Mediterr. J. Math. (2018) 15:12
https://doi.org/10.1007/s00009-017-1057-9
1660-5446/18/010001-22
published online December 29, 2017

Mediterranean Journal of Mathematics

Characterizations of Spacelike Submanifolds with Constant Scalar Curvature in the de Sitter Space

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Abstract. In this paper, we deal with complete spacelike submanifolds M^n immersed in the de Sitter space $\mathbb{S}_p^{n+p}$ of index p with parallel normalized mean curvature vector and constant scalar curvature R. Imposing a suitable restriction on the values of R, we apply a maximum principle for the so-called Cheng–Yau operator L, which enables us to show that either such a submanifold must be totally umbilical or it holds a sharp estimate for the norm of its total umbilicity tensor, with equality if and only the submanifold is isometric to a hyperbolic cylinder of the ambient space. In particular, when $n = 2$ this provides a nice characterization of the totally umbilical spacelike surfaces of $\mathbb{S}_p^{2+p}$ with codimension $p \geq 2$. Furthermore, we also study the case in which these spacelike submanifold are L-parabolic.

Mathematics Subject Classification. Primary 53C42; Secondary 53A10, 53C20, 53C50.

Keywords. De Sitter space, parallel normalized mean curvature vector, constant scalar curvature, totally umbilical submanifolds, hyperbolic cylinders.

1. Introduction

At the end of the 70s, Goddard conjectured in his seminal paper [15] that the unique complete spacelike hypersurfaces of de Sitter space $\mathbb{S}_1^{n+1}$ with constant mean curvature H were just the totally umbilical. Ten years have passed until Ramanathan [20] proved that Goddard's conjecture is true for $\mathbb{S}_1^3$ and $0 \leq H \leq 1$. However, for $H > 1$ he showed that the conjecture is false, as it can be seen from an example from Dajczer and Nomizu in [14]. Simultaneously and independently, Akutagawa [3] also proved that Goddard's conjecture is true when either $n = 2$ and $H^2 \leq 1$ or $n \geq 3$ and $H^2 < \frac{4(n-1)}{n^2}$. Moreover, he also constructed complete spacelike rotation surfaces in $\mathbb{S}_1^3$ with constant H satisfying $H > 1$ and which are not totally umbilical.

In [17], Montiel proved that Goddard's conjecture is true provided that M^n is compact. Furthermore, he exhibited examples of complete spacelike hypersurfaces in $\mathbb{S}_1^{n+1}$ with constant H satisfying $H^2 \geq \frac{4(n-1)}{n^2}$ and being non-totally umbilical, the so-called *hyperbolic cylinders*, which are isometric to a Riemannian product of the type $\mathbb{H}^1(r) \times \mathbb{S}^{n-1}(\sqrt{1+r^2})$, for some $r > 0$. In [18], Montiel characterized these hyperbolic cylinders as the only complete non-compact spacelike hypersurfaces in $\mathbb{S}_1^{n+1}$ with constant mean curvature $H^2 = 4(n-1)/n^2$ and having at least two ends. Later on, in [8], Brasil, Colares and Palmas obtained a sort of extension of Montiel's result, showing that the hyperbolic cylinders are the only complete spacelike hypersurfaces in $\mathbb{S}_1^{n+1}$ with constant mean curvature, nonnegative Ricci curvature and having at least two ends. They also characterized all complete spacelike hypersurfaces of constant mean curvature with two distinct principal curvatures as been rotation hypersurfaces or generalized hyperbolic cylinders $\mathbb{H}^k(r) \times \mathbb{S}^{n-k}(\sqrt{1+r^2})$.

Regarding higher codimension, Cheng [11] extended Akutagawa's result for complete spacelike submanifolds with parallel mean curvature vector in de Sitter space $\mathbb{S}_p^{n+p}$ of index p. Afterwards, Aiyama [2] studied compact spacelike submanifolds M^n in $\mathbb{S}_p^{n+p}$ with parallel mean curvature vector and proved that if the normal connection of M^n is flat, then M^n is totally umbilical. In the same work [2], she proved that compact spacelike submanifolds in $\mathbb{S}_p^{n+p}$ with parallel mean curvature vector and nonnegative sectional curvatures must also be totally umbilical. Next, Li [16] showed that Montiel's result still holds for higher codimensional spacelike submanifolds in $\mathbb{S}_p^{n+p}$. More recently, Camargo, Chaves and Sousa [9] studied complete spacelike submanifolds with parallel normalized mean curvature vector and constant scalar curvature immersed in a semi-Riemannian space form $\mathbb{Q}_p^{n+p}(c)$ of constant sectional curvature c and index p. In particular, they obtained characterization results concerning totaly umbilical spacelike submanifolds and hyperbolic cylinders of $\mathbb{S}_p^{n+p}$, under certain constraints on both the squared norm of the second fundamental form and on the mean curvature.

In this paper, we deal with complete spacelike submanifolds M^n immersed in the de Sitter space $\mathbb{S}_p^{n+p}$ of index p, with parallel normalized mean curvature vector and constant scalar curvature R. When $0 < R \leq 1$, we apply a maximum principle for the so-called Cheng–Yau operator L (Lemma 2), which enables us to show that either such a submanifold must be totally umbilical or it holds a sharp estimate for the norm of its total umbilicity tensor $|\Phi|^2$, with equality if and only the submanifold is isometric to a hyperbolic cylinder of the ambient space (Theorem 1). In particular, when $n = 2$ this characterizes the totally umbilical spacelike surfaces of $\mathbb{S}_p^{2+p}$, with codimension $p \geq 2$, as the only complete spacelike surfaces in $\mathbb{S}_p^{2+p}$ with parallel normalized mean curvature vector, constant Gaussian curvature $0 < K \leq 1$ and such that $\sup_M |\Phi|^2 < \frac{2p}{p-1} K$ (Corollary 3). Furthermore, we also study the case in which these spacelike submanifold are L-parabolic (see Theorem 2 and its Corollary 3).

The manuscript is organized as follows: initially, in Sect. 2 we develop a suitable Simons type formula concerning spacelike submanifolds immersed in $\mathbb{S}_p^{n+p}$ and having positive mean curvature function. In Sect. 3 we quote some auxiliary results which constitute our analytical and algebraic machineries. In particular, as application of Theorem 6.13 of [7] (see also Lemma 4.2 of [6]), we obtain a generalized maximum principle for the Cheng–Yau operator L (see Lemma 2). In Sect. 4, we use our Simons type formula to obtain an appropriated lower estimate to the operator L acting on the square of norm of total umbilicity tensor of a spacelike submanifold with constant scalar curvature (see Proposition 1) and, next, we establish our characterization theorems related to submanifolds totally umbilical and hyperbolic cylinders of $\mathbb{S}_p^{n+p}$ (see Theorems 1 and 2).

2. A Simons Type Formula in $\mathbb{S}_p^{n+p}$

We recall that a submanifold immersed into an indefinite ambient space is said to be *spacelike* if its induced metric is positive definite. So, let M^n be an n-dimensional connected spacelike submanifold immersed in the de Sitter space $\mathbb{S}_p^{n+p}$ of index p. We choose a local field of semi-Riemannian orthonormal frame $\{e_1, \ldots, e_{n+p}\}$ in $\mathbb{S}_p^{n+p}$, with dual co-frame $\{\omega_1, \ldots, \omega_{n+p}\}$, such that, at each point of M^n, $e_1, \ldots, e_n$ is tangent to M^n. We will use the following convention for the indices:

$$1 \leq A, B, C, \ldots \leq n+p, \quad 1 \leq i, j, k, \ldots \leq n \quad \text{and} \quad n+1 \leq \alpha, \beta, \gamma, \ldots \leq n+p.$$

In this setting, the indefinite metric of $\mathbb{S}_p^{n+p}$ of index p is given by

$$ds^2 = \sum_A \epsilon_A \, \omega_A^2 = \sum_i \omega_i^2 - \sum_\alpha \omega_\alpha^2,$$

where $\epsilon_i = 1$ and $\epsilon_\alpha = -1$, $1 \leq i \leq n$, $n+1 \leq \alpha \leq n+p$. Denoting by $\{\omega_{AB}\}$ the connection forms of $\mathbb{S}_p^{n+p}$, we have the structure equations of $\mathbb{S}_p^{n+p}$ are given by:

$$d\omega_A = \sum_B \epsilon_B \, \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0, \tag{2.1}$$

$$d\omega_{AB} = \sum_C \epsilon_C \, \omega_{AC} \wedge \omega_{CB} - \frac{1}{2} \sum_{C,D} \epsilon_C \epsilon_D K_{ABCD} \, \omega_C \wedge \omega_D, \tag{2.2}$$

where

$$K_{ABCD} = \epsilon_A \epsilon_B (\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}).$$

Next, we restrict all the tensors to M^n. First of all,

$$\omega_\alpha = 0, \quad n+1 \leq \alpha \leq n+p.$$

Thus, the Riemannian metric of M^n is written as $ds^2 = \sum_i \omega_i^2$. Since $\sum_i \omega_{\alpha i} \wedge \omega_i = d\omega_\alpha = 0$ and by Cartan's Lemma we can write

$$\omega_{\alpha i} = \sum_j h_{ij}^\alpha \omega_j, \quad h_{ij}^\alpha = h_{ji}^\alpha. \tag{2.3}$$

This gives the second fundamental form of M^n, $A = \sum_{\alpha,i,j} h_{ij}^\alpha \omega_i \otimes \omega_j e_\alpha$. Furthermore, we define the mean curvature vector h and the mean curvature function H of M^n, respectively, by

$$h = \frac{1}{n} \sum_\alpha \left(\sum_i h_{ii}^\alpha \right) e_\alpha \quad \text{and} \quad H = |h| = \sqrt{\sum_\alpha \left(\sum_i h_{ii}^\alpha \right)^2}.$$

From (2.1) and (2.2), the structure equations of M^n are divided into tangent part

$$d\omega_i = \sum_j \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0,$$

$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2} \sum_{k,l} R_{ijkl} \omega_k \wedge \omega_l,$$

where R_{ijkl} are the components of the curvature tensor of M^n.

Using the previous structure equations, we obtain the Gauss equation

$$R_{ijkl} = (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) - \sum_\alpha (h_{ik}^\alpha h_{jl}^\alpha - h_{il}^\alpha h_{jk}^\alpha). \tag{2.4}$$

The Ricci curvature and the (normalized) scalar curvature of M^n are given, respectively, by

$$R_{ij} = (n-1)\delta_{ij} - \sum_\alpha \left(\sum_k h_{kk}^\alpha \right) h_{ij}^\alpha + \sum_{\alpha,k} h_{ik}^\alpha h_{kj}^\alpha \tag{2.5}$$

and

$$R = \frac{1}{n(n-1)} \sum_i R_{ii}. \tag{2.6}$$

From (2.5) and (2.6) we obtain

$$|A|^2 = n^2 H^2 + n(n-1)(R-1), \tag{2.7}$$

where $|A|^2 = \sum_{\alpha,i,j} (h_{ij}^\alpha)^2$ is the square of the length of the second fundamental form A of M^n.

We also quote the structure equations of the normal bundle of M^n

$$d\omega_\alpha = -\sum_\beta \omega_{\alpha\beta} \wedge \omega_\beta, \quad \omega_{\alpha\beta} + \omega_{\beta\alpha} = 0,$$

$$d\omega_{\alpha\beta} = -\sum_\gamma \omega_{\alpha\gamma} \wedge \omega_{\gamma\beta} - \frac{1}{2} \sum_{k,l} R_{\alpha\beta kl} \omega_k \wedge \omega_l, \tag{2.8}$$

where $R_{\alpha\beta ij}$ satisfies the Ricci equation

$$R_{\alpha\beta ij} = \sum_l \left(h_{il}^\alpha h_{lj}^\beta - h_{jl}^\alpha h_{li}^\beta \right). \tag{2.9}$$

The components h^α_{ijk} of the covariant derivative ∇A satisfy

$$\sum_k h^\alpha_{ijk}\omega_k = dh^\alpha_{ij} + \sum_k h^\alpha_{ik}\omega_{kj} + \sum_k h^\alpha_{jk}\omega_{ki} + \sum_\beta h^\beta_{ij}\omega_{\alpha\beta}. \tag{2.10}$$

In this setting, from (2.3) and (2.10) we get the Codazzi equation

$$h^\alpha_{ijk} = h^\alpha_{ikj} = h^\alpha_{kij}. \tag{2.11}$$

The first and the second covariant derivatives of h^α_{ij} denoted by h^α_{ijk} and h^α_{ijkl}, respectively, satisfy

$$\sum_l h^\alpha_{ijkl}\omega_l = dh^\alpha_{ijk} + \sum_l h^\alpha_{ljk}\omega_{li} + \sum_l h^\alpha_{ilk}\omega_{lj} + \sum_l h^\alpha_{ijl}\omega_{lk} + \sum_\beta h^\beta_{ijk}\omega_{\alpha\beta}.$$

Thus, taking the exterior derivative in (2.10), we obtain the Ricci identity

$$h^\alpha_{ijkl} - h^\alpha_{ijlk} = \sum_m h^\alpha_{mj}R_{mikl} + \sum_m h^\alpha_{im}R_{mjkl} + \sum_{k,\beta} h^\beta_{ik}R_{\alpha\beta jk}. \tag{2.12}$$

The Laplacian Δh^α_{ij} of h^α_{ij} is defined by $\Delta h^\alpha_{ij} = \sum_k h^\alpha_{ijkk}$. From equations (2.11) and (2.12), we obtain that

$$\Delta h^\alpha_{ij} = \sum_k h^\alpha_{kkij} + \sum_{k,l} h^\alpha_{kl}R_{lijk} + \sum_{k,l} h^\alpha_{li}R_{lkjk} + \sum_{k,\beta} h^\beta_{ik}R_{\alpha\beta jk}. \tag{2.13}$$

In what follows, we will consider the case that $H > 0$. So, we can choose a local orthonormal frame $\{e_1, \ldots, e_{n+p}\}$ such that $e_{n+1} = \frac{h}{H}$. Thus,

$$H^{n+1} = \frac{1}{n}\mathrm{tr}(h^{n+1}) = H \quad \text{and} \quad H^\alpha = \frac{1}{n}\mathrm{tr}(h^\alpha) = 0,\ \alpha \geq n+2, \tag{2.14}$$

where $h^\alpha = (h^\alpha_{ij})$ denotes the second fundamental form of M^n in direction e_α for every $n+1 \leq \alpha \leq n+p$. Hence, from (2.4), (2.9), (2.13) and (2.14), we obtain

$$\begin{aligned}\Delta h^{n+1}_{ij} = {} & nH_{ij} + nh^{n+1}_{ij} - nH\delta_{ij} + \sum_{\beta,k,m} h^{n+1}_{km}h^\beta_{mk}h^\beta_{ij} - 2\sum_{\beta,k,m} h^{n+1}_{km}h^\beta_{mj}h^\beta_{ik} \\ & + \sum_{\beta,k,m} h^{n+1}_{mi}h^\beta_{mk}h^\beta_{kj} - nH\sum_m h^{n+1}_{mi}h^{n+1}_{mj} + \sum_{\beta,k,m} h^{n+1}_{jm}h^\beta_{mk}h^\beta_{ki},\end{aligned} \tag{2.15}$$

and

$$\begin{aligned}\Delta h^\alpha_{ij} = {} & nH^\alpha_{ij} + nh^\alpha_{ij} + \sum_{\beta,k,m} h^\alpha_{km}h^\beta_{mk}h^\beta_{ij} - 2\sum_{\beta,k,m} h^\alpha_{km}h^\beta_{mj}h^\beta_{ik} \\ & + \sum_{\beta,k,m} h^\alpha_{mi}h^\beta_{mk}h^\beta_{kj} - nH\sum_m h^\alpha_{mi}h^{n+1}_{mj} + \sum_{\beta,k,m} h^\alpha_{jm}h^\beta_{mk}h^\beta_{ki},\end{aligned} \tag{2.16}$$

for every $n+2 \leq \alpha \leq n+p$.

Since

$$\Delta|A|^2 = 2\left(\sum_{\alpha,i,j} h^\alpha_{ij}\Delta h^\alpha_{ij} + \sum_{\alpha,i,j,k} (h^\alpha_{ijk})^2\right), \tag{2.17}$$

inserting (2.15) and (2.16) into (2.17), we obtain the following Simons type formula

$$\frac{1}{2}\Delta|A|^2 = \sum_{\alpha,i,j,k}(h^\alpha_{ijk})^2 + n\sum_{\alpha,i,j} h^\alpha_{ij}H^\alpha_{ij} + n(|A|^2 - nH^2) + \sum_{\alpha,\beta}\big(\mathrm{tr}(h^\alpha h^\beta)\big)^2 - nH\sum_{\alpha}\mathrm{tr}\left(h^{n+1}(h^\alpha)^2\right) + \sum_{\alpha,\beta} N\big(h^\alpha h^\beta - h^\beta h^\alpha\big)\,, \quad (2.18)$$

where $N(B) = \mathrm{tr}(BB^t)$, for all matrix $B = (b_{ij})$.

3. Auxiliary Lemmas

We devote this section to present some auxiliary lemmas which will be used to prove our main results. For this, we define on M^n the symmetric tensor $\Psi = \sum_{i,j=1}^n \psi_{ij}\omega_i \otimes \omega_j$, where $\psi_{ij} = nH\delta_{ij} - h^{n+1}_{ij}$. According to Cheng and Yau [12], we consider an operator L associated with Ψ acting on any smooth function $f \in \mathcal{C}^2(M)$ in the following way

$$Lf = \sum_{i,j=1}^n \psi_{ij}f_{ij} = \sum_{i,j}(nH\delta_{ij} - h^{n+1}_{ij})f_{ij} = nH\Delta f - \sum_{i,j} h^{n+1}_{ij}f_{ij}, \quad (3.1)$$

where f_{ij} stands for a component of the Hessian of f. Thus, from (3.1) we have that

$$Lf = \mathrm{tr}(P \circ \nabla^2 f), \quad (3.2)$$

where

$$P = nHI - h^{n+1}, \quad (3.3)$$

I is the identity in the algebra of smooth vector fields on M^n, $h^{n+1} = (h^{n+1}_{ij})$ denotes the second fundamental form of M^n in direction e_{n+1} and $\nabla^2 f$ stands for the self-adjoint linear operator metrically equivalent to the Hessian of f.

The following lemma gives a sufficient criteria for the ellipticity of the operator L.

Lemma 1. *Let M^n be a spacelike submanifold in the de Sitter space $\mathbb{S}^{n+p}_p$ with $H > 0$. Let μ_- and μ_+ be, respectively, the minimum and the maximum of the eigenvalues of the operator P at every point $p \in M^n$. If $R < 1$ (resp., $R \le 1$ on M^n), then the operator L is elliptic (resp., semi-elliptic), with*

$$\mu_- > 0 \quad (resp., \mu_- \ge 0).$$

and

$$\mu_+ < 2nH \quad (resp., \mu_+ \le 2nH).$$

Proof. Let us choose a local orthonormal frame $\{e_1, \ldots, e_n\}$ on M^n such that $h^{n+1}_{ij} = \lambda^{n+1}_i\delta_{ij}$. Thus, for all $i = 1, \ldots, n$, from (2.7) it is not difficult check that

$$(\lambda^{n+1}_i)^2 \le |A|^2 = n^2H^2 + n(n-1)(R-1) < n^2H^2,$$

where we have used our assumption that $R < 1$ to obtain the last inequality. Consequently, for all $i = 1, \ldots, n$, we have

$$|\lambda^{n+1}_i| < |nH|.$$

Since $H > 0$, we get

$$-nH < \lambda_i^{n+1} < nH,$$

consequently, for every i

$$0 < nH - \lambda_i^{n+1} < 2nH.$$

However, $\mu_i := nH - \lambda_i^{n+1}$ are precisely the eigenvalues of operator P. In particular, we conclude that $\mu_- > 0$ and $\mu_+ < 2nH$. The case $R \leq 1$ follows in a similar way. □

Taking $f = nH$ in (3.1), we get

$$L(nH) = nH\Delta(nH) - n\sum_{i,j} h_{ij}^{n+1}H_{ij}. \tag{3.4}$$

On the other hand, once $\frac{1}{2}\Delta(nH)^2 = nH\Delta(nH) + n^2|\nabla H|^2$, from (2.7) and (3.4), we have

$$L(nH) = \frac{1}{2}\Delta|A|^2 - \frac{n(n-1)}{2}\Delta R - n^2|\nabla H|^2 - n\sum_{ij} h_{ij}^{n+1}H_{ij}. \tag{3.5}$$

Now, assume that R is constant; from (2.18) and (3.5), we get

$$\begin{aligned} L(nH) = &\sum_{\alpha,i,j,k}(h_{ijk}^{\alpha})^2 + n\sum_{i,j} h_{ij}^{n+1}H_{ij}^{n+1} - n\sum_{ij} h_{ij}^{n+1}H_{ij} - n^2|\nabla H|^2 \\ &+ \sum_{\alpha,\beta} N\big(h^{\alpha}h^{\beta} - h^{\beta}h^{\alpha}\big) + n(|A|^2 - nH^2) + n\sum_{\substack{i,j\\ \alpha\geq n+2}} h_{ij}^{\alpha}H_{ij}^{\alpha} \\ &+ \sum_{\alpha,\beta}\big(\mathrm{tr}(h^{\alpha}h^{\beta})\big)^2 - nH\sum_{\alpha}\mathrm{tr}(h^{n+1}(h^{\alpha})^2). \end{aligned} \tag{3.6}$$

Under the assumption of having $H > 0$, expression (3.6) can be rewritten in a simpler way. For this, choose $\{e_1, \ldots, e_{n+p}\}$ a local orthonormal frame to M^n such that $e_{n+1} = \frac{h}{H}$. Hence, $H^{n+1} = H$ and $H^{\alpha} = 0$ for every $\alpha \geq n+2$, which implies that $H_{ij}^{n+1} = H_{ij}$, $H_{ij}^{\alpha} = 0$ and hence

$$\begin{aligned} L(nH) = &\sum_{\alpha,i,j,k}(h_{ijk}^{\alpha})^2 - n^2|\nabla H|^2 + \sum_{\alpha,\beta} N\big(h^{\alpha}h^{\beta} - h^{\beta}h^{\alpha}\big) + n(|A|^2 - nH^2) \\ &+ \sum_{\alpha,\beta}\big(\mathrm{tr}(h^{\alpha}h^{\beta})\big)^2 - nH\sum_{\alpha}\mathrm{tr}\left(h^{n+1}(h^{\alpha})^2\right). \end{aligned} \tag{3.7}$$

Now, we consider the following symmetric tensor

$$\Phi = \sum_{\alpha,i,j}\Phi_{ij}^{\alpha}\omega_i \otimes \omega_j e_{\alpha},$$

where $\Phi_{ij}^{\alpha} = h_{ij}^{\alpha} - H^{\alpha}\delta_{ij}$, and H^{α} is defined in (2.14).

Let $|\Phi|^2 = \sum_{\alpha,i,j}(\Phi_{ij}^{\alpha})^2$ be the square of the length of Φ. It is easy to check that Φ is traceless and, from (2.7), we get the following relation:

$$|\Phi|^2 = |A|^2 - nH^2 = n(n-1)H^2 + n(n-1)(R-1). \tag{3.8}$$

Moreover, $|\Phi|^2 \geq 0$, with equality at the umbilical points of M. For that reason Φ is usually called the total umbilicity tensor of M.

According to [7], we say that *the Omori–Yau maximum principle* holds on M^n for the operator L if, for any function $u \in \mathcal{C}^2(M)$ with $u^* = \sup_M u < \infty$, there exists a sequence $\{p_k\}_{k\in\mathbb{N}} \subset M^n$ with the properties

$$u(p_k) > u^* - \frac{1}{k}, \quad |\nabla u(p_k)| < \frac{1}{k} \quad \text{and} \quad Lu(p_k) < \frac{1}{k}$$

for every $k \in \mathbb{N}$. As a consequence of Theorem 6.13 of [7] (see also Lemma 4.2 of [6]), we obtain the following.

Lemma 2. *Let M^n be a complete non-compact spacelike submanifold in $\mathbb{S}_p^{n+p}$, with constant scalar curvature satisfying $R \leq 1$. If $\sup_M |\Phi|^2 < +\infty$, then the Omori–Yau maximum principle holds on M^n for the operator L.*

Proof. From equation (2.7) and with the hypothesis on scalar curvature, we write

$$(h_{ij}^{\alpha})^2 \leq |A|^2 = n^2H^2 + n(n-1)(R-1) \leq n^2H^2,$$

for every α, i, j and, hence

$$h_{ii}^{\alpha}h_{jj}^{\alpha} \leq |h_{ii}^{\alpha}||h_{jj}^{\alpha}| \leq (nH)^2. \tag{3.9}$$

On the other hand, since we are assuming $\sup_M |\Phi|^2 < +\infty$ and that R is constant, from (3.8) it follows that $\sup_M H < +\infty$. Thus, from (2.4) and (3.9), we obtain

$$R_{ijij} = 1 - \sum_{\alpha}\left(h_{ii}^{\alpha}h_{jj}^{\alpha} - (h_{ij}^{\alpha})^2\right) \geq 1 - \sum_{\alpha} h_{ii}^{\alpha}h_{jj}^{\alpha} > -\infty, \tag{3.10}$$

that is, the sectional curvatures of M^n are bounded from below.

Moreover, from (3.3) and (2.14) we have

$$\mathrm{tr}(P) = n(n-1)H$$

and, hence,

$$\sup_M \mathrm{tr}(P) < +\infty. \tag{3.11}$$

Furthermore, Lemma 1 guarantees us that the operator L is semi-elliptic. Therefore, taking into account (3.2), (3.10), and (3.11), we can apply Theorem 6.13 of [7] to conclude the desired result. □

From Lemma 2.2 of [9] we get the following

Lemma 3. *Let M^n be a spacelike submanifold immersed in $\mathbb{S}_p^{n+p}$ with constant scalar curvature $R \leq 1$. Then*

$$|\nabla A|^2 = \sum_{\alpha,i,j,k} (h_{ijk}^{\alpha})^2 \geq n^2|\nabla H|^2. \tag{3.12}$$

Moreover, if $R < 1$ and the equality holds in (3.12) on M^n, then H is constant on M^n.

We will also need the following algebraic lemma, whose proof can be found in [21].

Lemma 4. *Let* $A, B : \mathbb{R}^n \to \mathbb{R}^n$ *be symmetric linear maps such that* $AB - BA = 0$ *and* $\mathrm{tr}(A) = \mathrm{tr}(B) = 0$. *Then*

$$-\frac{n-2}{\sqrt{n(n-1)}}N(A)\sqrt{N(B)} \le \mathrm{tr}(A^2B) \le \frac{n-2}{\sqrt{n(n-1)}}N(A)\sqrt{N(B)}.$$

Moreover, the equality holds on the right-hand side (resp. left-hand side) if and only if $(n-1)$ *of the eigenvalues* x_i *of* A *and corresponding eigenvalues* y_i *of* B *satisfy*

$$|x_i| = \sqrt{\frac{N(A)}{n(n-1)}},\; x_ix_j \ge 0 \quad \text{and} \quad y_i = \sqrt{\frac{N(B)}{n(n-1)}}\left(\text{resp.} -\sqrt{\frac{N(B)}{n(n-1)}}\right).$$

4. Characterizations of Spacelike Submanifolds in $\mathbb{S}_p^{n+p}$

Remark 1. From now on, we will consider spacelike submanifolds M^n of $\mathbb{S}_p^{n+p}$ having *parallel normalized mean curvature vector*, which means that $H > 0$ and the normalized mean curvature vector field h/H is parallel as a section of the normal bundle. The assumption about parallel normalized mean curvature vector was introduced by Chen in [10]. Submanifolds with nonzero parallel mean curvature vector also have parallel normalized mean curvature vector. But the condition of having parallel normalized mean curvature vector is much weaker than the condition of having parallel mean curvature vector. For instance, every hypersurface with non-vanishing mean curvature in a semi-Riemannian manifold always has parallel normalized mean curvature vector.

To establish our main results, a crucial point is to obtain a suitable lower estimate for the operator L acting on the square of the norm of the total umbilicity tensor of a spacelike submanifold. This is made in the following proposition.

Proposition 1. *Let* M^n *be a spacelike submanifold in* $\mathbb{S}_p^{n+p}$, *with parallel normalized mean curvature vector and constant scalar curvature* $R \le 1$. *Then*

$$\frac{1}{2}L(|\Phi|^2) \ge \frac{1}{\sqrt{n(n-1)}}|\Phi|^2 Q_R(|\Phi|)\sqrt{|\Phi|^2 + n(n-1)(1-R)},$$

where

$$Q_R(x) = \frac{(n-p-1)}{p}x^2 - (n-2)x\sqrt{x^2 + n(n-1)(1-R)} + n(n-1)R. \quad (4.1)$$

Proof. First of all, since R is constant it follows from (3.8) that

$$\frac{1}{n-1}L(|\Phi|^2) = 2HL(nH) + 2n\langle P(\nabla H), \nabla H\rangle \ge 2HL(nH), \quad (4.2)$$

once (3.2) guarantees that $L(u^2) = 2uL(u) + 2\langle P(\nabla u), \nabla u\rangle$ for every $u \in \mathcal{C}^2(M)$ and Lemma 1 guarantees that the operator L is semi-elliptic.

Since the normalized mean curvature vector of M^n is parallel, we may choose $\{e_1, \ldots, e_{n+p}\}$ a local orthonormal frame to M^n such that $e_{n+1} = \frac{h}{H}$.

In particular, $H^{n+1}=H$ and $H^\alpha=0$ for every $\alpha\geq n+2$. e_{n+1} being parallel and denoting by $\nabla^\perp$ the normal connection of M^n in $\mathbb{S}_p^{n+p}$, it follows that

$$0=\nabla^\perp e_{n+1}=\sum_\alpha \omega_{\alpha n+1}e_\alpha.$$

Thus,

$$\omega_{\alpha n+1}=0, \quad \text{for all} \quad \alpha\geq n+1.$$

Hence, from (2.8), it follows that $R_{n+1\alpha ij}=0$, for all α,i,j and, consequently, from Ricci equation (2.9), we have that $h^{n+1}h^\alpha-h^\alpha h^{n+1}=0$, for all α. This implies that the matrix h^{n+1} commutes with all the matrix h^α. Thus, being $\Phi^\alpha=(\Phi_{ij}^\alpha)$, we have that $\Phi^\alpha=h^\alpha-H^\alpha I$, where I stands for the identity. Hence, $\Phi^{n+1}=h^{n+1}-HI$ and $\Phi^\alpha=h^\alpha$, for $\alpha\geq n+2$. Therefore, Φ^{n+1} commutes with all the matrix Φ^α. Since the matrix Φ^α is traceless and symmetric, once the matrix h^α are symmetric, we can use Lemma 4, for $A=\Phi^\alpha$ and $B=\Phi^{n+1}$, to obtain

$$\left|\operatorname{tr}\left((\Phi^\alpha)^2\Phi^{n+1}\right)\right|\leq\frac{n-2}{\sqrt{n(n-1)}}N(\Phi^\alpha)\sqrt{N(\Phi^{n+1})}. \tag{4.3}$$

Summing (4.3) in α, we have

$$\sum_\alpha\left|\operatorname{tr}\left((\Phi^\alpha)^2\Phi^{n+1}\right)\right|\leq\frac{n-2}{\sqrt{n(n-1)}}\sum_\alpha N(\Phi^\alpha)\sqrt{N(\Phi^{n+1})}.$$

On the other hand, with a straightforward computation we guarantee that

$$\begin{aligned}&-nH\sum_\alpha\operatorname{tr}\left[h^{n+1}(h^\alpha)^2\right]+\sum_{\alpha,\beta}\left[\operatorname{tr}(h^\alpha h^\beta)\right]^2\\&\quad=-nH\sum_\alpha\operatorname{tr}\left[\Phi^{n+1}(\Phi^\alpha)^2\right]-nH^2|\Phi|^2+\sum_{\alpha,\beta}\left[\operatorname{tr}(\Phi^\alpha\Phi^\beta)\right]^2\end{aligned} \tag{4.4}$$

and

$$N(h^\alpha h^\beta-h^\beta h^\alpha)=N(\Phi^\alpha\Phi^\beta-\Phi^\beta\Phi^\alpha)\geq 0. \tag{4.5}$$

Moreover, $N(\Phi^{n+1})=\operatorname{tr}(\Phi^{n+1})^2\leq|\Phi|^2$ and $\sum_\alpha N(\Phi^\alpha)=|\Phi|^2$. Hence,

$$-nH\sum_\alpha|\operatorname{tr}(\Phi^{n+1}(\Phi^\alpha)^2)|\geq-\frac{n(n-2)}{\sqrt{n(n-1)}}H|\Phi|^3. \tag{4.6}$$

Using Cauchy–Schwarz inequality,

$$\begin{aligned}p\sum_{\alpha,\beta}[\operatorname{tr}(\Phi^\alpha\Phi^\beta)]^2&\geq p\sum_\alpha[\operatorname{tr}(\Phi^\alpha)^2]^2=p\sum_\alpha[N(\Phi^\alpha)]^2\\&\geq\left(\sum_\alpha N(\Phi^\alpha)\right)^2=|\Phi|^4,\end{aligned} \tag{4.7}$$

Hence, from (3.7), (3.8), (3.12), (4.4), (4.5), (4.6) and (4.7) we obtain

$$L(nH)\geq|\Phi|^2\left(\frac{|\Phi|^2}{p}-\frac{n(n-2)}{\sqrt{n(n-1)}}|\Phi|H-n\left(H^2-1\right)\right). \tag{4.8}$$

Thus, from (4.2) and (4.8) we get

$$\frac{1}{2(n-1)}L(|\Phi|^2) \geq H|\Phi|^2\left(\frac{|\Phi|^2}{p} - \frac{n(n-2)}{\sqrt{n(n-1)}}H|\Phi| - n(H^2-1)\right). \quad (4.9)$$

Besides, from (3.8) we have

$$H^2 = \frac{1}{n(n-1)}|\Phi|^2 + (1-R). \quad (4.10)$$

Consequently, taking into account that $H > 0$, we can write

$$H = \frac{1}{\sqrt{n(n-1)}}\sqrt{|\Phi|^2 + n(n-1)(1-R)}. \quad (4.11)$$

Therefore, inserting (4.10) and (4.11) in (4.9) we conclude that desired result. □

Now, we are in a position to present our first theorem.

Theorem 1. *Let M^n be a complete spacelike submanifold immersed in $\mathbb{S}_p^{n+p}$ with parallel normalized mean curvature vector and constant scalar curvature $0 < R \leq 1$. Then*

(i) *either $\sup_M |\Phi|^2 = 0$ and M^n is a totally umbilical submanifold,*
(ii) *or*

$$\sup_M |\Phi|^2 \geq \alpha(n,p,R) > 0,$$

where $\alpha(n,p,R)$ is a positive constant depending only on n, p, R (see Remark 3).

Moreover, the equality $\sup_M |\Phi|^2 = \alpha(n,p,R)$ holds and this supremum is attained at some point of M^n if and only if $p = 1$, $n \geq 3$ and M^n is isometric to a hyperbolic cylinder $\mathbb{H}^1(r) \times \mathbb{S}^{n-1}(\sqrt{1+r^2})$ of radius $r > 0$.

Remark 2. Geometrically, Theorem 1 can be seen as a gap theorem for the total umbilicity tensor of M, close in spirit to other similar gap theorems either for the second fundamental form, as in the classical papers on minimal submanifolds by Simons [22] and Chern, do Carmo and Kobayashi [13], or for the total umbilicity tensor itself, as in [4].

Proof of Theorem 1. If $\sup_M |\Phi|^2 = 0$, then M^n is totally umbilical and, hence, item (i) holds. If $\sup_M |\Phi|^2 = +\infty$, then (ii) is trivially satisfied. So, let us suppose that $0 < \sup_M |\Phi|^2 < +\infty$ and let us take $u = |\Phi|^2$. Then, from Proposition 1 we get

$$L(u) \geq f(u), \quad (4.12)$$

where

$$f(u) = \frac{2}{\sqrt{n(n-1)}}uQ_R(\sqrt{u})\sqrt{u + n(n-1)(1-R)}$$

and $Q_R(x)$ is given by (4.1).

If M^n is compact, there exists a point $p_0 \in M^n$ such that $u(p_0) = u^*$. Consequently, $\nabla u(p_0) = 0$ and $Lu(p_0) \leq 0$. Therefore, from (4.12) we get $f(u^*) \leq 0$. Now, assume that M^n is complete and non-compact. Since $u^* <$

$+\infty$, Lemma 2 guarantees that there exists a sequence of points $\{p_k\}_{k\in\mathbb{N}} \subset M^n$ satisfying

$$u(p_k) > u^* - \frac{1}{k} \quad \text{and} \quad Lu(p_k) < \frac{1}{k} \tag{4.13}$$

for every $k \in \mathbb{N}$. Therefore, from (4.12) and (4.13), we get

$$\frac{1}{k} > Lu(p_k) \geq f(u(p_k)). \tag{4.14}$$

Taking into (4.14) the limit when $k \to +\infty$, by continuity, we have

$$0 \geq f(u^*) = \frac{2}{\sqrt{n(n-1)}} u^* Q_R(\sqrt{u^*})\sqrt{u^* + n(n-1)(1-R)}.$$

Hence, in any case we obtain that, when $0 < u < +\infty$, it must be $f(u^*) \leq 0$. Since $u^* > 0$ and $R \leq 1$, this implies

$$Q_R(\sqrt{u^*}) \leq 0. \tag{4.15}$$

Note that the hypothesis $R > 0$ guarantees us that

$$Q_R(0) = n(n-1)R > 0.$$

At this point, we observe from (4.1) that if $p = 1$ and $(n-2)/n \leq R \leq 1$, $Q_R(x) > 0$ for every $x \geq 0$. Therefore, (4.15) cannot hold and we conclude in this case that M^n must be a totally umbilical hypersurface. In particular, this happens when $p = 1$ and $n = 2$. On the other hand, if $p = 1$, $n \geq 3$ and $0 < R < (n-2)/n$ it is easy to see that $Q_R(x)$ has a unique positive root x_0 determined by

$$x_0^2 = \alpha(n, 1, R) = \frac{n(n-1)R^2}{(n-2)(n-2-nR)}.$$

The same happens when $p \geq 2$, $n \geq 2$ and $0 < R \leq 1$, but in this case the unique positive root x_0 of $Q_R(x) = 0$ does not have a so simple expression, unless $n = 2$ (see Remark 3 for the details).

Therefore, either if $p = 1$, $n \geq 3$ and $0 < R < (n-2)/n$ or if $p \geq 2$, $n \geq 2$ and $0 < R \leq 1$, inequality (4.15) implies

$$u^* \geq x_0^2 = \alpha(n, p, R),$$

that is,

$$\sup_M |\Phi|^2 \geq \alpha(n, p, R).$$

This proves the inequality in (ii).

Moreover, equality $\sup_M |\Phi|^2 = \alpha(n, p, R)$ holds if, and only if, $\sqrt{u^*} = x_0$. Thus $Q_R(\sqrt{u}) \geq 0$ on M^n, which jointly with (4.12) implies that

$$L(u) \geq 0 \quad \text{on} \quad M^n.$$

Now, suppose that $R < 1$. Hence, Lemma 1 assures that the operator L is elliptic. Therefore, if there exists a point $p_0 \in M^n$ such that $|\Phi(p_0)| = \sup_M |\Phi|$, from the maximum principle the function $u = |\Phi|^2$ must be constant and, consequently, $|\Phi| \equiv x_0$. Thus,

$$0 = \frac{1}{2}L(|\Phi|^2) \geq \frac{1}{\sqrt{n(n-1)}}|\Phi|^2 Q_R(|\Phi|)\sqrt{|\Phi|^2 + n(n-1)(1-R)}. \tag{4.16}$$

Hence, the inequality (4.2) become equality. In particular, since L is elliptic if and only if P is positive definite, from (4.2) we obtain that H is constant. Since $|\Phi| > 0$ and $R < 1$, from (4.16) we must have $Q_R(|\Phi|) = 0$. Thus, all inequalities obtained along the proof of Proposition 1 are, in fact, equalities. In particular, from inequality (4.6) we conclude that

$$\mathrm{tr}(\Phi^{n+1})^2 = |\Phi|^2.$$

So, from (3.8) we get

$$\mathrm{tr}(\Phi^{n+1})^2 = |\Phi|^2 = |A|^2 - nH^2. \tag{4.17}$$

On the other hand, we also have that

$$\mathrm{tr}(\Phi^{n+1})^2 = |A|^2 - \sum_{\alpha>n+1}\sum_{i,j}(h_{ij}^{\alpha})^2 - nH^2. \tag{4.18}$$

Thus, from (4.17) and (4.18) we conclude that $\sum_{\alpha>n+1}\sum_{i,j}(h_{ij}^{\alpha})^2 = 0$. But, from inequality (4.7) we also have that

$$|\Phi|^4 = p\sum_{\alpha\geq n+1}[N(\Phi^{\alpha})]^2 = pN(\Phi^{n+1})^2 = p|\Phi|^4. \tag{4.19}$$

Since $|\Phi| > 0$, we must have that $p = 1$.

In this setting, from (3.12) and (4.19) we get

$$\sum_{i,j,k}(h_{ijk}^{n+1})^2 = n^2|\nabla H|^2 = 0,$$

that is, $h_{ijk}^{n+1} = 0$ for all i, j. Hence, we obtain that M^n is an isoparametric hypersurface of $\mathbb{S}_1^{n+1}$.

Hence, since the equality occurs in (4.3), we have that also occurs the equality in Lemma 4. Consequently, M^n has at most two distinct constant principal curvatures. Therefore, we can apply classical congruence theorem due to Abe, Koike and Yamaguchi (see Theorem 5.1 of [1]) we conclude that M^n must be one of the two following standard product embeddings into $\mathbb{S}_1^{n+1}$: (a) $\mathbb{H}^1(r)\times\mathbb{S}^{n-1}(\sqrt{1+r^2})$, or (b) $\mathbb{H}^{n-1}(r)\times\mathbb{S}^1(\sqrt{1+r^2})$, of positive radius $r > 0$. In case (a), for a given radius $r > 0$ the standard product embedding $\mathbb{H}^1(r)\times\mathbb{S}^{n-1}(\sqrt{1+r^2}) \hookrightarrow \mathbb{S}_1^{n+1}$ has constant principal curvatures given by

$$\lambda_1 = \frac{\sqrt{1+r^2}}{r}, \quad \lambda_2 = \cdots = \lambda_n = \frac{r}{\sqrt{1+r^2}}.$$

Therefore,

$$nH = \frac{1+nr^2}{r\sqrt{1+r^2}}, \quad |A|^2 = \frac{1+2r^2+nr^4}{r^2(1+r^2)}, \quad \text{and} \quad |\Phi|^2 = \frac{n-1}{nr^2(1+r^2)},$$

and its constant scalar curvature is given

$$R = \frac{n-2}{n(1+r^2)},$$

which satisfies our hypothesis, since

$$0 < R < \frac{n-2}{n} < 1$$

for every $r > 0$. On the other hand, in case (b) and for a given radius $r > 0$ the standard product embedding $\mathbb{H}^{n-1}(r) \times \mathbb{S}^1(\sqrt{1+r^2}) \hookrightarrow \mathbb{S}_1^{n+1}$ has constant principal curvatures given by

$$\lambda_1 = \cdots = \lambda_{n-1} = \frac{\sqrt{1+r^2}}{r}, \quad \lambda_n = \frac{r}{\sqrt{1+r^2}}.$$

Therefore,

$$nH = \frac{(n-1)+nr^2}{r\sqrt{1+r^2}}, \quad |A|^2 = \frac{n-1+2(n-1)r^2+nr^4}{r^2(1+r^2)}, \quad \text{and}$$
$$|\Phi|^2 = \frac{n-1}{nr^2(1+r^2)},$$

and its constant scalar curvature is given

$$R = -\frac{n-2}{nr^2} < 0,$$

which does not satisfy our hypothesis. □

Remark 3. A direct computation from (4.1) shows that when $p \geq 2$ and $0 < R \leq 1$ the unique positive root x_0 of $Q_R(x) = 0$ is determined by

$$x_0^2 = \alpha(n,p,R)$$

where $\alpha(n,p,R)$ is the unique positive root of the following quadratic equation:

$$aY^2 + bY + c = 0, \tag{4.20}$$

with

$$a = \left(\frac{n-p-1}{p}\right)^2 - (n-2)^2 = -\frac{(n-1)(p-1)((n-1)(p+1)-2p)}{p^2} < 0,$$

$$b = \frac{n(n-1)}{p}\left(2(n-p-1)R - (n-2)^2p(1-R)\right),$$

and

$$c = n^2(n-1)^2R^2 > 0.$$

Actually, since $a < 0$ and $c > 0$ it then follows that the discriminant of (4.20) is $D = b^2 - 4ac > b^2$. Hence, $-b - \sqrt{D} < 0$ and the unique positive root of (4.20) is given by

$$\begin{aligned}\alpha(n,p,R) &= \frac{-b-\sqrt{D}}{2a}\\ &= \frac{n(n-1)p}{2((n-p-1)^2-(n-2)^2p^2)}\beta(n,p,R),\end{aligned}$$

where

$$\begin{aligned}\beta(n,p,R) = {}&(n-2)^2p(1-R) - 2(n-p-1)R\\ &-(n-2)\sqrt{p(1-R)((n-2)^2p(1-R)-4(n-p-1)R)+4p^2R^2}.\end{aligned}$$

In particular, when $n = 2$ the expression for $\alpha(n, p, R)$ reduces to

$$\alpha(2, p, R) = \frac{2pR}{p-1}.$$

We recall that, in the context of spacelike surfaces, the Gaussian curvature of M^2 satisfies the relation $K = R$. Hence, as a consequence of the proof of Theorem 1, we also obtain the following.

Corollary 1. *The only complete spacelike surfaces immersed in* $\mathbb{S}^{2+p}_p$*,* $p \geq 2$*, with parallel normalized mean curvature vector, constant Gaussian curvature* $0 < K \leq 1$ *and such that* $\sup_M |\Phi|^2 < \frac{2p}{p-1} K$*, are the totally umbilical ones.*

5. Some Remarks and Applications to L-Parabolic Manifolds

We recall that a Riemannian manifold M^n is said to be parabolic (with respect to the Laplacian operator) if the constant functions are the only subharmonic functions on M^n which are bounded from above; that is, for a function $u \in \mathcal{C}^2(M)$

$$\Delta u \geq 0 \quad \text{and} \quad u \leq u^* < +\infty \quad \text{implies} \quad u = \text{constant}.$$

More generally, let M^n be a Riemannian manifold and consider a general class of second-order differential operators on M given by

$$\mathcal{L}(u) = \text{tr}(\mathcal{P} \circ \nabla^2 u) \tag{5.1}$$

for every $u \in \mathcal{C}^2(M)$, where $\mathcal{P} : TM \to TM$ is a symmetric operator on M^n. In this setting, M^n is said to be $\mathcal{L}$-parabolic (or parabolic with respect to the operator $\mathcal{L}$) if the constant functions are the only functions $u \in \mathcal{C}^2(M)$ which are bounded from above and satisfying $\mathcal{L}u \geq 0$. That is, for a function $u \in \mathcal{C}^2(M)$

$$\mathcal{L}u \geq 0 \quad \text{and} \quad u \leq u^* < +\infty \quad \text{implies} \quad u = \text{constant}.$$

The differential operator $\mathcal{L}$ is elliptic (resp. semi-elliptic) if and only if $\mathcal{P}$ is positive definite (resp. positive semi-definite).

By a standard tensor computation, it is not difficult to see that

$$\mathcal{L}(u) = \text{div}(\mathcal{P}(\nabla u)) - \langle \text{div}\mathcal{P}, \nabla u \rangle \tag{5.2}$$

for every function $u \in \mathcal{C}^2(M)$, where

$$\text{div}\mathcal{P} = \text{tr}(\nabla \mathcal{P}) = \sum_{i=1}^{n} \nabla \mathcal{P}(e_i, e_i)$$

with

$$\nabla \mathcal{P}(X, Y) = (\nabla_Y \mathcal{P})X = \nabla_Y(\mathcal{P}X) - \mathcal{P}(\nabla_Y X)$$

for every $X, Y \in TM$. In particular, when $\text{div}\mathcal{P} = 0$

$$\mathcal{L}(u) = \text{tr}(\mathcal{P} \circ \nabla^2 u) = \text{div}(\mathcal{P}(\nabla u)) \tag{5.3}$$

and the operator $\mathcal{L}$ can be seen as a divergence type operator. This happens, for instance, for the Cheng–Yau operator L given in (3.1) in the case of spacelike submanifolds M^n of $\mathbb{S}_p^{n+p}$ with parallel normalized mean curvature vector, as shown below.

Lemma 5. *Let M^n be a spacelike submanifold in $\mathbb{S}_p^{n+p}$ with parallel normalized mean curvature vector and let L be the Cheng–Yau operator on M^n given in (3.1). Then* $\mathrm{div}P = 0$. *Equivalently,* $Lu = \mathrm{div}(P(\nabla u)$ *for every* $u \in \mathcal{C}^2(M)$.

Proof. In our case, we have $P = nHI - h^{n+1}$, where I denotes the identity on TM and h^{n+1} stands for the second fundamental form on M^n in the direction of $e_{n+1} = h/H$. Thus, for every tangent vector fields $X, Y \in TM$, we have

$$\nabla P(X,Y) = nY(H)X - \nabla h^{n+1}(X,Y). \tag{5.4}$$

Since e_{n+1} is parallel, it follows from the Codazzi equation (2.11) that ∇h^{n+1} is symmetric, that is

$$\nabla h^{n+1}(X,Y) = \nabla h^{n+1}(Y,X)$$

for every $X, Y \in TM$. Therefore, for every $X \in TM$ and for every e_i, $1 \leq i \leq n$, we have

$$\langle \nabla h^{n+1}(e_i,e_i), X\rangle = \langle (\nabla_{e_i} h^{n+1})e_i, X\rangle = \langle (\nabla_{e_i} h^{n+1})X, e_i\rangle = \langle (\nabla_X h^{n+1})e_i, e_i\rangle,$$

which implies

$$\begin{aligned}\langle \mathrm{tr}(\nabla h^{n+1}), X\rangle &= \sum_{i=1}^{n} \langle \nabla h^{n+1}(e_i,e_i), X\rangle = \sum_{i=1}^{n} \langle (\nabla_X h^{n+1})e_i, e_i\rangle \\ &= \mathrm{tr}(\nabla_X h^{n+1}) = \nabla_X(\mathrm{tr}\, h^{n+1}) \\ &= n\langle \nabla H, X\rangle\end{aligned}$$

for every $X \in TM$. In other words,

$$\mathrm{tr}(\nabla h^{n+1}) = n\nabla H.$$

Using this in (5.4) we easily conclude

$$\mathrm{div}P = \mathrm{tr}(\nabla P) = n\nabla H - n\nabla H = 0$$

as desired. □

Our objective in Theorem 2 below is to characterize the hyperbolic cylinders of the form $\mathbb{H}^1(r) \times \mathbb{S}^{n-1}(\sqrt{1+r^2})$ as the only L-parabolic complete spacelike submanifolds in $\mathbb{S}_p^{n+p}$ with parallel normalized mean curvature vector and constant scalar curvature $0 < R \leq 1$ satisfying $\sup_M |\Phi|^2 = \alpha(n,p,R)$. To this aim, we first need to prove the L-parabolicity of the hyperbolic cylinders. Observe that, for every positive radius r, $\mathbb{H}^1(r) \times \mathbb{S}^{n-1}$ $(\sqrt{1+r^2})$ is canonically embedded in $\mathbb{S}_p^{n+p}$ as a spacelike hypersurface of the totally geodesic submanifold $\mathbb{S}_1^{n+1} \subset \mathbb{S}_p^{n+p}$, so that

$$\mathbb{H}^1(r) \times \mathbb{S}^{n-1}(\sqrt{1+r^2}) \hookrightarrow_{\text{spacelike hypersurface}} \mathbb{S}_1^{n+1} \hookrightarrow_{\text{totally geodesic}} \mathbb{S}_p^{n+p}.$$

Therefore, $h^{n+1} = S$, where S stands for the shape operator of $\mathbb{H}^1(r) \times \mathbb{S}^{n-1}(\sqrt{1+r^2})$ as a spacelike hypersurface of $\mathbb{S}_1^{n+1}$, and $nH = \mathrm{tr}\, S$.

Recall that S has constant principal curvatures given by

$$\lambda_1 = \frac{\sqrt{1+r^2}}{r}, \quad \lambda_2 = \cdots = \lambda_n = \frac{r}{\sqrt{1+r^2}},$$

with

$$nH = \frac{1+nr^2}{r\sqrt{1+r^2}},$$

and it splits as

$$S(U,V) = \left(\frac{\sqrt{1+r^2}}{r} U, \frac{r}{\sqrt{1+r^2}} V \right)$$

for every $U \in T\mathbb{H}^1(r)$ and $V \in T\mathbb{S}^{n-1}(\sqrt{1+r^2})$. Thus, the operator $P = nHI - S$ also splits as

$$P(U,V) = \left(\frac{(n-1)r}{\sqrt{1+r^2}} U, \frac{1+(n-1)r^2}{r\sqrt{1+r^2}} V \right)$$

for every $U \in T\mathbb{H}^1(r)$ and $V \in T\mathbb{S}^{n-1}(\sqrt{1+r^2})$. Then, the L-parabolicity of $\mathbb{H}^1(r) \times \mathbb{S}^{n-1}(\sqrt{1+r^2})$ is a consequence of the following general result, inspired by Proposition 4.2 in [5].

Proposition 2. *Let $M = M_1 \times M_2$ be a Riemannian (connected) product manifold, where M_1 is parabolic (with respect to the Laplacian operator) and M_2 is compact. Let $\mathcal{P} : TM \to TM$ be a positive definite symmetric operator on M which splits as*

$$\mathcal{P}(U,V) = (\lambda U, \mu V)$$

for every $U \in TM_1$ and $V \in TM_2$, with positive constants $\lambda, \mu \in \mathbb{R}$. Then M^n is $\mathcal{L}$-parabolic.

Corollary 2. *The hyperbolic cylinders $\mathbb{H}^1(r) \times \mathbb{S}^{n-1}(\sqrt{1+r^2})$, as spacelike submanifolds of $\mathbb{S}_p^{n+p}$, are parabolic with respect to the Cheng–Yau operator L.*

For the proof of Proposition 2 we will need the following lemma, which extends Lemma 4.1 in [5] to the case of semi-elliptic operators.

Lemma 6. *Assume that $\mathcal{L}$ is semi-elliptic on a connected Riemannian manifold M^n. M^n is $\mathcal{L}$-parabolic if and only if every positive, bounded function u satisfying $\mathcal{L}(u) \geq 0$ is constant.*

Proof. The "if" part follows directly from the definition, without the positivity of the function. Therefore, it suffices to prove the "only if" part.

Let $u \in \mathcal{C}^2(M)$ be a function which is bounded from above and satisfying $\mathcal{L}(u) \geq 0$. Consider the positive function $v = e^u$, which is also bounded from above with $v^* = \sup_M v = e^{u^*}$, $u^* = \sup_M u < +\infty$. Moreover, from (5.1) an easy computation gives

$$\mathcal{L}(v) = e^u \mathcal{L}(u) + e^u \langle \mathcal{P}(\nabla u), \nabla u \rangle \geq 0, \tag{5.5}$$

since $\mathcal{P}$ is positive semi-definite. Then, our assumptions show that v is constant and, since $\nabla v = e^u \nabla u$, it follows that $\nabla u = 0$. Therefore, u is also constant, and M^n is $\mathcal{L}$-parabolic. □

Proof of Proposition 2. First of all, since λ and μ are both constants, it easily follows that div$\mathcal{P} = 0$, so that (5.3) holds. Let $u \in \mathcal{C}^2(M)$ be a positive, bounded function satisfying $\mathcal{L}(u) \geq 0$. According to Lemma 6, it suffices to prove that u is constant.

We observe that if $v = u^2$, then $v : M \to \mathbb{R}$ is also positive and bounded. Using again (5.3), we have

$$\mathcal{L}(v) = 2u\mathcal{L}(u) + 2\langle \mathcal{P}(\nabla u), \nabla u\rangle \geq 0. \tag{5.6}$$

For every $x \in M_1$ and every $y \in M_2$, let us denote by $v^x : M_2 \to \mathbb{R}$ and by $v_y : M_1 \to \mathbb{R}$ the functions given by

$$v^x(y) = v_y(x) = v(x, y).$$

An easy computation gives

$$\nabla v(x, y) = (\nabla^1 v_y(x), \nabla^2 v^x(y)), \qquad \text{for every } (x, y) \in M_1 \times M_2$$

where ∇^1 and ∇^2 denote the gradient operators on M_1 and M_2, respectively. Therefore,

$$\mathcal{P}(\nabla v(x, y)) = (\lambda \nabla^1 v_y(x), \mu \nabla^2 v^x(y)), \qquad \text{for every } (x, y) \in M_1 \times M_2,$$

and from (5.3) we obtain

$$\mathcal{L}v(x, y) = \lambda \Delta_1 v_y(x) + \mu \Delta_2 v^x(y), \qquad \text{for every } (x, y) \in M_1 \times M_2, \tag{5.7}$$

where Δ_1 and Δ_2 denote the Laplacian operators on M_1 and M_2, respectively.

Since M_2 is compact, integrating this expression (5.7) over M_2 we have, from the divergence theorem,

$$\int_{M_2} \mathcal{L}v(x, y)\mathrm{d}y = \lambda \int_{M_2} \Delta_1 v_y(x)\mathrm{d}y + \mu \int_{M_2} \Delta_2 v^x(y)\mathrm{d}y = \lambda \int_{M_2} \Delta_1 v_y(x)\mathrm{d}y. \tag{5.8}$$

Therefore, from $\mathcal{L}v \geq 0$ and $\lambda > 0$ we have

$$\int_{M_2} \Delta_1 v_y(x)\mathrm{d}y \geq 0. \tag{5.9}$$

Now, the compactness of M_2 allows us to compute derivatives under the integral sign to get

$$\begin{aligned}\int_{M_2} \Delta_1 v_y(x)\mathrm{d}y &= \int_{M_2} \sum_{i=1}^{k} (e_i e_i - \nabla^1_{e_i} e_i) v_y(x)\mathrm{d}y \\ &= \sum_{i=1}^{k} (e_i e_i - \nabla^1_{e_i} e_i) \left(\int_{M_2} v_y(x)\mathrm{d}y \right) \\ &= \Delta_1 h(x)\end{aligned}$$

where $\{e_1 \ldots, e_k\}$ is a local orthonormal frame on M_1 with $k = \dim M_1$, and

$$h(x) = \int_{M_2} v_y(x)\mathrm{d}y.$$

Hence, it follows from (5.9) that $h : M_1 \to \mathbb{R}$ is a subharmonic function. Moreover, for every $x \in M_1$

$$h(x) = \int_{M_2} v_y(x)\mathrm{d}y \le \sup_M v \left(\int_{M_2} \mathrm{d}y\right) = \sup_M v \,\mathrm{vol}(M_2) < +\infty.$$

As a consequence, h is a subharmonic function on M_1 which is bounded from above. Since M_1 is parabolic, we conclude that h is constant and, in particular, $\Delta_1 h = 0$.

Returning to (5.8), we obtain that

$$\int_{M_2} \mathcal{L}v(x,y)\mathrm{d}y = \lambda\Delta_1 h(x) = 0 \quad \text{for every } x \in M_1. \tag{5.10}$$

Since $\mathcal{L}(v) \ge 0$, this implies $\mathcal{L}(v) = 0$. Thus, from (5.6) we obtain that $\langle \mathcal{P}(\nabla u), \nabla u\rangle = 0$ and, since $\mathcal{P}$ is positive definite, we conclude $\nabla u = 0$, as desired.

We are now ready to prove the following result.

Theorem 2. *Let M^n, $n \ge 3$, be a complete spacelike submanifold immersed in $\mathbb{S}_p^{n+p}$, with parallel normalized mean curvature vector and constant scalar curvature $0 < R \le 1$. Suppose that M^n is not totally umbilical. If M^n is L-parabolic, then*

$$\sup_M |\Phi|^2 \ge \alpha(n,p,R) > 0, \tag{5.11}$$

with equality if and only if $p = 1$ and M^n is isometric to a hyperbolic cylinder $\mathbb{H}^1(r) \times \mathbb{S}^{n-1}(\sqrt{1+r^2})$ of radius $r > 0$.

Proof. If $\sup_M |\Phi|^2 = +\infty$, then there is nothing to prove. Suppose then that $0 < \sup_M |\Phi|^2 < +\infty$. In this case, we can proceed as in the first part of the proof of Theorem 1, to guarantee that $\sup_M |\Phi|^2 \ge \alpha(n,p,R)$. Moreover, if equality holds in (5.11), then we have $Q_R(|\Phi|) \ge 0$ and, consequently, $L(|\Phi|^2) \ge 0$ on M^n. Therefore, from the L-parabolicity of M^n we conclude that the function $u = |\Phi|^2$ must be constant and equal to $\alpha(n,p,R)$. At this point, we can reason as in the proof of the Theorem 1 to conclude the result. □

Observe that from the proof of Theorem 2 we also obtain the following rigidity result

Corollary 3. *The only L-parabolic complete spacelike surfaces M^2 immersed in $\mathbb{S}_p^{2+p}$, $p \ge 2$, with parallel normalized mean curvature vector, constant Gaussian curvature $0 < K \le 1$ and such that $\sup_M |\Phi|^2 \le \frac{2p}{p-1}K$, are the totally umbilical ones.*

We closed our paper establishing the following L-parabolicity criterium.

Proposition 3. *Let M^n be a complete spacelike submanifold immersed in $\mathbb{S}_p^{n+p}$ with parallel normalized mean curvature vector and constant scalar curvature $0 < R \le 1$. If $\sup_M |\Phi|^2 < +\infty$ and, for some reference point $o \in M^n$,*

$$\int_0^{+\infty} \frac{\mathrm{d}r}{\mathrm{vol}(\partial B_r)} = +\infty, \tag{5.12}$$

then M^n is L-parabolic. Here B_r denotes the geodesic ball of radius r in M^n centered at the origin o.

Proof. From Lemma 5 we know that

$$L(u) = \mathrm{div}(P(\nabla u)), \tag{5.13}$$

for any $u \in \mathcal{C}^2(M)$, where P is defined in (3.3).

Now, we consider on M^n the symmetric $(0,2)$ tensor field ξ given by $\xi(X,Y) = \langle PX, Y\rangle$, or, equivalently, $\xi(\nabla u, \cdot)^\sharp = P(\nabla u)$, where $^\sharp : T^*M \to TM$ denotes the musical isomorphism. Thus, from (5.13) we get

$$L(u) = \mathrm{div}\left(\xi(\nabla u, \cdot)^\sharp\right).$$

On the other hand, as $\sup_M |\Phi|^2 < +\infty$ and M^n has constant scalar curvature, from equation (3.8), we have that $\sup_M H < +\infty$. So, we can define a positive continuous function ξ_+ on $[0, +\infty)$, by

$$\xi_+(r) = 2n \sup_{\partial B_r} H. \tag{5.14}$$

Thus, from (5.14) we have

$$\xi_+(r) = 2n \sup_{\partial B_r} H \leq 2n \sup_M H < +\infty. \tag{5.15}$$

Hence, from (5.12) and (5.15) we get

$$\int_0^{+\infty} \frac{\mathrm{d}r}{\xi_+(r)\mathrm{vol}(\partial \mathrm{B_r})} = +\infty.$$

Therefore, we can apply Theorem 2.6 of [19] to conclude the proof. □

Acknowledgements

The authors would like to thank the referee for reading the manuscript in great detail and giving several valuable suggestions and useful comments which improved the paper. This work is a result of the activity developed within the framework of the Programme in Support of Excellence Groups of the Región de Murcia, Spain, by Fundación Séneca, Science and Technology Agency of the Región de Murcia. The first author was partially supported by MINECO/FEDER project reference MTM2015-65430-P and Fundación Séneca project reference 19901/GERM/15, Spain, and Ciência sem Fronteiras, Programa PVE, project A012/2013, CAPES, Brazil. The second author was partially supported by CNPq, Brazil, Grant 303977/2015-9.

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Received: May 19, 2017.
Revised: November 24, 2017.
Accepted: December 21, 2017.