i-s)^{\alpha-1}\left(\frac{K}{1-L}\right)|x(s)-z(s)|\,\mathrm{d}s\\ &\leq \left(\frac{K}{1-L}\right)\frac{|Z|}{\Gamma(\alpha)}t^{\gamma-1}\sum_{i=1}^{m} c_i I^{\alpha}_{0^+}\left|x(\tau_i)-z(\tau_i)\right|\\ &= 0.\end{aligned}$$

Thus,

$$A_x = A_z.$$

Then, we have

$$x(t) = A_z + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} K_x(s) \mathrm{d}s.$$

By integration of the inequality (25) and applying Lemma 4.7, we obtain

$$\left| z(t) - A_z - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} K_z(s) \mathrm{d}s \right| \leq \left(\frac{|Z|\,(mc)T^{\gamma+\alpha-1}}{\Gamma(\alpha+1)} + \frac{T^\alpha}{\Gamma(\alpha+1)} \right) \epsilon. \tag{26}$$

We have for any $t \in J$

$$\begin{aligned} |z(t) - x(t)| &\leq \left| z(t) - A_z - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} K_z(s) \mathrm{d}s \right| \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |K_z(s) - K_x(s)| \, \mathrm{d}s \\ &\leq \left| z(t) - A_z - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} K_z(s) \mathrm{d}s \right| \\ &\quad + \left(\frac{K}{1-L} \right) \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |z(s) - x(s)| \, \mathrm{d}s. \end{aligned}$$

By using (26), we have

$$\begin{aligned} |z(t) - x(t)| &\leq \left(\frac{|Z|\,(mc)T^{\gamma+\alpha-1}}{\Gamma(\alpha+1)} + \frac{T^\alpha}{\Gamma(\alpha+1)} \right) \epsilon \\ &\quad + \left(\frac{K}{1-L} \right) \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |z(s) - x(s)| \, \mathrm{d}s \end{aligned}$$

and to apply Lemma 4.9, we obtain

$$\begin{aligned} |z(t) - x(t)| &\leq \left(\frac{|Z|\,(mc)T^{\gamma+\alpha-1}}{\Gamma(\alpha+1)} + \frac{T^\alpha}{\Gamma(\alpha+1)} \right) \left[1 + \frac{\nu K}{(1-L)\Gamma(\alpha+1)} T^\alpha \right] \epsilon \\ &:= C_f \epsilon, \end{aligned}$$

where $\nu = \nu(\alpha)$ is a constant, which completes the proof of the theorem. Moreover, if we set $\psi(\epsilon) = C_f \epsilon; \psi(0) = 0$, then the problem (1) is generalized Ulam–Hyers stable. □

First we introduce the following assumption:

(H3) There exists an increasing function $\varphi \in C_{1-\gamma}[J, \mathbb{R}]$ and there exists $\lambda_\varphi > 0$ such that for any $t \in J$

$$I_{0^+}^\alpha \varphi(t) \leq \lambda_\varphi \varphi(t).$$

Theorem 4.11. *Assume that* (H2), (H3) *and* (16) *are satisfied, then the problem* (1) *is Ulam–Hyers–Rassias stable.*

Proof. Let $\epsilon > 0$ and let $z \in C^{\gamma}_{1-\gamma}[J,\mathbb{R}]$ be a function which satisfies the inequality:

$$\left| D^{\alpha,\beta}_{0^+} z(t) - f(t, z(t), D^{\alpha,\beta}_{0^+} z(t)) \right| \leq \epsilon\varphi(t) \quad \text{for any } t \in J, \tag{27}$$

and let $x \in C^{\gamma}_{1-\gamma}[J,\mathbb{R}]$ the unique solution of the following implicit differential equation

$$D^{\alpha,\beta}_{0^+} x(t) = f(t, x(t), D^{\alpha,\beta}_{0^+} x(t)), \quad t \in J := [0,T],$$
$$I^{1-\gamma}_{0^+} x(0) = I^{1-\gamma}_{0^+} z(0) = \sum_{i=1}^{m} c_i x(\tau_i), \quad \tau_i \in [0,T], \ \gamma = \alpha + \beta - \alpha\beta$$

where $0 < \alpha < 1, \quad 0 \leq \beta \leq 1$.

Using Lemma 2.12, we obtain

$$x(t) = A_z + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} K_x(s)\mathrm{d}s,$$

where

$$A_z = \frac{|Z|}{\Gamma(\alpha)} t^{\gamma-1} \sum_{i=1}^{m} c_i \int_0^{\tau_i} (\tau_i - s)^{\alpha-1} K_z(s)\mathrm{d}s.$$

By integration of (27) and applying Lemma 4.8, we obtain

$$\left| z(t) - A_z - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} K_z(s)\mathrm{d}s \right| \leq \left(|Z|\, t^{\gamma-1}(mc) + 1 \right) \epsilon\lambda_\varphi \varphi(t).$$

On the other hand, we have

$$\begin{aligned} |z(t) - x(t)| \leq{}& \left| z(t) - A_z - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} K_z(s)\mathrm{d}s \right| \\ &+ \left(\frac{K}{1-L} \right) \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |z(s) - x(s)|\, \mathrm{d}s. \end{aligned}$$

By using (26), we have

$$\begin{aligned} |z(t) - x(t)| \leq{}& \left(|Z|\, t^{\gamma-1}(mc) + 1 \right) \epsilon\lambda_\varphi \varphi(t) \\ &+ \left(\frac{K}{1-L} \right) \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |z(s) - x(s)|\, \mathrm{d}s \end{aligned}$$

and to apply Lemma 4.9, we obtain

$$|z(t) - x(t)| \leq \left[\left(|Z|\, t^{\gamma-1}(mc) + 1 \right) \left(1 + \frac{K\nu_1\lambda_\varphi}{1-L} \right) \lambda_\varphi \right] \epsilon\varphi(t),$$

where $\nu_1 = \nu_1(\alpha)$ is a constant, then for any $t \in J$:

$$|z(t) - x(t)| \leq C_f \epsilon\varphi(t),$$

which completes the proof of the theorem. □

In next section, we give some examples to illustrate the usefulness of our main results.

5. Examples

Example 5.1. Consider FIDE with Hilfer fractional derivative

$$D_{1^+}^{\alpha,\beta}x(t) = \frac{1}{5e^{t+2}\left(1+|x(t)|+\left|D_{1^+}^{\alpha,\beta}x(t)\right|\right)},\ \text{for},\ t \in J := (1,2], \quad (28)$$

$$I_{1^+}^{1-\gamma}x(1) = 2x\left(\frac{2}{3}\right),\ \gamma = \alpha+\beta-\alpha\beta. \quad (29)$$

Now

$$f(t,u,v) = \frac{1}{5e^{t+2}(1+|u|+|v|)},\ t \in J, \quad u,v \in [0,\infty),$$

and we see that $\alpha = \frac{1}{2}$, $\beta = \frac{1}{2}$ and $\gamma = \frac{3}{4}$. Clearly, the function f is continuous, and for $u, v, \overline{u}, \overline{v} \in [0,\infty)$ and $t \in J$,

$$|f(t,u,v) - f(t,\overline{u},\overline{v})| \leq \frac{1}{5e^3}\left(|u-\overline{u}|+|v-\overline{v}|\right).$$

Hence, the condition (H2) is satisfied with $K = L = \frac{1}{5e^3}$.

Thus,

$$\left(\frac{K}{1-L}\right)\frac{B(\gamma,\alpha)}{\Gamma(\alpha)}\left(|Z|\sum_{i=1}^{m}c_i(\tau_i)^{\alpha+\gamma-1}+T^\alpha\right) < 1.$$

It follows from Theorem 3.2 that the problem (28) and (29) has a unique solution on J. For $t \in J$, let $\varphi(t) = t$. Since

$$\begin{aligned} I_{1^+}^{\frac{1}{2}}\varphi(t)\frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_1^t (t-s)^{\frac{1}{2}-1}(s)\mathrm{d}s \\ \leq \frac{t}{\Gamma\left(\frac{1}{2}\right)}\int_1^t (t-s)^{\frac{1}{2}-1}\mathrm{d}s \\ \leq \frac{2\varphi(t)}{\sqrt{\pi}}, \end{aligned}$$

condition (H3) is satisfied with $\lambda_\varphi = \frac{2}{\sqrt{\pi}}$. It follows from Theorem 4.11 that the problem (28), (29) is Ulam–Hyers–Rassias stable.

Example 5.2. Consider the FIDE with Hilfer fractional derivative

$$D_{1^+}^{\alpha,\beta}x(t) = \frac{2+|x(t)|+\left|D_{1^+}^{\alpha,\beta}x(t)\right|}{108e^{t+3}\left(1+|x(t)|+\left|D_{1^+}^{\alpha,\beta}x(t)\right|\right)},\ \text{for}\ t \in J := (1,2], \quad (30)$$

$$I_{1^+}^{1-\gamma}x(1) = 3x\left(\frac{6}{5}\right)+2x\left(\frac{3}{2}\right). \quad (31)$$

Here

$$f(t,u,v) = \frac{2+|u|+|v|}{108e^{t+3}(1+|u|+|v|)}, \quad t \in J,\ u,v \in [0,\infty),$$

and $\alpha = \frac{1}{2}$, $\beta = \frac{2}{3}$ and $\gamma = \frac{5}{6}$. Clearly, f is continuous, and for $u, v, \overline{u}, \overline{v} \in [0, \infty)$ and $t \in J$,

$$|f(t,u,v) - f(t,\overline{u},\overline{v})| \le \frac{1}{108e^4} \left(|u-\overline{u}| + |v-\overline{v}|\right).$$

Hence, condition (H2) is satisfied with $K = L = \frac{1}{108e^4}$. In addition, for $t \in J$,

$$|f(t,u,v)| \le \frac{1}{108e^{t+3}} \left(2 + |u| + |v|\right),$$

so condition (H1) is satisfied with $p(t) = \frac{1}{54e^{t+3}}$, $q(t) = r(t) = \frac{1}{108e^{t+3}}$, and $q^*(t) = r^*(t) = \frac{1}{108e^4}$. Thus, condition (H2) is satisfied with $K = L = \frac{1}{108e^4}$. We see that (16) holds with $|Z| \approx 0.19844$. So it follows from Theorem 3.2 that the problem (30), (31) has at least one solution on J.

Example 5.3. Consider the FIDE with Hilfer fractional derivative

$$\begin{aligned} D_{1^+}^{\alpha,\beta} x(t) &= \frac{1}{10}\left[\frac{|x(t)|}{1+|x(t)|} + \frac{\left|D_{1^+}^{\alpha,\beta} x(t)\right|}{1+\left|D_{1^+}^{\alpha,\beta} x(t)\right|}\right], \quad t \in J := (1,2], \\ I_{1^+}^{1-\gamma} x(1) &= 2x\left(\frac{3}{2}\right), \quad \gamma = \alpha + \beta - \alpha\beta. \end{aligned} \tag{32}$$

Notice that this problem is a particular case of (1), where $\alpha = \frac{2}{3}$, $\beta = \frac{1}{2}$ and choose $\gamma = \frac{5}{6}$.
Set

$$f(t,u,v) = \frac{1}{10}\left[\frac{u}{1+u} + \frac{v}{1+v}\right], \quad \text{for any } u, v \in [0,\infty).$$

Clearly, the function f satisfies the conditions of Theorem 3.1.
For any $u, v, \overline{u}, \overline{v} \in [0, \infty)$ and $t \in J$,

$$|f(t,u,v) - f(t,\overline{u},\overline{v})| \le \frac{1}{10}|u-\overline{u}| + \frac{1}{10}|v-\overline{v}|.$$

Hence the condition (H2) is satisfied with $K = L = \frac{1}{10}$.
Thus condition from (16)

$$\frac{K}{(1-L)} \frac{B(\gamma,\alpha)}{\Gamma(\alpha)} \left(|Z| \sum_{i=1}^{m} c_i (\tau_i)^{\alpha+\gamma-1} + T^\alpha\right) = 0.2295 < 1,$$

where $|Z| = 0.8959$.

It follows from Theorem 3.2 that the problem (32) has a unique solution. Moreover, Theorem 4.11 implies that the problem (32) is Ulam–Hyers stable.

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