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Groups with the Weak Maximal Condition on Non-permutable Subgroups

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Abstract. Let G be a group with the weak maximal condition on non-permutable subgroups. We prove that if G is a generalized radical group then G is either quasihamiltonian or a soluble-by-finite minimax group.

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1. Introduction

A group G is said to have the weak maximal condition, denoted by max-∞ if, in every ascending chain of subgroups

$$H_1 \lneq H_2 \lneq H_3 \lneq \cdots \lneq H_n \lneq \cdots$$

at most finitely many of the indices $|H_{i+1} : H_i|$ are infinite. There is a corresponding weak minimal condition, min-∞, connected with descending chains, these being natural generalizations of the maximal and minimal conditions respectively. These concepts of weak maximal and weak minimal condition were first discussed in the papers of Zaitsev [16–18] and Baer [1] who showed that for soluble groups the conditions max-∞, min-∞ and minimax are equivalent. Zaitsev [17, Theorem 5] also showed that a locally soluble group with the weak minimal condition is a soluble minimax group.

More general conditions than the weak maximal condition have been defined. For a property $\mathcal{P}$, a group G is said to satisfy the *weak maximal condition on $\mathcal{P}$-subgroups* if there is no infinite ascending chain $H_1 < H_2 < H_3 < \cdots H_n < \cdots$ of $\mathcal{P}$-subgroups of G with each $|H_{i+1} : H_i|$ infinite. Similarly, we can define the weak minimal condition for $\mathcal{P}$-subgroups by considering descending chains. The structure of groups satisfying the weak maximal (respectively minimal) condition on non-normal subgroups was investigated by Kurdachenko and Goretskii [9] who showed that a locally (soluble-by-finite)

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group G satisfies the weak maximal condition (respectively, weak minimal condition) on non-normal subgroups if and only if G is either a Dedekind group or is an almost soluble minimax group. Also L. A. Kurdachenko and H. Smith studied the structure of groups satisfying the weak minimal and weak maximal conditions on non-subnormal subgroups in [10,11]. Groups satisfying the minimal condition on non-quasinormal subgroups were studied in [3].

In this paper, we investigate generalized radical groups satisfying the weak maximal condition on non-permutable subgroups and recall that a group is called *generalized radical* if it is the last term of an ascending normal series whose factors are locally nilpotent or locally finite. We recall also that a subgroup H of a group G is said to be *permutable* or *quasinormal* in G if $HK = KH$ for every subgroup K of G, a concept introduced by Ore [13]. We shall often denote the weak maximal condition on non-permutable subgroups by max-∞-$\overline{\text{qn}}$. Of course the class of groups satisfying max-∞-$\overline{\text{qn}}$ contains the class of soluble minimax groups and also the class of *quasihamiltonian* groups, those groups in which every subgroup is permutable. Such groups were classified quite precisely by Iwasawa [7].

The layout of the paper is as follows: In Sect. 2 we obtain several preliminary results that will be useful in the proof of our main theorem. In Sect. 3 we prove the main result of our paper which is as follows:

Theorem 1.1. *Let G be a generalized radical group satisfying max-∞-$\overline{qn}$. Then either G is quasihamiltonian, or G is a soluble-by-finite minimax group.*

We note that similar results can be obtained for groups with the weak minimal condition on non-permutable subgroups, some details of which appear in [4,5]. Our notation is generally that in standard use, where not explained, and can be found in [14]. The authors would like to thank the referee for several suggestions that improved this paper.

2. Preliminary Results

It is clear that every subgroup and factor group of a group satisfying max-∞-$\overline{\text{qn}}$ also satisfies max-∞-$\overline{\text{qn}}$.

We note the following easily proved fact. Its proof appeared in [5] but we give its simple proof for the sake of completeness:

Lemma 2.1. *Let $C \leq B \leq A$ be subgroups of a group G and suppose A, B, C are permutable where $|A : B|$, $|B : C|$ are infinite. Let $x \in G$. Then at least one of $|A\langle x\rangle : B\langle x\rangle|$, $|B\langle x\rangle : C\langle x\rangle|$ is infinite.*

Proof. It is easy to see that $|A\langle x\rangle : B\langle x\rangle| = |A : B(A \cap \langle x\rangle)|$ and that $|B(A \cap \langle x\rangle) : B| = |A \cap \langle x\rangle : B \cap \langle x\rangle|$, so

$$|A : B| = |A\langle x\rangle : B\langle x\rangle| \cdot |A \cap \langle x\rangle : B \cap \langle x\rangle|.$$

We may suppose that $|A\langle x\rangle : B\langle x\rangle|$ is finite. Then $|A \cap \langle x\rangle : B \cap \langle x\rangle|$ is infinite, so $B \cap \langle x\rangle = 1$. However, $|B\langle x\rangle : C\langle x\rangle||B \cap \langle x\rangle : C \cap \langle x\rangle| = |B : C|$ also and hence $|B\langle x\rangle : C\langle x\rangle|$ infinite. This completes the proof. $\square$

One consequence of this lemma is the following frequently used corollary:

Corollary 2.2. *Suppose $A_1 \le A_2 \le A_3 \le \cdots$ is an ascending chain of permutable subgroups of G with $|A_{i+1} : A_i|$ infinite for all i. Let $x \in G$. Then there is a subsequence $\{i_j\}_{j\ge 1}$ such that $|A_{i_{j+1}}\langle x\rangle : A_{i_j}\langle x\rangle|$ is infinite for all $j \ge 1$.*

The next result is also very useful.

Lemma 2.3. *Let G be a group satisfying max-∞-$\overline{qn}$ and suppose that there are subgroups X, Y with $Y \lhd X$ such that X/Y is a direct product of infinitely many nontrivial subgroups. Then*

(i) *X is permutable in G.*
(ii) *$X\langle x\rangle$ is permutable in G, for all $x \in G$.*

Proof. (i) It is easy to see that $X/Y = Dr_{i\ge 1}(C_i/Y)$, where each of the subgroups C_i/Y is itself an infinite direct product of nontrivial subgroups. Clearly, $X = \prod_{i\ge 1} C_i$, the product of the groups C_i.

Let $H = \prod_{i\ge 1} C_{2i}, D_n = \prod_{j\le n} C_{2i}$ and let $K = \prod_{i\ge 1} C_{2i-1}, E_n = \prod_{i\le n} C_{2i-1}$. Then $H \lneq HE_1 \lneq HE_2 \lneq \cdots$ is an ascending chain of subgroups of G and $|E_{n+1} : E_n|$ is infinite for all n. Since G satisfies max-∞-$\overline{\text{qn}}$ there is a natural number k such that HE_k is permutable in G. Similarly there is a natural number l such that KD_l is permutable in G. It follows that $X = HK = HE_kKD_l$ is permutable in G.

(ii) By Part (i), C_k is permutable in G, for each $k \ge 1$. If we write $L_n = C_1C_2\cdots C_n$ we have an ascending chain $L_1 \lneq L_2 \lneq L_3 \lneq \cdots$ of permutable subgroups of G with $|L_{n+1} : L_n|$ infinite for all n.

Fix $x \in G$. By Corollary 2.2 there is a subsequence $L_{k_1} \lneq L_{k_2} \lneq L_{k_3} \lneq \cdots$ such that $|L_{k_{l+1}}\langle x\rangle : L_{k_l}\langle x\rangle|$ is infinite for all $l \ge 1$. Since G satisfies max-∞-$\overline{\text{qn}}$ there exists a positive integer m such that $L_{k_m}\langle x\rangle$ is permutable in G. Since $\prod_{r\ne k_m} L_r$ is permutable in G, we deduce that $X\langle x\rangle$ is permutable in G, as required. □

We continue this section of preliminary results with the following useful lemma. In general, even the intersection of two permutable subgroups need not be permutable.

Lemma 2.4. *Let G be a group satisfying max-∞-$\overline{qn}$ and suppose that G contains subgroups L, B with $L \lhd B$. Suppose that $B/L = Dr_{i\ge 1} C_i/L$ and $C_i/L = Dr_{j\ge 1} B_{i,j}/L$. Then $\cap_{i\ge 1} C_i\langle x\rangle = L\langle x\rangle$ for all $x \in G$. In particular, L is permutable in G.*

Proof. We note that, by Lemma 2.3, C_i is permutable in G and hence we can form the subgroups $C_i\langle x\rangle$, for each $x \in G$. Let $d \in \cap_{i\ge 1} C_i\langle x\rangle$. Then $d \in C_i\langle x\rangle$ for all i and we may write

$$d = c_n x^{i_n} \quad \text{for all } n, \tag{1}$$

where $c_n \in C_n$ and $i_n \in \mathbb{Z}$ for each n. This implies $c_{k+1}^{-1} c_k = x^{i_{k+1}} x^{-i_k} \in B \cap \langle x\rangle$ for all k. If $B \cap \langle x\rangle = 1$, then $c_k = c_j$ for all k, j, so $c_j \in \cap_{k\ge 1} C_k = L$ and hence $d \in L\langle x\rangle$. Therefore, $L\langle x\rangle = \cap_{i\ge 1} C_i\langle x\rangle$ in this case.

Assume that $B \cap \langle x \rangle \neq 1$. Then there exists $l \neq 0$ such that $x^l \in B$. Then, for some natural number $k \neq 0$, we have $x^l L \in \mathrm{Dr}_{i=1}^{k} C_i/L$. Let $W/L = \mathrm{Dr}_{i>k} C_i/L$. If $W \cap \langle x \rangle \neq 1$, then there exists $m \in \mathbb{Z}$ such that $x^m \in W$ so that $x^m L \in \mathrm{Dr}_{i>k} C_i/L$. This implies that $x^{lm} \in L$, so there exists a positive integer μ such that $L \cap \langle x \rangle = \langle x^\mu \rangle$.

Now, in Eq. (1), we write $i_j = \mu q_j + r_j$ with $0 \leq r_j < \mu$ and observe that $d_j = c_j x^{q_j \mu} \in C_j$. Equation (1) then becomes

$$d = d_n x^{r_n} \quad \text{for all } n.$$

Since $0 \leq r_i < \mu$, there are infinitely many r_i with a common value t. Therefore,

$$dx^{-t} \in \cap_{i \geq 1} C_i = L$$

and consequently $d \in L\langle x \rangle$. Therefore, $\cap_{i \geq 1} C_i \langle x \rangle = L\langle x \rangle$, so $L\langle x \rangle$ is a subgroup of G, for each $x \in G$ and hence L is permutable in G. This completes the proof. $\square$

We conclude this section with a further elementary lemma which helps us handle elements of finite order:

Lemma 2.5. *Let G be a group satisfying max-∞-$\overline{qn}$ and let $x, y \in G$, where y has finite order. Let $L \leq G$ and suppose that $\{A_k\}_{k \geq 1}$ is a collection of subgroups of G such that L is a normal subgroup of A_k for all k. Suppose that $\langle A_k : k \geq 1 \rangle / L = \mathrm{Dr}_{k \geq 1}\, A_j/L$. and $A_k \langle x \rangle \langle y \rangle$ is a subgroup for all k. Then $L\langle x \rangle \langle y \rangle$ is a subgroup of G.*

Proof. First suppose that x has finite order. It suffices to show that $\cap_{k \geq 1} A_k \langle x \rangle \langle y \rangle \subseteq L\langle x \rangle \langle y \rangle$. Suppose that $d \in \cap_{k \geq 1} A_k \langle x \rangle \langle y \rangle$. Then $d \in A_k \langle x \rangle \langle y \rangle$ for all k and we have

$$d = a_n x^{i_n} y^{j_n} \quad \text{for all } n \geq 1, \tag{2}$$

where $a_k \in A_k$ and i_k, j_k are non-negative integers for all k. Since both x and y have finite order we may assume that $i_n = i_{n+1}$ and $j_n = j_{n+1}$, for all $n \geq 1$. Denote these common values by r, s respectively. Then, from (2), we have $d = a_n x^r y^s$, for all n, where $a_n \in A_n$. Then $a_n = a_{n+1}$, for all $n \geq 1$ and it follows that $a_1 \in \cap_{i \geq 1} A_i = L$. Hence $d \in L\langle x \rangle \langle y \rangle$, as required.

Now let x have infinite order and write $\mathrm{Dr}_{j \geq 1} A_j/L$ as $\mathrm{Dr}_{i \geq 1} (\mathrm{Dr}_{j \geq 1} B_{i,j}/L)$, where for each i, j, $B_{i,j} = A_k$ for some k and then set $C_i/L = \mathrm{Dr}_{j \geq 1} B_{i,j}/L$.

Suppose that $d \in \cap_{i \geq 1} C_i \langle x \rangle \langle y \rangle$. Then $d \in C_i \langle x \rangle \langle y \rangle$ for all i and hence we may write

$$d = a_n x^{i_n} y^{j_n} \quad \text{for all } n \geq 1,$$

where $a_i \in C_i$, $i_k \in \mathbb{Z}$, j_k is a natural number. As earlier, we may assume that $j_n = j_{n+1} = r$, for all $n \geq 1$ and some fixed natural number r. Then it follows that $dy^{-r} \in \cap_{i \geq 1} C_{k_i} \langle x \rangle$. However, $\cap_{i \geq 1} C_i \langle x \rangle = L\langle x \rangle$ by Lemma 2.4. Therefore, $d \in L\langle x \rangle \langle y \rangle$ and hence $L\langle x \rangle \langle y \rangle$ is a subgroup. $\square$

3. The Proof of the Main Theorem

Theorem 3.1. *Let G be a group satisfying max-∞-$\overline{qn}$. Suppose G contains a subgroup B of the form $B_1 \times B_2 \times B_3 \times \cdots$, with $B_i \neq 1$. Then G is quasihamiltonian.*

Proof. Since B is an infinite direct product we can write $B = \mathrm{Dr}_{j\geq 1}\, C_j$, where $C_j = \mathrm{Dr}\,_{i\geq 1} B_{i,j}$ and $|\mathrm{Dr}_{j\leq i+1}\, C_j : \mathrm{Dr}_{j\leq i}\, C_j|$ is infinite. By Lemma 2.3, each C_j is permutable in G. Clearly $\cap_{j\geq 1} C_j = 1$.

Since C_k is an infinite direct product it contains an ascending chain

$$D_1 \lneq D_2 \lneq D_3 \lneq \cdots \lneq C_k$$

of permutable subgroups D_i such that $|D_{i+1} : D_i|$ is infinite for all i. Clearly we may choose D_i to also be a direct product of infinitely many non-trivial factors.

Fix k and $x, y \in G$. An easy argument involving Corollary 2.2 allows us to assume that $|D_{i+1}\langle x\rangle : D_i\langle x\rangle|, |D_{i+1}\langle y\rangle : D_i\langle y\rangle|$ are both infinite and since G satisfies max-∞-$\overline{\mathrm{qn}}$, there exists a positive integer l such that $D_l\langle x\rangle$, $D_l\langle y\rangle$ per G. We let $D_l = E_k$, so $E_k\langle x\rangle$, $E_k\langle y\rangle$ are permutable in G and E_k is a direct product of infinitely many non-trivial factors. This implies that $E_k\langle x\rangle\langle y\rangle \leq G$ for all k and hence $\cap_{k\geq 1} E_k\langle x\rangle\langle y\rangle \leq G$. Of course $\cap_{k\geq 1} E_k = 1$.

We claim that $\cap_{k\geq 1} E_k\langle x\rangle\langle y\rangle = \langle x\rangle\langle y\rangle$ and Lemma 2.5 shows that this is true when at least one of x, y has finite order. Thus we assume that x, y both have infinite order and it is then easy to see that we may assume $\langle x\rangle \cap E_k = \langle y\rangle \cap E_k = 1$ for all k. Since each E_k is permutable a theorem of Stonehewer [15, 13.2.3] shows that $x, y \in N_G(E_k)$ for all k, so $E_k \lhd E_k\langle x\rangle\langle y\rangle$. Furthermore, for each k, either $E_k\langle x\rangle \cap \langle y\rangle = 1$ or $E_k\langle x\rangle \cap \langle y\rangle \neq 1$, so, by deleting terms as necessary we may assume that either $E_k\langle x\rangle \cap \langle y\rangle = 1$ for all k, or $E_k\langle x\rangle \cap \langle y\rangle \neq 1$ for all k.

In the former case [15, 13.2.3] implies that $E_k\langle x\rangle \lhd E_k\langle x\rangle\langle y\rangle$ for all k, so $\cap_{k\geq 1} E_k\langle x\rangle \lhd \cap_{k\geq 1} E_k\langle x\rangle\langle y\rangle$. Lemma 2.4 shows that $\cap_{k\geq 1} E_k\langle x\rangle = \langle x\rangle$, so $\langle x\rangle \lhd E_k\langle x\rangle\langle y\rangle$ and $\langle x\rangle\langle y\rangle$ is a subgroup. Thus $\langle x\rangle\langle y\rangle = \langle y\rangle\langle x\rangle$ in this case.

In the latter case we first assume that $\langle x\rangle \cap \langle y\rangle \neq 1$. Let $d \in \cap_{k\geq 1} E_k \langle x\rangle\langle y\rangle$, so

$$d = e_n x^{i_n} y^{j_n} \quad \text{for all } n \in \mathbb{N}.$$

Since $\langle x\rangle \cap \langle y\rangle \neq 1$, we have $x^r = y^s$ for some $r, s \in \mathbb{Z}$. Also $j_n = sq_n + r_n$ where $0 \leq r_n < s$. Hence $x^{i_n} y^{j_n} = x^{i_n + rq_n} y^{r_n} = x^{l_n} y^{r_n}$ for all $n \in \mathbb{N}$ where $l_n = i_n + r\,q_n$. Since, $r_n < s$, we may assume that $r_n = r_{n+1} = t$, for some fixed natural number t, for all $n \in \mathbb{N}$. We deduce that $dy^{-t} \in \cap_{k\geq 1} E_k\langle x\rangle = \langle x\rangle$, using Lemma 2.4. Hence $\cap_{k\geq 1} E_k\langle x\rangle\langle y\rangle = \langle x\rangle\langle y\rangle$, a subgroup of G.

Finally, we suppose that $E_k\langle x\rangle \cap \langle y\rangle \neq 1$ for all k and $\langle x\rangle \cap \langle y\rangle = 1$. We let $E = E_1 \times E_2 \times \cdots$ and note that we may assume $E \cap \langle y\rangle = 1 = E \cap \langle x\rangle$.

Suppose that $d \in \cap_{k\geq 1} E_k\langle x\rangle\langle y\rangle$ and write

$$d = x^{i_n} e_n y^{j_n} \quad \text{for all } n \in \mathbb{N},$$

where $e_k \in E_k$. We recall that E_k is normalized by $\langle x, y\rangle$. Notice that if $i_r = i_s$ for $r \neq s$, then $e_r y^{j_r} = e_s y^{j_s}$ and since $E \cap \langle y\rangle = 1$ we deduce that $e_r = e_s = 1$, so $d \in \langle x\rangle\langle y\rangle$. Similarly if $j_r = j_s$, then $d \in \langle x\rangle\langle y\rangle$.

For all distinct r, s the equation $x^{i_r} e_r y^{j_r} = x^{i_s} e_s y^{j_s}$ implies that

$$x^{(i_r - i_s)} = e_s f_r y^{(j_s - j_r)},$$

for some element $f_r \in E_r$, since $\langle x, y\rangle \leq N_G(E_r)$. Thus if r, s, m, n are distinct we have

$$\left(x^{i_r - i_s}\right)^{i_m - i_n} = a_s a_r \left(y^{j_s - j_r}\right)^{i_m - i_n}$$
$$\text{and} \quad \left(x^{i_m - i_n}\right)^{i_r - i_s} = a_m a_n \left(y^{j_n - j_m}\right)^{i_r - i_s},$$

for certain elements $a_i \in E_i$. It follows from these two equations that $a_i = 1$, for $i = r, s, m, n$, since $E \cap \langle y\rangle = 1$. This implies that $(j_s - j_r)(i_m - i_n) = (j_n - j_m)(i_r - i_s) = 0$, since $\langle x\rangle \cap \langle y\rangle = 1$, so $i_r = i_s$ or $j_n = j_m$. It follows that $\cap_{k\geq 1} E_k\langle x\rangle\langle y\rangle = \langle x\rangle\langle y\rangle$ is a subgroup in this case also. Consequently G is quasihamiltonian. □

The proof of our next result is very similar to that given in [3].

Theorem 3.2. *Let G be a locally finite group satisfying max-∞-$\overline{qn}$. Then either G is quasihamiltonian or it is a Chernikov group.*

Proof. Suppose that G is not a Chernikov group. Then, by a Theorem of Shunkov [8, Theorem 5.8], G does not satisfy the minimal condition on abelian subgroups. Therefore, G contains an infinite abelian subgroup A, which is not Chernikov and it is easy to see that $A = \mathrm{Dr}_{i\geq 1}\ \langle a_i\rangle$, where each a_i has prime power order. The result now follows from Theorem 3.1. □

Theorem 3.2 allows us to easily prove the following proposition:

Proposition 3.3. *Let G be a generalized radical group satisfying max-∞-$\overline{qn}$. Then G is radical-by-finite.*

Proof. Let R be the maximal normal radical subgroup of G and suppose $R \neq G$. Then there exists $N \lhd G$ such that $R \lneq N$ and N/R is locally finite. Let $L = G/R$ and suppose K is the maximal normal locally finite subgroup of L. Since K satisfies max-∞-$\overline{\text{qn}}$ it is either quasihamiltonian or Chernikov by Theorem 3.2. If K is quasihamiltonian, then K is locally nilpotent, so K is trivial by the choice of R. Thus K is Chernikov and hence finite.

If L is infinite then $K \neq L$ and the locally finite radical of L/K is trivial. Hence there is a nontrivial normal torsion-free locally nilpotent subgroup M/K of L/K. Since

$$C_M(K) \simeq C_M(K)/C_M(K) \cap K \simeq C_M(K)K/K$$

it follows that $C_M(K)$ is locally nilpotent. Also $L/C_L(K)$ is finite so $C_M(K)$ is infinite, contradicting the choice of R. The result follows: □

Our next result generalizes Theorem 3.1. There is a more general version of this result, but the one given is suitable for our purposes.

Proposition 3.4. *Let G be a generalized radical group satisfying max-∞-$\overline{qn}$. Suppose that G contains an abelian subgroup A that has a subgroup K such that A/K is periodic and $\pi(A/K)$ is infinite. Then $K\langle x\rangle\langle y\rangle$ is a subgroup for all $x, y \in G$.*

Proof. Let $x, y \in G$. As in the proof of Theorem 3.1 we may write $A/K = \mathrm{Dr}_{j\geq 1}\, C_j/K$ where each C_j/K is an infinite direct product of non-trivial groups, for all $j \geq 1$. By Lemma 2.3, each C_j is permutable in G and, as in the proof of Theorem 3.1, C_j contains a subgroup E_j such that $K \leq E_j$ and $E_j, E_j\langle x\rangle, E_j\langle y\rangle$ are permutable. Furthermore, we may assume that E_j/K is a direct product of infinitely many non-trivial factors. Then $E_j\langle x\rangle\langle y\rangle$ is a subgroup for all j and hence $\cap_{j\geq 1}E_j\langle x\rangle\langle y\rangle \leq G$.

If $x, y \in G$ and at least one of x, y has finite order then we use Lemma 2.5 to deduce that $K\langle x\rangle\langle y\rangle = \cap_{i\geq 1}E_i\langle x\rangle\langle y\rangle$. Hence $K\langle x\rangle\langle y\rangle$ is a subgroup of G, so we may assume that x and y are both elements of infinite order.

If $E_i\langle x\rangle \cap \langle y\rangle = 1$ for all $i \geq 1$, then by [15, 13.2.3], $E_i\langle x\rangle \lhd E_i\langle x\rangle\langle y\rangle$ for all i. Therefore, $\cap_{i\geq 1}E_i\langle x\rangle \lhd \cap_{i\geq 1}E_i\langle x\rangle\langle y\rangle$. Since $\cap_{i\geq 1}E_i\langle x\rangle = K\langle x\rangle$, by Lemma 2.4, it follows that $K\langle x\rangle\langle y\rangle$ is a subgroup in this case.

Consequently we may suppose that $E_i\langle x\rangle \cap \langle y\rangle \neq 1$ for all i. Suppose first that $\langle x\rangle \cap \langle y\rangle \neq 1$.

In this case we have $x^r = y^s$ for some r, $s \neq 0$. If $d \in \cap_{i\geq 1}E_i\langle x\rangle\langle y\rangle$, we have $d = b_n x^{i_n} y^{j_n}$, for all n, where $b_i \in B_i, i_k, j_k \in \mathbb{Z}$. We write $j_k = q_k s + r_k,\ 0 \leq r_k < s$ and $l_k = i_k + rq_k$ to reduce this, using $x^r = y^s$, to

$$d = b_n x^{l_n} y^{r_n}. \tag{3}$$

Since $r_k < s$, we may assume that $r_m = r_{m+1} = r$, for all m. Then Eq. (3) reduces to

$$d = b_{k_n} x^{l_n} y^r,$$

so $dy^{-r} \in \cap_{n\geq 1}E_n\langle x\rangle$. Now Lemma 2.4 implies that $dy^{-r} \in K\langle x\rangle$ and, therefore, $d \in K\langle x\rangle\langle y\rangle$. Hence $\cap_{i\geq 1}E_i\langle x\rangle\langle y\rangle = K\langle x\rangle\langle y\rangle$ is a subgroup in this case.

Hence we may suppose that $E_i\langle x\rangle \cap \langle y\rangle \neq 1$, for all i and $\langle x\rangle \cap \langle y\rangle = 1$. In this case suppose that $E_1 \cap \langle y\rangle \neq 1$ and that $d \in \cap_{i\geq 1}E_i\langle x\rangle\langle y\rangle$. As usual, write

$$d = x^{i_n} b_n y^{j_n} \quad \text{for all } n. \tag{4}$$

Since $E_1 \cap \langle y\rangle \neq 1$ and E_1/K is periodic, it follows that $K \cap \langle y\rangle \neq 1$ Hence $K \cap \langle y\rangle = \langle y^\mu\rangle$ for some $\mu \neq 0$. Thus $y^\mu \in E_i$ for all i. We write $j_k = q_k\mu + r_k$ where $0 < r_k < \mu$, for some $q_k \in \mathbb{Z}$ and $c_k = b_k y^{\mu q_k} \in E_k$. Then, from Eq. (4), we have

$$d = x^{i_n} c_n y^{r_n}.$$

Since, $0 < r_k < \mu$, we may assume that $r_m = r_{m+1} = r$, for all $m \in \mathbb{N}$. Then $dy^{-r} \in \cap_{i\geq 1}E_i\langle x\rangle$. By Lemma 2.4, $dy^{-r} \in K\langle x\rangle$ so $d \in K\langle x\rangle\langle y\rangle$ and $K\langle x\rangle\langle y\rangle$ is a subgroup of G in this case.

Finally, we suppose also that $E_1 \cap \langle x\rangle = 1 = E_1 \cap \langle y\rangle$ so, by [15, 13.2.3], $x, y \in N_G(E_i)$ for all i. Hence $x, y \in N_G(K)$ and we can form the groups $E_i\langle x\rangle\langle y\rangle/K$. By Theorem 3.1, $\cap_{i\geq 1}E_i\langle x\rangle\langle y\rangle/K = \langle Kx\rangle\langle Ky\rangle$. Therefore $K\langle x\rangle\langle y\rangle/K$ is a subgroup of $E_i\langle x\rangle\langle y\rangle/K$ and hence $K\langle x\rangle\langle y\rangle$ is a subgroup of G. □

We use this to prove the following theorem:

Theorem 3.5. *Let G be a generalized radical group satisfying max-∞-$\overline{qn}$. Then either G is quasihamiltonian, or G is soluble-by-finite of finite rank.*

Proof. Since G is a generalized radical group satisfying max-∞-$\overline{\text{qn}}$, it follows, by Proposition 3.3, that G is radical-by-finite. Let N be a normal radical subgroup of G such that G/N is finite. Now, we consider the abelian subgroups of G. If G contains an abelian subgroup A of infinite rank, then A contains a subgroup of the form $A_1 \times A_2 \times A_3 \times \cdots$ it follows, by Theorem 3.1, that G is quasihamiltonian.

If all the abelian subgroups of G have finite rank, then, by the Baer–Heineken Theorem [2], G also has finite rank. We prove that G is soluble-by-finite in this case. Clearly we may assume that G is non-quasihamiltonian. Since G has finite rank [6, Theorem A] implies that there exist normal subgroups $1 \leq T \leq L \leq M \leq G$ such that T is locally finite, L/T is a torsion-free nilpotent group, M/L is a finitely generated torsion-free abelian group and G/M is finite.

Since T satisfies max-∞-$\overline{\text{qn}}$, Theorem 3.2 shows that T is either quasihamiltonian or Chernikov. If T is quasihamiltonian, then T is locally nilpotent, so $T = \text{Dr}_{p\in\pi}\, T_p$, where π is a set of primes. If $|\pi|$ is infinite then G is quasihamiltonian by Theorem 3.1. Therefore, $|\pi|$ is finite. Since a locally finite p-group of finite rank is Chernikov each T_p is Chernikov and hence so is T.

Without loss of generality, we may assume that T is finite. Since

$$C_M(T)/\zeta(T) = C_M(T)/C_M(T) \cap T \simeq C_M(T)T/T$$

is soluble, $C_M(T)$ is soluble. Moreover, G/M and $M/C_M(T)$ are finite, so G is soluble-by-finite. This completes the proof. □

We now prove our main theorem, mentioned in the Introduction.

Proof of Theorem 1.1. It follows by Theorem 3.5 that if G is not quasihamiltonian, then G is a soluble-by-finite group of finite rank. To prove the theorem we may assume that G is soluble and, for a contradiction, not minimax. Then G contains an abelian subgroup that is not minimax, by a result of Baer [14, Theorem 10.35]. Let A be such an abelian group and note that A contains a finitely generated torsion-free subgroup K such that A/K is periodic and $\pi(A/K)$ is infinite. Hence A contains a collection of subgroups $K_i \leq K$ such that K/K_i is finite and $\cap_{i\geq 1}K_i = 1$. It follows that A/K_i is a periodic group such that $\pi(A/K_i)$ is infinite. Using the first part of the argument of Proposition 3.4 we see that G has a torsion subgroup. Indeed if $x, y \in G$ are both of finite order, then we can form the groups $K_i\langle x\rangle\langle y\rangle$ and then their intersection $D = \cap_{i\geq 1}K_i\langle x\rangle\langle y\rangle$. If $d \in D$, then $d = k_i x^{m_i} y^{n_i}$ for certain $k_i \in K_i, m_i, n_i \in \mathbb{Z}$. Since x, y have finite order we may assume that $m_i = r, n_i = s$, for fixed integers r, s and then we see that $k_i = k_j$ for all i, j.

Since $\cap_{i\geq 1}K_i = 1$ this implies that $k_i = 1$ for all i and hence $D = \langle x\rangle\langle y\rangle$, so that $\langle x\rangle\langle y\rangle = \langle y\rangle\langle x\rangle$. (Indeed using an argument similar to that given in the proof of Lemma 2.4 it is possible to show that $\langle x\rangle\langle y\rangle$ is a subgroup when only one of these elements has finite order.)

Thus G is generated by its elements of infinite order and we let x, y be two such elements. We claim that K contains a subgroup F of finite index such that $x, y \in N_G(F)$. Proposition 3.4 implies that each subgroup of K of finite index is permutable in G. Since K is permutable in G, [15, 13.2.3] implies that if $K \cap \langle x\rangle = 1$ and $K \cap \langle y\rangle = 1$, then $x, y \in N_G(K)$ and we let $F = K$ in this case. If $K \cap \langle x\rangle \neq 1$, then $|K\langle x\rangle : K|$ is finite, so there is a normal subgroup F of $K\langle x\rangle$ contained in K such that $K\langle x\rangle/F$ is finite. Since F is permutable, if $F \cap \langle y\rangle = 1$, then F is also normalized by y and our claim follows. If $F \cap \langle y\rangle \neq 1$, then,

$$|F\langle x\rangle\langle y\rangle : F| = |F\langle x\rangle\langle y\rangle : F\langle x\rangle| \cdot |F\langle x\rangle : F|$$

is finite and there is a normal subgroup X of $F\langle x\rangle\langle y\rangle$, lying in F, such that $x, y \in N_G(X)$ and $|F\langle x\rangle\langle y\rangle : X|$ is finite. We note that the group $X\langle x\rangle\langle y\rangle$ is polycyclic. Furthermore X^r is normalized by x, y, for all natural numbers r. Let N be a normal subgroup of $X\langle x\rangle\langle y\rangle$ of finite index. Since $X\langle x\rangle\langle y\rangle/N$ is finite, there is an integer k such that $X^k \leq N$. Of course $X^k\langle x\rangle\langle y\rangle$ is a group, so $X^k\langle x\rangle\langle y\rangle = X^k\langle y\rangle\langle x\rangle$. Hence $N\langle x\rangle\langle y\rangle = N\langle y\rangle\langle x\rangle$. We now invoke [12, Theorem A] to deduce that $\langle x\rangle\langle y\rangle = \langle y\rangle\langle x\rangle$. Since x, y were arbitrary elements of G of infinite order, we deduce that G is quasihamiltonian, which is the contradiction sought. The result follows. □

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