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Nontrivial Solutions of Systems of Hammerstein Integral Equations with First Derivative Dependence

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Abstract. By means of classical fixed point index, we prove new results on the existence, non-existence, localization and multiplicity of nontrivial solutions for systems of Hammerstein integral equations where the nonlinearities are allowed to depend on the first derivative. As a byproduct of our theory, we discuss the existence of positive solutions of a system of third order ODEs subject to nonlocal boundary conditions. Some examples are provided to illustrate the applicability of the theoretical results.

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Keywords. Nontrivial solutions, derivative dependence, fixed point index, cone.

1. Introduction

Motivated by earlier work of do Ó et al. [2] on radial solutions of elliptic systems, Infante and Pietramala [5] studied the existence, multiplicity and non-existence of *nontrivial* solutions of systems of Hammerstein integral equations of the type

$$\begin{cases} u(t) = \int_0^1 k_1(t,s)g_1(s)f_1(s,u(s),v(s))\,\mathrm{d}s, \\ v(t) = \int_0^1 k_2(t,s)g_2(s)f_2(s,u(s),v(s))\,\mathrm{d}s. \end{cases}$$

The methodology of [5] is based on classical fixed point index theory and the authors work in a suitable cone in $(C[0,1])^2$. Due to the choice of the space involved, the setting of [5] does not allow derivative dependence in the nonlinearities.

On the other hand, Minhós and de Sousa [10] studied the system of third order ordinary differential equations subject to nonlocal boundary conditions

$$\begin{cases} -u'''(t) = f_1(t, v(t), v'(t)), \\ -v'''(t) = f_2(t, u(t), u'(t)), \\ u(0) = u'(0) = 0, u'(1) = \alpha u'(\eta), \\ v(0) = v'(0) = 0, v'(1) = \alpha v'(\eta), \end{cases} \tag{1.1}$$

where $0 < \eta < 1$ and $1 < \alpha < 1/\eta$. The approach of [10] relies on the celebrated Krasnosel'skiĭ–Guo fixed point theorem and on the rewriting the system (1.1) in the form

$$\begin{cases} u(t) = \int_0^1 k(t,s) f_1(s, v(s), v'(s))\, \mathrm{d}s, \\ v(t) = \int_0^1 k(t,s) f_2(s, u(s), u'(s))\, \mathrm{d}s. \end{cases} \tag{1.2}$$

Minhós and de Sousa proved the existence of *one* positive solution of the system (1.2), by assuming suitable superlinear/sublinear behaviours of the nonlinearities. A key ingredient in [10] is the use of the cone

$$\hat{K} := \left\{ w \in C^1[0,\,1] : w(t) \geq 0, \min_{t \in [\frac{\eta}{\alpha}, \eta]} w(t) \geq c\|w\|_C, \min_{t \in [\frac{\eta}{\alpha}, \eta]} w'(t) \geq d\|w'\|_C \right\}, \tag{1.3}$$

where $c, d \in (0, 1]$ and $\|w\|_C := \max_{t\in[0,\,1]} |w(t)|$. The cone (1.3) is similar to a cone of *non-negative* functions first used by Krasnosel'skiĭ, see, e.g., [7], and Guo, see, e.g., [4] in the space $C[0,\,1]$. Note that the functions in (1.3) are non-negative and their derivatives are non-negative on a subset of $[0, 1]$.

Here we make use of a new cone of functions that are allowed to *change sign*, similar to one introduced, in the space of continuous functions, by Infante and Webb [6]. With this ingredient we prove existence, multiplicity and non-existence results for *nontrivial* solutions of the systems of integral equations of the kind

$$\begin{cases} u(t) = \int_0^1 k_1(t,s) g_1(s) f_1(s, u(s), u'(s), v(s), v'(s))\, \mathrm{d}s, \\ v(t) = \int_0^1 k_2(t,s) g_2(s) f_2(s, u(s), u'(s), v(s), v'(s))\, \mathrm{d}s, \end{cases}$$

extending the results of [5] to this different setting.

We note that our approach can be also used to prove the existence of *non-negative* solutions; we highlight this fact by considering a generalization of the system (1.1), that is,

$$\begin{cases} -u'''(t) = g_1(t) f_1(t,\, u(t),\, u'(t), v(t),\, v'(t)), \\ -v'''(t) = g_2(t) f_2(t,\, u(t),\, u'(t), v(t),\, v'(t)), \\ u(0) = u'(0) = 0, u'(1) = \alpha_1 u'(\eta_1), \\ v(0) = v'(0) = 0, v'(1) = \alpha_2 v'(\eta_2), \end{cases} \tag{1.4}$$

where $0 < \eta_i < 1$, $1 < \alpha_i < \frac{1}{\eta_i}$. Note that the boundary conditions in (1.4) can generate two different kernels and the nonlinearities are allowed to have a stronger coupling with respect to the ones present in (1.1).

Some examples are given to show that the constants that occur in our theoretical results can be computed.

2. The System of Integral Equations

We begin by stating some assumptions on the terms that occur in the system of Hammerstein integral equations

$$\begin{cases} u(t) = \int_0^1 k_1(t,s)g_1(s)f_1(s,u(s),u'(s),v(s),v'(s))\,\mathrm{d}s, \\ v(t) = \int_0^1 k_2(t,s)g_2(s)f_2(s,u(s),u'(s),v(s),v'(s))\,\mathrm{d}s, \end{cases} \tag{2.1}$$

namely

$(A1)$ For $i = 1, 2$, $f_i : [0,1] \times \mathbb{R}^4 \to [0,+\infty)$ is a L^∞-Carathéodory function, that is, $f_i(\cdot, u_1, u_2, v_1, v_2)$ is measurable for each fixed (u_1, u_2, v_1, v_2), $f_i(t,\cdot,\cdot,\cdot,\cdot)$ is continuous for almost every (a.e.) $t \in [0,1]$, and for each $r > 0$ there exists $\varphi_{i,r} \in L^\infty[0,1]$ such that

$$f_i(t, u_1, u_2, v_1, v_2) \le \varphi_{i,r}(t) \;\text{ for }\; u_1, u_2, v_1, v_2 \in [-r, r] \;\text{ and a. e. } t \in [0,1].$$

$(A2)$ For every $i = 1, 2$, $k_i : [0,1]^2 \to \mathbb{R}$ is such that k_i are measurable, and for all $\tau \in [0,1]$, we have

$$\lim_{t\to\tau} |k_i(t,s) - k_i(\tau,s)| = 0, \quad \text{for a. e. } s \in [0,1]$$

and

$$\lim_{t\to\tau} \left| \frac{\partial k_i}{\partial t}(t,s) - \frac{\partial k_i}{\partial t}(\tau,s) \right| = 0, \quad \text{for a. e. } s \in [0,1].$$

$(A3)$ For every $i = 1, 2$, there exist subintervals $[a_i, b_i], [\gamma_i, \delta_i] \subseteq [0,1]$, functions $\phi_i, \psi_i \in L^\infty[0,1]$, and constants $c_i, d_i \in (0,1]$ such that

$$\begin{aligned} |k_i(t,s)| &\le \phi_i(s) \text{ for } t \in [0,1] \text{ and a. e. } s \in [0,1], \\ \left| \frac{\partial k_i}{\partial t}(t,s) \right| &\le \psi_i(s) \text{ for } t \in [0,1] \text{ and a. e. } s \in [0,1], \\ k_i(t,s) &\ge c_i\phi_i(s) \text{ for } t \in [a_i, b_i] \text{ and a. e. } s \in [0,1], \\ \frac{\partial k_i}{\partial t}(t,s) &\ge d_i\psi_i(s) \text{ for } t \in [\gamma_i, \delta_i] \text{ and a. e. } s \in [0,1]. \end{aligned}$$

$(A4)$ For every $i = 1, 2$, we have $g_i \in L^1[0,1]$, $g_i(t) \ge 0$ a.e. $t \in [0,1]$, $\int_{a_i}^{b_i} \phi_i(s)g_i(s)\,\mathrm{d}s > 0$ and $\int_{\gamma_i}^{\delta_i} \psi_i(s)g_i(s)\,\mathrm{d}s > 0$.

Forward in the paper, we use the space $\left(C^1[0,1]\right)^2$ equipped with the norm

$$\|(u,v)\| := \max\{\|u\|_{C^1}, \|v\|_{C^1}\},$$

where $\|w\|_{C^1} := \max\{\|w\|_C, \|w'\|_C\}$.

For the reader's convenience, we recall that a *cone* K in a Banach space X is a closed convex set such that $\lambda x \in K$ for $x \in K$ and $\lambda \ge 0$ and $K \cap (-K) = \{0\}$.

Consider, in the space $C^1[0,\,1]$, the cones

$$\tilde{K}_i := \left\{ w \in C^1[0,\,1] : \min_{t\in[a_i,b_i]} w(t) \ge c_i\|w\|_C, \min_{t\in[\gamma_i,\delta_i]} w'(t) \ge d_i\|w'\|_C \right\}, \tag{2.2}$$

and their product in $\left(C^1[0,1]\right)^2$ defined by

$$K := \{(u,v) \in \tilde{K_1} \times \tilde{K_2}\}. \tag{2.3}$$

By a *nontrivial* solution of the system (2.1) we mean a solution $(u,v) \in K$ of (2.1) such that $\|(u,v)\| \neq 0$. Note that the functions in $\tilde{K_i}$ are non-negative on the sub-intervals $[a_i, b_i]$ and non-decreasing on $[\gamma_i, \delta_i]$, but nevertheless, they can change sign or have a different variation in $[0,1]$.

We define the integral operator

$$T(u,v)(t) := \begin{pmatrix} T_1(u,v)(t) \\ T_2(u,v)(t) \end{pmatrix} = \begin{pmatrix} \int_0^1 k_1(t,s)g_1(s)f_1(s,u(s),u'(s),v(s),v'(s))\,\mathrm{d}s \\ \int_0^1 k_2(t,s)g_2(s)f_2(s,u(s),u'(s),v(s),v'(s))\,\mathrm{d}s \end{pmatrix}, \tag{2.4}$$

and prove that T leaves the cone K invariant and is compact.

Lemma 2.1. *The operator T given by* (2.4) *maps K into K and is compact.*

Proof. Take $(u,v) \in K$. Then, by (A3),

$$\|T_1(u,v)\|_C \leq \int_0^1 \phi_1(s)g_1(s)f_1(s,u(s),u'(s),v(s),v'(s))\,\mathrm{d}s,$$

and

$$\begin{aligned} \min_{t\in[a_1,b_1]} T_1(u,v)(t) &= \min_{t\in[a_1,b_1]} \int_0^1 k_1(t,s)g_1(s)f_1(s,u(s),u'(s),v(s),v'(s))\,\mathrm{d}s \\ &\geq c_1 \int_0^1 \phi_1(s)g_1(s)f_1(s,u(s),u'(s),v(s),v'(s))\,\mathrm{d}s \\ &\geq c_1\|T_1(u,v)\|_C. \end{aligned}$$

Moreover,

$$\|\left(T_1(u,v)\right)'\|_C \leq \int_0^1 \psi_1(s)g_1(s)f_1(s,u(s),u'(s),v(s),v'(s))\,\mathrm{d}s,$$

and

$$\begin{aligned} \min_{t\in[\gamma_1,\delta_1]} \left(T_1(u,v)(t)\right)' &= \min_{t\in[\gamma_1,\delta_1]} \int_0^1 \frac{\partial k_1}{\partial t}(t,s)g_1(s)f_1(s,u(s),u'(s),v(s),v'(s))\,\mathrm{d}s \\ &\geq d_1 \int_0^1 \psi_1(s)g_1(s)f_1(s,u(s),u'(s),v(s),v'(s))\,\mathrm{d}s \\ &\geq d_1\|\left(T_1(u,v)\right)'\|_C. \end{aligned}$$

Therefore, $T_1\tilde{K_1} \subset \tilde{K_1}$. By similar arguments it can be proved that $T_2\tilde{K_2} \subset \tilde{K_2}$.

The compactness of T follows, in a routine way, by the Ascoli–Arzelà Theorem. □

To specify our notation, for Ω an open bounded subset with $\Omega \subset K$ (endowed with the relative topology), we denote by $\overline{\Omega}$ and $\partial\Omega$ the closure and the boundary relative to K, respectively. If Ω is an open bounded subset of X then we write $\Omega_K = \Omega \cap K$, an open subset of K.

The next Lemma summarizes some classical results on fixed point index (more details can be seen in the books [1,4]).

Lemma 2.2. *Let Ω be an open bounded set with $0 \in \Omega_K$ and $\overline{\Omega}_K \neq K$. Assume that $F : \overline{\Omega}_K \to K$ is a compact map such that $x \neq Fx$ for all $x \in \partial\Omega_K$. Then the fixed point index $i_K(F, \Omega_K)$ has the following properties:*

(1) *If there exists $e \in K \setminus \{0\}$ such that $x \neq Fx + \lambda e$ for all $x \in \partial\Omega_K$ and all $\lambda > 0$, then $i_K(F, \Omega_K) = 0$.*
(2) *If $\mu x \neq Fx$ for all $x \in \partial\Omega_K$ and for every $\mu \geq 1$, then $i_K(F, \Omega_K) = 1$.*
(3) *If $i_K(F, \Omega_K) \neq 0$, then F has a fixed point in Ω_K.*
(4) *Let Ω^1 be open in X with $\overline{\Omega^1_K} \subset \Omega_K$. If $i_K(F, \Omega_K) = 1$ and $i_K(F, \Omega^1_K) = 0$, then F has a fixed point in $\Omega_K \setminus \overline{\Omega^1_K}$. The same result holds if $i_K(F, \Omega_K) = 0$ and $i_K(F, \Omega^1_K) = 1$.*

Along the paper, we use the following (relative) open bounded sets in K:

$$K_{\rho_1,\rho_2} = \{(u, v) \in K : \|u\|_{C^1} < \rho_1 \ \text{ and } \|v\|_{C^1} < \rho_2\}, \tag{2.5}$$

For our index calculations we make use of the following Lemma, similar to Lemma 5 of [3]. The novelty here is that we take into account the derivative. We omit the simple proof.

Lemma 2.3. *For the set defined by (2.5) we have that $(w_1, w_2) \in \partial K_{\rho_1,\rho_2}$ iff $(w_1, w_2) \in K$, and for $i = 1, 2$,*

$$\max_{t\in[0,1]} w_1(t) = \rho_1,\ -\rho_1 \leq w_1'(t) \leq \rho_1,\ -\rho_2 \leq w_2(t) \leq \rho_2,\ -\rho_2 \leq w_2'(t) \leq \rho_2,$$

or

$$-\rho_1 \leq w_1(t) \leq \rho_1,\ \max_{t\in[0,1]} w_1'(t) = \rho_1,\ -\rho_2 \leq w_2(t) \leq \rho_2,\ -\rho_2 \leq w_2'(t) \leq \rho_2,$$

or

$$-\rho_1 \leq w_1(t) \leq \rho_1,\ -\rho_1 \leq w_1'(t) \leq \rho_1,\ \max_{t\in[0,1]} w_2(t) = \rho_2,\ \ -\rho_2 \leq w_2'(t) \leq \rho_2,$$

or

$$-\rho_1 \leq w_1(t) \leq \rho_1,\ -\rho_1 \leq w_1'(t) \leq \rho_1,\ -\rho_2 \leq w_2(t) \leq \rho_2,\ \max_{t\in[0,1]} w_2'(t) = \rho_2.$$

3. Existence Results and Non-existence Results

The existence results are obtained via the fixed point index on the set K_{ρ_1,ρ_2} given by (2.5). First, we obtain sufficient conditions for the fixed point index on the set K_{ρ_1,ρ_2} to be 1.

Lemma 3.1. *Assume that*

$(\mathrm{I}^1_{\rho_1,\rho_2})$ *there exist $\rho_1, \rho_2 > 0$ such that for every $i = 1, 2$,*

$$f_i^{\rho_1,\rho_2} < \min\{m_i, m_i^*\}, \tag{3.1}$$

where

$$f_i^{\rho_1,\rho_2} := \sup\left\{\frac{f_i(t,u_1,u_2,v_1,v_2)}{\rho_i} : (t,u_1,u_2,v_1,v_2) \in [0,1]\times[-\rho_1,\rho_1]^2\times[-\rho_2,\rho_2]^2\right\}, \tag{3.2}$$

$$\frac{1}{m_i} := \max_{t\in[0,1]}\int_0^1 |k_i(t,s)|g_i(s)\,\mathrm{d}s \tag{3.3}$$

and

$$\frac{1}{m_i^*} := \max_{t\in[0,1]}\int_0^1 \left|\frac{\partial k_i}{\partial t}(t,s)\right| g_i(s)\,\mathrm{d}s. \tag{3.4}$$

Then $i_K(T,K_{\rho_1,\rho_2}) = 1$.

Proof. We claim that $\lambda(u,v) \neq T(u,v)$ for every $(u,v) \in \partial K_{\rho_1,\rho_2}$ and for every $\lambda \geq 1$, which implies that the index is 1 on K_{ρ_1,ρ_2}, by Lemma 2.2 (3).

Assume this is not true. Then there exist $\lambda \geq 1$ and $(u,v) \in \partial K_{\rho_1,\rho_2}$ such that $\lambda(u,v) = T(u,v)$.

Consider that

$$\|u\|_C = \rho_1, \|u'\|_C \leq \rho_1, \|v\|_C \leq \rho_2 \quad \text{and} \quad \|v'\|_C \leq \rho_2 \tag{3.5}$$

holds. Then we have

$$\lambda\,|u(t)| \leq \int_0^1 |k_1(t,s)|\,g_1(s)f_1(s,u(s),u'(s),v(s),v'(s))\,\mathrm{d}s,$$

and taking the maximum over $[0,1]$, by (3.2) and (3.3)

$$\begin{aligned}\lambda\rho_1 &\leq \max_{t\in[0,1]}\int_0^1 |k_1(t,s)|\,g_1(s)f_1(s,u(s),u'(s),v(s),v'(s))\,\mathrm{d}s\\ &\leq \max_{t\in[0,1]}\int_0^1 |k_1(t,s)|\,g_1(s)\rho_1 f_1^{\rho_1,\rho_2}\,\mathrm{d}s\\ &\leq \rho_1 f_1^{\rho_1,\rho_2}\frac{1}{m_1}.\end{aligned}$$

By (3.1), $\lambda\rho_1 < \rho_1$, which contradicts the fact that $\lambda \geq 1$.

If

$$\|u\|_C \leq \rho_1, \|u'\|_C = \rho_1, \|v\|_C \leq \rho_2 \text{ and } \|v'\|_C \leq \rho_2,$$

then we have

$$\lambda\,|u'(t)| \leq \int_0^1 \left|\frac{\partial k_1}{\partial t}(t,s)\right| g_1(s)f_1(s,u(s),u'(s),v(s),v'(s))\,\mathrm{d}s.$$

By (3.2) and (3.4), and taking the maximum in $[0,1]$,

$$\begin{aligned}\lambda\rho_1 &\leq \max_{t\in[0,1]}\int_0^1 \left|\frac{\partial k_i}{\partial t}(t,s)\right| g_1(s)f_1(s,u(s),u'(s),v(s),v'(s))\,\mathrm{d}s\\ &\leq \max_{t\in[0,1]}\int_0^1 \left|\frac{\partial k_i}{\partial t}(t,s)\right| g_1(s)\rho_1 f_1^{\rho_1,\rho_2}\,\mathrm{d}s\\ &\leq \rho_1 f_1^{\rho_1,\rho_2}\frac{1}{m_1^*},\end{aligned}$$

we obtain a similar contradiction as above.

The other cases follow the same arguments. □

Second, we provide a condition to have a null fixed point index on K_{ρ_1,ρ_2}.

Lemma 3.2. *Assume that*

$(\mathrm{I}^0_{\rho_1,\rho_2})$ *there exist* $\rho_1, \rho_2 > 0$ *such that for every* $i = 1, 2$,

$$f_{1,(\rho_1,\rho_2)} > M_1,\ f^*_{1,(\rho_1,\rho_2)} > M^*_1,\ f_{2,(\rho_1,\rho_2)} > M_2,\ f^*_{2,(\rho_1,\rho_2)} > M^*_2, \tag{3.6}$$

where

$$f_{1,(\rho_1,\rho_2)} := \inf\left\{ \begin{array}{l} \dfrac{f_1(t,u_1,u_2,v_1,v_2)}{\rho_1} : \\ (t,u_1,u_2,v_1,v_2) \in [a_1,b_1]\times[c_1\rho_1,\rho_1]\times[-\rho_1,\rho_1]\times[-\rho_2,\rho_2]^2 \end{array}\right\},$$

$$f^*_{1,(\rho_1,\rho_2)} := \inf\left\{ \begin{array}{l} \dfrac{f_1(t,u_1,u_2,v_1,v_2)}{\rho_1} : \\ (t,u_1,u_2,v_1,v_2) \in [\gamma_1,\delta_1]\times[-\rho_1,\rho_1]\times[d_1\rho_1,\rho_1]\times[-\rho_2,\rho_2]^2 \end{array}\right\},$$

$$f_{2,(\rho_1,\rho_2)} := \inf\left\{ \begin{array}{l} \dfrac{f_2(t,u_1,u_2,v_1,v_2)}{\rho_2} : \\ (t,u_1,u_2,v_1,v_2) \in [a_2,b_2]\times[-\rho_1,\rho_1]^2\times[c_2\rho_2,\rho_2]\times[-\rho_2,\rho_2] \end{array}\right\},$$

$$f^*_{2,(\rho_1,\rho_2)} := \inf\left\{ \begin{array}{l} \dfrac{f_2(t,u_1,u_2,v_1,v_2)}{\rho_2} : \\ (t,u_1,u_2,v_1,v_2) \in [\gamma_2,\delta_2]\times[-\rho_1,\rho_1]^2\times[-\rho_2,\rho_2]\times[d_2\rho_2,\rho_2] \end{array}\right\},$$

and

$$\frac{1}{M_i} := \min_{t\in[a_i,b_i]} \int_{a_i}^{b_i} k_i(t,s)g_i(s)\,\mathrm{d}s, \tag{3.7}$$

$$\frac{1}{M^*_i} := \min_{t\in[\gamma_i,\delta_i]} \int_{\gamma_i}^{\delta_i} \frac{\partial k_i}{\partial t}(t,s)g_i(s)\,\mathrm{d}s\,. \tag{3.8}$$

Then $i_K(T, K_{\rho_1,\rho_2}) = 0$.

Proof. Consider $e(t) \equiv 1$ for $t \in [0,1]$, and note that $(e,e) \in K$.

We claim that

$$(u,v) \neq T(u,v) + \lambda(e,e) \quad \text{for } (u,v) \in \partial K_{\rho_1,\rho_2} \quad \text{and } \lambda \geq 0.$$

Assume, by contradiction, that there exist $(u,v) \in \partial K_{\rho_1,\rho_2}$ and $\lambda \geq 0$ such that $(u,v) = T(u,v) + \lambda(e,e)$.

Consider that (3.5) holds. Then we can assume that for all $t \in [a_1,b_1]$ we have

$$c_1\rho_1 \leq u(t) \leq \rho_1, -\rho_1 \leq u'(t) \leq \rho_1, -\rho_2 \leq v(t) \leq \rho_2 \text{ and } -\rho_2 \leq v'(t) \leq \rho_2.$$

Then, for $t \in [a_1, b_1]$, we obtain, by (3.6),

$$\begin{aligned} u(t) &= \int_0^1 k_1(t,s) g_1(s) f_1(s, u(s), u'(s), v(s), v'(s))\, \mathrm{d}s + \lambda e(t) \\ &\geq \int_{a_1}^{b_1} k_1(t,s) g_1(s) f_1(s, u(s), u'(s), v(s), v'(s))\, \mathrm{d}s + \lambda \\ &\geq \int_{a_1}^{b_1} k_1(t,s) g_1(s) \rho_1 f_{1,(\rho_1,\rho_2)}\, \mathrm{d}s + \lambda. \end{aligned}$$

Taking the maximum over $[a_1, b_1]$ gives

$$\rho_1 \geq \max_{t \in [a_1,b_1]} u(t) \geq \rho_1 f_{1,(\rho_1,\rho_2)} \frac{1}{M_1} + \lambda.$$

By (3.6), we obtain the following contradiction: $\rho_1 > \rho_1 + \lambda$.

Suppose that

$$-\rho_1 \leq u(t) \leq \rho_1, \max_{t\in[0,1]} u'(t) = \rho_1, -\rho_2 \leq v(t) \leq \rho_2, -\rho_2 \leq v'(t) \leq \rho_2,$$

holds. Then, that for all $t \in [\gamma_1, \delta_1]$, we have

$$\begin{aligned} u'(t) &= \int_0^1 \frac{\partial k_1}{\partial t}(t,s) g_1(s) f_1(s, u(s), u'(s), v(s), v'(s))\, \mathrm{d}s + \lambda e(t) \\ &\geq \int_{\gamma_1}^{\delta_1} \frac{\partial k_1}{\partial t}(t,s) g_1(s) f_1(s, u(s), u'(s), v(s), v'(s))\, \mathrm{d}s + \lambda \\ &\geq \int_{\gamma_1}^{\delta_1} \frac{\partial k_1}{\partial t}(t,s) g_1(s) \rho_1 f^*_{1,(\rho_1,\rho_2)} \mathrm{d}s + \lambda. \end{aligned}$$

Taking the maximum over $[\gamma_1, \delta_1]$ gives

$$\rho_1 \geq \max_{t\in[\gamma_1,\delta_1]} u'(t) \geq \rho_1 f^*_{1,(\rho_1,\rho_2)} \frac{1}{M_1^*} + \lambda,$$

and by (3.7), a similar contradiction is achieved.

For the other cases, the procedure is analogous. □

In the following Theorem, we provide a result valid for up to three nontrivial solutions, but it is possible to prove the existence of four or more nontrivial solutions; see for example [8] for the kind of results that may be stated. We omit the proof that follows, in a routine manner, by means of the properties of fixed point index.

Theorem 3.3. *The system* (2.1) *has at least one nontrivial solution in K if one of the following conditions holds:*

- (S_1) *For $i = 1, 2$ there exist $\rho_i, r_i \in (0, \infty)$ with $\rho_i/c_i < r_i$ such that $(\mathrm{I}^0_{\rho_1,\rho_2})$, $(\mathrm{I}^1_{r_1,r_2})$ hold.*
- (S_2) *For $i = 1, 2$ there exist $\rho_i, r_i \in (0, \infty)$ with $\rho_i < r_i$ such that $(\mathrm{I}^1_{\rho_1,\rho_2})$, $(\mathrm{I}^0_{r_1,r_2})$ hold.*

The system (2.1) *has at least two nontrivial solutions in K if one of the following conditions holds:*

(S_3) *For* $i=1,2$ *there exist* $\rho_i, r_i, s_i \in (0,\infty)$ *with* $\rho_i/c_i < r_i < s_i$ *such that* $(\mathrm{I}^0_{\rho_1,\rho_2})$, $(\mathrm{I}^1_{r_1,r_2})$ *and* $(\mathrm{I}^0_{s_1,s_2})$ *hold.*

(S_4) *For* $i=1,2$ *there exist* $\rho_i, r_i, s_i \in (0,\infty)$ *with* $\rho_i < r_i$ *and* $r_i/c_i < s_i$ *such that* $(\mathrm{I}^1_{\rho_1,\rho_2})$, $(\mathrm{I}^0_{r_1,r_2})$ *and* $(\mathrm{I}^1_{s_1,s_2})$ *hold.*

The system (2.1) *has at least three nontrivial solutions in* K *if one of the following conditions holds:*

(S_5) *For* $i=1,2$ *there exist* $\rho_i, r_i, s_i, \sigma_i \in (0,\infty)$ *with* $\rho_i/c_i < r_i < s_i$ *and* $s_i/c_i < \sigma_i$ *such that* $(\mathrm{I}^0_{\rho_1,\rho_2})$, $(\mathrm{I}^1_{r_1,r_2})$, $(\mathrm{I}^0_{s_1,s_2})$ *and* $(\mathrm{I}^1_{\sigma_1,\sigma_2})$ *hold.*

(S_6) *For* $i=1,2$ *there exist* $\rho_i, r_i, s_i, \sigma_i \in (0,\infty)$ *with* $\rho_i < r_i$ *and* $r_i/c_i < s_i < \sigma_i$ *such that* $(\mathrm{I}^1_{\rho_1,\rho_2})$, $(\mathrm{I}^0_{r_1,r_2})$, $(\mathrm{I}^1_{s_1,s_2})$ *and* $(\mathrm{I}^0_{\sigma_1,\sigma_2})$ *hold.*

In the next example, we illustrate the applicability of Theorem 3.3.

Example 3.4. Consider the system

$$\begin{cases} u(t) = \int_0^1 s(\frac{7}{8}t - t^2)\left((u(t))^2 + (u'(t))^2\right)(2+\cos(v(t)\,v'(t)))\,\mathrm{d}s, \\ v(t) = \int_0^1 s(\frac{11}{10}t - t^2 - \frac{1}{10})\left((v(t))^2 + (v'(t))^2\right)(2-\sin(u(t)\,u'(t)))\,\mathrm{d}s. \end{cases} \tag{3.9}$$

In this case, we have

$$\begin{aligned} & k_1(t,s) = s\left(\frac{7}{8}t - t^2\right), k_2(t,s) = s\left(\frac{11}{10}t - t^2 - \frac{1}{10}\right), \\ & \frac{\partial k_1}{\partial t}(t,s) = s\left(\frac{7}{8} - 2t\right), \frac{\partial k_2}{\partial t}(t,s) = s\left(\frac{11}{10} - 2t\right), \\ & g_1(t) \equiv 1,\ g_2(t) \equiv 1, \\ & f_1(t,u_1,u_2,v_1,v_2) = \left((u_1)^2 + (u_2)^2\right)(2+\cos(v_1\,v_2)), \\ & f_2(t,u_1,u_2,v_1,v_2) = \left((v_1)^2 + (v_2)^2\right)(2-\sin(u_1\,u_2)). \end{aligned}$$

Note that k_1, k_2, $\dfrac{\partial k_1}{\partial t}$ and $\dfrac{\partial k_2}{\partial t}$ change sign on $[0,1]^2$. The assumption ($A3$) is satisfied with the choices

$$\phi_1(s) = \frac{49}{256}s, \phi_2(s) = \frac{81}{400}s,$$
$$a_1 = \frac{7}{32}, b_1 = \frac{21}{32}, c_1 = \frac{3}{4}, a_2 = \frac{13}{40}, b_2 = \frac{31}{40}, c_2 = \frac{3}{4}$$
$$\psi_1(s) = \frac{9}{8}s, \psi_2(s) = \frac{11}{10}s,$$
$$\gamma_1 = 0,\ \delta_1 = \frac{7}{32},\ d_1 = \frac{7}{18}, \gamma_2 = 0,\ \delta_2 = \frac{11}{40},\ d_2 = \frac{13}{44},$$

Furthermore, ($A4$) is satisfied since

$$\int_{\frac{7}{32}}^{\frac{21}{32}} \frac{49}{256}s\,\mathrm{d}s = \frac{2401}{65536}, \int_{\frac{13}{40}}^{\frac{31}{40}} \frac{81}{400}s\,\mathrm{d}s = \frac{8019}{160000}, \int_0^{\frac{7}{32}} \frac{9}{8}s\,\mathrm{d}s$$
$$= \frac{441}{16384}, \int_0^{\frac{11}{40}} \frac{11}{10}s\,\mathrm{d}s = \frac{1331}{32000}.$$

By direct calculation, we have

$$\frac{1}{m_1}=\max_{t\in[0,1]}\int_0^1\left|s\left(\frac{7}{8}t-t^2\right)\right|\,\mathrm{d}s=\frac{49}{512},$$
$$\frac{1}{m_2}=\max_{t\in[0,1]}\int_0^1\left|s\left(\frac{11}{10}t-t^2-\frac{1}{10}\right)\right|\,\mathrm{d}s=\frac{81}{800},$$
$$\frac{1}{m_1^*}=\max_{t\in[0,1]}\int_0^1\left|s\left(\frac{7}{8}-2t\right)\right|$$
$$\mathrm{d}s=\frac{9}{16},\ \frac{1}{m_2^*}=\max_{t\in[0,1]}\int_0^1\left|s\left(\frac{11}{10}-2t\right)\right|\,\mathrm{d}s=\frac{11}{20},$$
$$\frac{1}{M_1}=\min_{t\in[\frac{7}{32},\frac{21}{32}]}\int_{\frac{7}{32}}^{\frac{21}{32}}s\left(\frac{7}{8}t-t^2\right)\,\mathrm{d}s=\frac{7203}{262144},\ \frac{1}{M_2}$$
$$=\min_{t\in[\frac{13}{40},\frac{31}{40}]}\int_{\frac{13}{40}}^{\frac{31}{40}}s\left(\frac{11}{10}t-t^2-\frac{1}{10}\right)\,\mathrm{d}s=\frac{24057}{640000},$$
$$\frac{1}{M_1^*}=\min_{t\in[0,\frac{7}{32}]}\int_0^{\frac{7}{32}}s\left(\frac{7}{8}-2t\right)\,\mathrm{d}s=\frac{343}{32768},$$
$$\frac{1}{M_2^*}=\min_{t\in[0,\frac{11}{40}]}\int_0^{\frac{11}{40}}s\left(\frac{11}{10}-2t\right)\,\mathrm{d}s=\frac{1331}{64000}.$$

Now we need

$$f_1^{\rho_1,\rho_2}\le 6\rho_1<\min\{m_1,m_1^*\}=\frac{16}{9}\quad\left(\text{true if }\rho_1<\frac{1}{27}\right),$$

and

$$f_2^{\rho_1,\rho_2}\le 6\rho_2<\min\{m_2,m_2^*\}=\frac{20}{11}\quad\left(\text{true if }\rho_2<\frac{10}{33}\right).$$

Furthermore, we need

$$f_{1,(\rho_1,\rho_2)}\ge\frac{9}{16}\rho_1>M_1=\frac{262144}{7203}\quad\left(\text{true if }\rho_1>\frac{4194304}{64827}\right),$$
$$f_{1,(\rho_1,\rho_2)}^*\ge\frac{49}{324}\rho_1>M_1^*=\frac{32768}{343}\quad\left(\text{valid if }\rho_1>\frac{10616832}{16807}\right),$$
$$f_{2,(\rho_1,\rho_2)}\ge\frac{9}{16}\rho_2>M_2=\frac{640000}{24057}\quad\left(\text{true if }\rho_2>\frac{10240000}{216513}\right),$$
$$f_{2,(\rho_1,\rho_2)}^*\ge\frac{169}{1936}\rho_2>M_2^*=\frac{64000}{1331}\quad\left(\text{true if }\rho_2>\frac{1024000}{1859}\right).$$

Thus, if we fix

$$0<\rho_1<\frac{1}{27},\ 0<\rho_2<\frac{10}{33},$$
$$r_1>\max\left\{\frac{4194304}{64827},\frac{10616832}{16807}\right\}=\frac{10616832}{16807},$$
$$r_2>\max\left\{\frac{10240000}{216513},\frac{1024000}{1859}\right\}=\frac{1024000}{1859}$$

the conditions $(\mathrm{I}^1_{\rho_1,\rho_2})$, $(\mathrm{I}^0_{r_1,r_2})$ hold and we obtain, by Theorem 3.3, the existence of one nontrivial solution of the system (3.9).

Remark 3.5. Note that in the case of *non-negative* kernels, the same reasoning as above provides the existence of *positive* solutions. In this case, one may use the smaller cones (with abuse of notation)

$$\tilde{K}_i := \left\{ w \in C^1[0,\, 1] : w \geq 0,\, \min_{t\in[a_i,b_i]} w(t) \geq c_i \|w\|_C,\, \min_{t\in[\gamma_i,\delta_i]} w'(t) \geq d_i \|w'\|_C \right\}.$$

If, additionally, the derivative with respect to t of the kernels is non-negative, one may seek solutions in the even smaller cone (again with abuse of notation) given by

$$\tilde{K}_i := \left\{ w \in C^1[0,\, 1] : w \geq 0, w' \geq 0,\, \min_{t\in[a_i,b_i]} w(t) \geq c_i \|w\|_C,\, \min_{t\in[\gamma_i,\delta_i]} w'(t) \geq d_i \|w'\|_C \right\}.$$

For brevity we do not re-state all the results within these frameworks, but we illustrate the latter situation in Sect. 4, when discussing the system (1.4).

We now give sufficient conditions for the non-existence of nontrivial solutions for the system (2.1).

Theorem 3.6. *Let m_i be given by (3.3), M_i be given by (3.7) and a_i, b_i, c_i as in (A3) and suppose that the following conditions $(N1)$ and $(N2)$ are satisfied:*

$(N1)$ *Either*

$$f_1(t, u_1, u_2, v_1, v_2) < m_1|u_1| \text{ for every } t \in [0, 1], u_1 \neq 0 \text{ and } u_2, v_1, v_2 \in \mathbb{R}; \tag{3.10}$$

or

$$f_1(t, u_1, u_2, v_1, v_2) > \frac{M_1}{c_1} u_1 \text{ for every } t \in [a_1, b_1],\ u_1 > 0 \text{ and } u_2, v_1, v_2 \in \mathbb{R}, \tag{3.11}$$

holds.

$(N2)$ *Either*

$$f_2(t, u_1, u_2, v_1, v_2) < m_2|v_1| \text{ for every } t \in [0, 1], v_1 \neq 0 \text{ and } u_1, u_2, v_2 \in \mathbb{R};$$

or

$$f_2(t, u_1, u_2, v_1, v_2) > \frac{M_2}{c_2} v_1 \text{ for every } t \in [a_2, b_2],\ v_1 > 0 \text{ and } u_1, u_2, v_2 \in \mathbb{R},$$

holds.

Then there is no nontrivial solution of the system (2.1) in the cone K given by (2.3).

Proof. Suppose, by contradiction, that there exists a nontrivial solution of (2.1) in K, that is, $(u, v) \in K$ such that $(u, v) = T(u, v)$ and $(u, v) \neq (0, 0)$. Assume, without loss of generality, that $\|u\|_C \neq 0$. If (3.10) holds, then, for $t \in [0, 1]$, we have

$$
\begin{aligned}
|u(t)| &\le \int_0^1 |k_1(t,s)|g_1(s) f_1(s,u(s),u'(s),v(s),v'(s))\,\mathrm{d}s \\
&< m_1 \int_0^1 |k_1(t,s)|g_1(s)|u(s)|\,\mathrm{d}s \le m_1\|u\|_C \int_0^1 |k_1(t,s)|g_1(s)\,\mathrm{d}s.
\end{aligned}
$$

Taking the maximum for $t \in [0,1]$, we have, by (3.3), the following contradiction:

$$
\|u\|_C < m_1\|u\|_C \sup_{t\in[0,1]} \int_0^1 |k_1(t,s)|g_1(s)\,\mathrm{d}s = \|u\|_C.
$$

If (3.11) holds, then, for $t \in [a_1, b_1]$, we have

$$
\begin{aligned}
u(t) &= \int_0^1 k_1(t,s)g_1(s) f_1(s,u(s),u'(s),v(s),v'(s))\,\mathrm{d}s \\
&> \int_{a_1}^{b_1} k_1(t,s)g_1(s)\frac{M_1}{c_1}u(s)\,\mathrm{d}s.
\end{aligned}
$$

Taking the minimum for $t \in [a_1, b_1]$, we obtain, for some $\xi_1 > 0$, the following contradiction, by (3.7) and (2.2),

$$
\begin{aligned}
\xi_1 = \min_{t\in[a_1,b_1]} u(t) &> \frac{M_1}{c_1} \inf_{t\in[a_1,b_1]} \int_{a_1}^{b_1} k_1(t,s)g_1(s) \min_{s\in[a_1,b_1]} u(s)\,\mathrm{d}s \\
&\ge M_1\|u\|_C \inf_{t\in[a_1,b_1]} \int_{a_1}^{b_1} k_1(t,s)g_1(s)\,\mathrm{d}s = \|u\|_C \ge \xi_1.
\end{aligned}
$$

The proof in the case of $\|v\|_C \neq 0$ follows as above, using the condition $(N2)$. □

4. Positive Solutions of Some Third Order Systems

We turn back our attention to the system of third order ODEs with three point boundary conditions

$$
\begin{cases}
-u'''(t) = g_1(t) f_1(t,\, u(t),\, u'(t), v(t),\, v'(t)),\\
-v'''(t) = g_2(t) f_2(t,\, u(t),\, u'(t), v(t),\, v'(t)),\\
u(0) = u'(0) = 0, u'(1) = \alpha_1 u'(\eta_1),\\
v(0) = v'(0) = 0, v'(1) = \alpha_2 v'(\eta_2),
\end{cases}
\tag{4.1}
$$

where for $i = 1, 2$, $f_i : [0,\, 1] \times [0,\, +\infty)^4 \to [0,\, +\infty)$ is a L^∞-Carathéodory function, $g_i \in L^1[0,1]$ with $g_i(t) \ge 0$ for a.e. $t \in [0,\, 1]$, $0 < \eta_i < 1$ and $1 < \alpha_i < \frac{1}{\eta_i}$.

By routine calculation, we can associate to the system (4.1) the system of Hammerstein integral equations

$$
\begin{cases}
u(t) = \int_0^1 k_1(t,s)g_1(s) f_1(s,\, u(s),\, u'(s), v(s),\, v'(s))\,\mathrm{d}s,\\
v(t) = \int_0^1 k_2(t,s)g_2(s) f_2(s,\, u(s),\, u'(s), v(s),\, v'(s))\,\mathrm{d}s,
\end{cases}
\tag{4.2}
$$

where $k_i(t,s)$ are the Green's function given by

$$k_i(t,s) = \frac{1}{2(1-\alpha\eta_i)} \begin{cases} (2ts-s^2)(1-\alpha_i\eta_i)+t^2s(\alpha_i-1), & s \le \min\{\eta_i, t\}, \\ t^2(1-\alpha_i\eta_i)+t^2s(\alpha_i-1), & t \le s \le \eta_i, \\ (2ts-s^2)(1-\alpha_i\eta_i)+t^2(\alpha_i\eta_i-s), & \eta_i \le s \le t, \\ t^2(1-s), & \max\{\eta_i,t\} \le s. \end{cases} \tag{4.3}$$

The derivatives of the Green's functions (4.3) are given by

$$\frac{\partial k_i}{\partial t}(t,s) = \frac{1}{(1-\alpha_i\eta_i)} \begin{cases} s(1-\alpha_i\eta_i)+ts(\alpha_i-1), & s \le \min\{\eta_i, t\}, \\ t(1-\alpha_i\eta_i)+ts(\alpha_i-1), & t \le s \le \eta_i, \\ s(1-\alpha_i\eta_i)+t(\alpha_i\eta_i-s), & \eta_i \le s \le t, \\ t(1-s), & \max\{\eta_i,t\} \le s, \end{cases} \tag{4.4}$$

The following Lemmas provide some useful properties of the Green's functions and their derivatives.

Lemma 4.1. [9] *Take* $0<\eta_i<1$, $1<\alpha_i<\frac{1}{\eta_i}$ *and* k_i *as in* (4.3). *Then we have*

$$0 \le k_i(t,s) \le \phi_i(s), \ (t,s) \in [0,\,1]\times[0,\,1],$$

where

$$\phi_i(s) = \frac{1+\alpha_i}{1-\alpha_i\eta_i}s(1-s).$$

Furthermore, we have

$$k_i(t,s) \ge c_i\phi_i(s), \quad (t,s) \in \left[\frac{\eta_i}{\alpha_i},\,\eta_i\right]\times[0,\,1],$$

where

$$0 < c_i = \frac{\eta_i^2}{2\alpha_i^2(1+\alpha_i)}\min\{\alpha_i-1,\,1\} < 1. \tag{4.5}$$

Lemma 4.2. [10] *Take* $0<\eta_i<1$, $1<\alpha_i<\frac{1}{\eta_i}$, $\frac{\partial k_i}{\partial t}$ *as in* (4.4). *Then we have*

$$0 \le \frac{\partial k_i}{\partial t}(t,s) \le \psi_i(s), \ (t,s) \in [0,\,1]\times[0,\,1],$$

where

$$\psi_i(s) = \frac{(1-s)}{(1-\alpha_i\eta_i)}.$$

Furthermore, we have

$$\frac{\partial k_i}{\partial t}(t,s) \ge d_i\psi_i(s), \ (t,s) \in \left[\frac{\eta_i}{\alpha_i},\,\eta_i\right]\times[0,\,1],$$

with

$$0 < d_i = \min\{\alpha_i\eta_i,\,\eta_i\} < 1. \tag{4.6}$$

From Lemmas 4.1 and 4.2 we obtain that k_i satisfies a stronger positivity requirement than $(A3)$. This setting enables us to work in the cone

$$K := \{(u,v) \in \tilde{K_1} \times \tilde{K_2}\}, \tag{4.7}$$

where

$$\tilde{K_i} := \left\{ w \in C^1[0,\,1] : w \geq 0, w' \geq 0, \min_{t\in[\frac{\eta_i}{\alpha_i},\,\eta_i]} w(t) \geq c_i\|w\|_C, \min_{t\in[\frac{\eta_i}{\alpha_i},\,\eta_i]} w'(t) \geq d_i\|w'\|_C \right\}.$$

The condition $(\mathrm{I}^1_{\rho_1,\rho_2})$ in this case reads as follows:

$(\mathrm{I}^1_{\rho_1,\rho_2})$ there exist $\rho_1, \rho_2 > 0$ such that for every $i = 1, 2$, $f_i^{\rho_1,\rho_2} < \min\{m_i, m_i^*\}$, where

$$f_i^{\rho_1,\rho_2} := \sup\left\{ \frac{f_i(t,u_1,u_2,v_1,v_2)}{\rho_i} : (t,u_1,u_2,v_1,v_2) \in [0,1] \times [0,\rho_1]^2 \times [0,\rho_2]^2 \right\},$$

$$\frac{1}{m_i} = \max_{t\in[0,1]} \int_0^1 k_i(t,s)g_i(s)\,\mathrm{d}s, \quad \frac{1}{m_i^*} = \max_{t\in[0,1]} \int_0^1 \frac{\partial k_i}{\partial t}(t,s)g_i(s)\,\mathrm{d}s.$$

On the other hand, the condition $(\mathrm{I}^0_{\rho_1,\rho_2})$ reads as follows:

$(\mathrm{I}^0_{\rho_1,\rho_2})$ there exist $\rho_1, \rho_2 > 0$ such that for every $i = 1, 2$,

$$f_{1,(\rho_1,\rho_2)} > M_1,\ f^*_{1,(\rho_1,\rho_2)} > M_1^*,\ f_{2,(\rho_1,\rho_2)} > M_2,\ f^*_{2,(\rho_1,\rho_2)} > M_2^*, \tag{4.8}$$

where

$$f_{1,(\rho_1,\rho_2)} := \inf\left\{ \begin{array}{l} \dfrac{f_1(t,u_1,u_2,v_1,v_2)}{\rho_1} : \\ (t,u_1,u_2,v_1,v_2) \in [a_1,b_1] \times [c_1\rho_1,\rho_1] \times [0,\rho_1] \times [0,\rho_2]^2 \end{array} \right\},$$

$$f^*_{1,(\rho_1,\rho_2)} := \inf\left\{ \begin{array}{l} \dfrac{f_1(t,u_1,u_2,v_1,v_2)}{\rho_1} : \\ (t,u_1,u_2,v_1,v_2) \in [\gamma_1,\delta_1] \times [0,\rho_1] \times [d_1\rho_1,\rho_1] \times [0,\rho_2]^2 \end{array} \right\},$$

$$f_{2,(\rho_1,\rho_2)} := \inf\left\{ \begin{array}{l} \dfrac{f_2(t,u_1,u_2,v_1,v_2)}{\rho_2} : \\ (t,u_1,u_2,v_1,v_2) \in [a_2,b_2] \times [0,\rho_1]^2 \times [c_2\rho_2,\rho_2] \times [0,\rho_2] \end{array} \right\},$$

$$f^*_{2,(\rho_1,\rho_2)} := \inf\left\{ \begin{array}{l} \dfrac{f_2(t,u_1,u_2,v_1,v_2)}{\rho_2} : \\ (t,u_1,u_2,v_1,v_2) \in [\gamma_2,\delta_2] \times [0,\rho_1]^2 \times [0,\rho_2] \times [d_2\rho_2,\rho_2] \end{array} \right\}.$$

We can now state an existence result for one nontrivial solution for the System (4.1). Note that it is possible to state a result for two or more nontrivial solutions, in the spirit of Theorem 3.3.

Theorem 4.3. *For $i = 1, 2$, let $f_i : [0,\,1] \times [0,\,+\infty)^4 \to [0,\,+\infty)$ be a L^∞-Carathéodory function and let $g_i \in L^1[0,1]$ be such that $g_i(t) \geq 0$ for a.e. $t \in [0,1]$ and*

(A^*4)

$$\int_{\frac{\eta_i}{\alpha_i}}^{\eta_i} \frac{1+\alpha_i}{1-\alpha_i\eta_i} s(1-s)g_i(s)\,\mathrm{d}s > 0, \quad \int_{\frac{\eta_i}{\alpha_i}}^{\eta_i} \frac{(1-s)}{(1-\alpha_i\eta_i)} g_i(s)\,\mathrm{d}s > 0.$$

The system (4.1) admits a nontrivial solution with non-negative, non-decreasing components if one of the following conditions hold.

- $(\hat{S}_1)$ *For $i = 1, 2$ there exist $\rho_i, r_i \in (0, \infty)$ with $\rho_i/c_i < r_i$ such that $(\mathrm{I}^0_{\rho_1,\rho_2})$, $(\mathrm{I}^1_{r_1,r_2})$ hold.*
- $(\hat{S}_2)$ *For $i = 1, 2$ there exist $\rho_i, r_i \in (0, \infty)$ with $\rho_i < r_i$ such that $(\mathrm{I}^1_{\rho_1,\rho_2})$, $(\mathrm{I}^0_{r_1,r_2})$ hold.*

Example 4.4. Consider the following third order nonlinear system:

$$\begin{cases} -u'''(t) = t\left((u(t))^2 + (u'(t))^2\right)(2 + \cos(v(t)\, v'(t))), \\ -v'''(t) = t\left((v(t))^2 + (v'(t))^2\right)(2 - \sin(u(t)\, u'(t))), \\ u(0) = u'(0) = 0, u'(1) = \frac{3}{2}u'\left(\frac{1}{2}\right), \\ v(0) = v'(0) = 0, v'(1) = 2v'\left(\frac{1}{3}\right). \end{cases} \tag{4.9}$$

The system (4.9) is a particular case of the system (4.1) with

$$\begin{aligned} g_1(t) &\equiv 1,\ g_2(t) \equiv 1, \\ f_1(t, u_1, u_2, v_1, v_2) =& t\left((u_1)^2 + (u_2)^2\right)(2 + \cos(v_1\, v_2)), \\ f_2(t, u_1, u_2, v_1, v_2) =& t\left((v_1)^2 + (v_2)^2\right)(2 - \sin(u_1\, u_2)), \\ \eta_1 =& \frac{1}{2},\ \alpha_1 = \frac{3}{2}, \eta_2 = \frac{1}{3},\ \alpha_2 = 2. \end{aligned}$$

Note that f_1 and f_2 are continuous and non-negative.

Furthermore, we may take

$$\begin{gathered} \phi_1(s) = 10s(1-s), \phi_2(s) = 9s(1-s), \\ \psi_1(s) = 4(1-s), \psi_2(s) = 3(1-s), \\ c_1 = \frac{1}{45}, c_2 = \frac{1}{216}, d_1 = \frac{1}{2}, d_2 = \frac{1}{3}, \\ a_1 = \gamma_1 = \frac{1}{3}, b_1 = \delta_1 = \frac{1}{2}, \\ a_2 = \gamma_2 = \frac{1}{6}, b_2 = \delta_2 = \frac{1}{3}. \end{gathered}$$

Moreover, as

$$\begin{gathered} \int_{\frac{1}{3}}^{\frac{1}{2}} 10s(1-s)\, \mathrm{d}s = \frac{65}{162}, \int_{\frac{1}{6}}^{\frac{1}{3}} 9s(1-s)\, \mathrm{d}s = \frac{5}{18}, \\ \int_{\frac{1}{3}}^{\frac{1}{2}} 4(1-s)\, \mathrm{d}s = \frac{7}{18}, \int_{\frac{1}{6}}^{\frac{1}{3}} 3(1-s)\, \mathrm{d}s = \frac{3}{8}, \end{gathered}$$

assumption (A^*4) holds.

We have

$$
\begin{aligned}
\frac{1}{m_1} &= \max_{t\in[0,1]} \int_0^1 k_1(t,s)\,\mathrm{d}s \\
&\le \max_{t\in[0,1]} \left(\begin{array}{c} \int_0^{\frac{1}{2}} \left(t^2 s + ts - s^2\right)\,\mathrm{d}s + \int_{\frac{1}{2}}^{1-\frac{\sqrt{2}}{2}t} \left(-2t^2 s + ts + \frac{3}{2}t^2 - s^2\right)\mathrm{d}s \\ + \int_{1-\frac{\sqrt{2}}{2}t}^{1} \left(-2t^2 s + 2t^2 - s^2\right)\mathrm{d}s \end{array} \right) \\
&= \frac{1}{24} + \frac{\sqrt{2}}{3}, \\
\frac{1}{m_1^*} &= \max_{t\in[0,1]} \int_0^1 \frac{\partial k_1}{\partial t}(t,s) g_1(s)\,\mathrm{d}s \le \max_{t\in[0,1]} \left(\int_0^{\frac{1}{2}} 2ts + s\,\mathrm{d}s + \int_{\frac{1}{2}}^1 -4ts + 3t + s\,\mathrm{d}s \right) \\
&= \frac{3}{4}, \\
\frac{1}{m_2} &= \max_{t\in[0,1]} \int_0^1 k_2(t,s)\,\mathrm{d}s \\
&\le \max_{t\in[0,1]} \left(\int_0^{\frac{1}{3}} \frac{3}{2}t^2 s + ts - \frac{s^2}{2}\,\mathrm{d}s + \int_{\frac{1}{3}}^1 -\frac{3}{2}t^2 s + ts - \frac{s^2}{2} + 3t^2\mathrm{d}s \right) = \frac{43}{324}, \\
\frac{1}{m_2^*} &= \max_{t\in[0,1]} \int_0^1 \frac{\partial k_2}{\partial t}(t,s)\,\mathrm{d}s \\
&\le \max_{t\in[0,1]} \left(\int_0^{\frac{1}{3}} 3ts + s\,\mathrm{d}s + \int_{\frac{1}{3}}^1 -3ts + s + 6t\,\mathrm{d}s \right) = \frac{10}{3}, \\
\frac{1}{M_1} &= \min_{t\in[\frac{1}{3},\frac{1}{2}]} \int_{\frac{1}{3}}^{\frac{1}{2}} k_1(t,s)\,\mathrm{d}s = \min_{t\in[\frac{1}{3},\frac{1}{2}]} \int_{\frac{1}{3}}^{\frac{1}{2}} \left(t^2 s + \frac{t^2}{2}\right)\mathrm{d}s = \frac{11}{648}, \\
\frac{1}{M_1^*} &= \min_{t\in[\frac{1}{3},\frac{1}{2}]} \int_{\frac{1}{3}}^{\frac{1}{2}} \frac{\partial k_1}{\partial t}(t,s)\,\mathrm{d}s = \min_{t\in[\frac{1}{3},\frac{1}{2}]} \int_{\frac{1}{3}}^{\frac{1}{2}} (2ts + t)\,\mathrm{d}s = \frac{11}{108}, \\
\frac{1}{M_2} &= \min_{t\in[\frac{1}{6},\frac{1}{3}]} \int_{\frac{1}{6}}^{\frac{1}{3}} k_2(t,s)\,\mathrm{d}s = \min_{t\in[\frac{1}{6},\frac{1}{3}]} \int_{\frac{1}{6}}^{\frac{1}{3}} \left(\frac{3}{2}t^2 s + ts - \frac{s^2}{2}\right)\mathrm{d}s = \frac{17}{5184}, \\
\frac{1}{M_2^*} &= \min_{t\in[\frac{1}{6},\frac{1}{3}]} \int_{\frac{1}{6}}^{\frac{1}{3}} \frac{\partial k_2}{\partial t}(t,s)\,\mathrm{d}s = \min_{t\in[\frac{1}{6},\frac{1}{3}]} \int_{\frac{1}{6}}^{\frac{1}{3}} (3ts + t)\,\mathrm{d}s = \frac{7}{144},
\end{aligned}
$$

and therefore, we obtain

$$
\begin{aligned}
m_1 &= \frac{1}{\frac{1}{24} + \frac{\sqrt{2}}{3}},\ m_1^* = \frac{4}{3},\ m_2 = \frac{324}{43},\ m_2^* = \frac{3}{10}, \\
M_1 &= \frac{648}{11},\ M_1^* = \frac{108}{11},\ M_2 = \frac{5184}{17},\ M_2^* = \frac{144}{7}.
\end{aligned}
$$

Moreover, for

$$
\rho_1 < \frac{2}{9} \quad \text{and} \quad \rho_2 < \frac{1}{20},
$$

we obtain

$$
f_1^{\rho_1,\rho_2} \le 6\rho_1 < \min\{m_1, m_1^*\} = \frac{4}{3},
$$

$$f_2^{\rho_1,\rho_2} \leq 6\rho_2 < \min\{m_2, m_2^*\} = \frac{3}{10}.$$

Taking

$$\rho_1 > \frac{3936\,600}{11} \text{ and } \rho_2 > \frac{279\,936}{17},$$

we obtain

$$\begin{aligned} f_{1,(\rho_1,\rho_2)} &> \tfrac{\rho_1}{6075} > M_1 = \tfrac{648}{11}, \\ f^*_{1,(\rho_1,\rho_2)} &> \tfrac{\rho_1}{12} > M_1^* = \tfrac{108}{11}, \\ f_{2,(\rho_1,\rho_2)} &> \tfrac{\rho_2}{54} > M_2 = \tfrac{5184}{17}, \\ f^*_{2,(\rho_1,\rho_2)} &> \tfrac{\rho_2}{54} > M_2^* = \tfrac{144}{7}, \end{aligned}$$

that is, assumption $(\hat{S}_2)$ holds.

Therefore, all the assumptions of Theorem 4.3 are satisfied.

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