Mediterr. J. Math. (2017) 14:230
https://doi.org/10.1007/s00009-017-1032-5
1660-5446/17/060001-11
published online October 29, 2017

Mediterranean Journal of Mathematics

Derivations, Local and 2-Local Derivations on Some Algebras of Operators on Hilbert C*-Modules

Jun He, Jiankui Li and Danjun Zhao

Abstract. For a commutative C*-algebra $\mathcal{A}$ with unit e and a Hilbert $\mathcal{A}$-module $\mathcal{M}$, denote by $\mathrm{End}_{\mathcal{A}}(\mathcal{M})$ the algebra of all bounded $\mathcal{A}$-linear mappings on $\mathcal{M}$, and by $\mathrm{End}^*_{\mathcal{A}}(\mathcal{M})$ the algebra of all adjointable mappings on $\mathcal{M}$. We prove that if $\mathcal{M}$ is full, then each derivation on $\mathrm{End}_{\mathcal{A}}(\mathcal{M})$ is $\mathcal{A}$-linear, continuous, and inner, and each 2-local derivation on $\mathrm{End}_{\mathcal{A}}(\mathcal{M})$ or $\mathrm{End}^*_{\mathcal{A}}(\mathcal{M})$ is a derivation. If there exist x_0 in $\mathcal{M}$ and f_0 in $\mathcal{M}^{'}$, such that $f_0(x_0) = e$, where $\mathcal{M}^{'}$ denotes the set of all bounded $\mathcal{A}$-linear mappings from $\mathcal{M}$ to $\mathcal{A}$, then each $\mathcal{A}$-linear local derivation on $\mathrm{End}_{\mathcal{A}}(\mathcal{M})$ is a derivation.

Mathematics Subject Classification. 47B47, 47L10.

Keywords. Derivation, Hilbert C*-module, inner derivation, local derivation, 2-local derivation.

1. Introduction and Preliminaries

The structure of derivations on operator algebras is an important part of the theory of operator algebras.

Let $\mathcal{A}$ be an algebra and $\mathcal{M}$ be an $\mathcal{A}$-bimodule. Recall that a *derivation* is a linear mapping d from $\mathcal{A}$ into $\mathcal{M}$ such that $d(xy) = d(x)y + xd(y)$, for all x, y in $\mathcal{A}$. For each m in $\mathcal{M}$, one can define a derivation D_m by $D_m(x) = mx - xm$, for all x in $\mathcal{A}$. Such derivations are called *inner derivations*.

It is a classical problem to identify those algebras on which all derivations are inner derivations. Several authors investigate this topic. The following two results are classical. Sakai [17] proves that all derivations from a W*-algebra into itself are inner derivations. Christensen [3] proves that all derivations from a nest algebra into itself are inner derivations.

In 1990, Kadison [10] and Larson and Sourour [13] independently introduced the concept of local derivation in the following sense: a linear mapping δ from $\mathcal{A}$ into $\mathcal{M}$ such that for every $a \in \mathcal{A}$, there exists a derivation $d_a : \mathcal{A}$

$\rightarrow \mathcal{M}$, depending on a, satisfying $\delta(a) = d_a(a)$. In [10], Kadison proves that each continuous local derivation from a von Neumann algebra into its dual Banach module is a derivation. In [13], Larson and Sourour prove that each local derivation from $B(\mathcal{X})$ into itself is a derivation, where $\mathcal{X}$ is a Banach space. Johnson [8] proves that each local derivation from a C*-algebra into its Banach bimodule is a derivation. Pan and the second author of this paper [14] prove that each local derivation from the algebra $\mathcal{M} \bigcap alg\mathcal{L}$ into $B(\mathcal{H})$ is a derivation, where $\mathcal{H}$ is a Hilbert space, $\mathcal{M}$ is a von Neumann algebra acting on $\mathcal{H}$, and $\mathcal{L}$ is a commutative subspace lattice in $\mathcal{M}$. For more information about this topic, we refer to [2,4,6].

In 1997, Šemrl [18] introduced the concept of 2-local derivations. Recall that a mapping $\delta : \mathcal{A} \rightarrow \mathcal{M}$ (not necessarily linear) is called a *2-local derivation* if for each $a, b \in \mathcal{A}$, there exists a derivation $d_{a,b} : \mathcal{A} \rightarrow \mathcal{M}$ such that $\delta(a) = d_{a,b}(a)$ and $\delta(b) = d_{a,b}(b)$. Moreover, the author proves that every 2-local derivation on $B(\mathcal{H})$ is a derivation for a separable Hilbert space $\mathcal{H}$. Zhang and Li [19] extend the above result for arbitrary symmetric digraph matrix algebras and construct an example of 2-local derivation which is not a derivation on the algebra of all upper triangular complex 2×2 matrices. Ayupov and Kudaybergenov [1] prove that each 2-local derivation on a von Neumann algebra is a derivation. For more information about this topic, we refer to [2,7,11].

In this paper, we study derivations, local derivations and 2-local derivations on some algebras of operators on Hilbert C*-modules. There are few results in this topic. Li et al. [15] prove that each derivation on $\mathrm{End}^*_{\mathcal{A}}(\mathcal{M})$ is inner, where $\mathcal{M}$ is a full Hilbert C*-module over a commutative unital C*-algebra $\mathcal{A}$. Moghadam et al. [16] prove that each $\mathcal{A}$-linear derivation on $\mathrm{End}_{\mathcal{A}}(\mathcal{M})$ is inner, where $\mathcal{M}$ is a full Hilbert C*-module over a commutative unital C*-algebra $\mathcal{A}$ with the property that there exist x_0 in $\mathcal{M}$ and f_0 in $\mathcal{M}^{'}$ such that $f_0(x_0) = e$.

Hilbert C*-modules provide a natural generalization of Hilbert spaces by replacing the complex field $\mathbb{C}$ with an arbitrary C*-algebra. The theory of Hilbert C*-modules plays an important role in the theory of operator algebras, as it can be applied in many fields, such as index theory of elliptic operators, K- and K K-theory, noncommutative geometry, and so on.

In the following, we would first review some properties of Hilbert C*-modules [12]:

Let $\mathcal{A}$ be a C*-algebra and $\mathcal{M}$ be a left $\mathcal{A}$-module.

$\mathcal{M}$ is called a *Pre-Hilbert $\mathcal{A}$-module* if there exists a mapping $\langle \cdot, \cdot \rangle : \mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{A}$ with the following properties: for each $\lambda \in \mathbb{C}, a \in \mathcal{A}, x, y, z \in \mathcal{M}$,

(1) $\langle x, x \rangle \geq 0$, and $\langle x, x \rangle = 0$ implies that $x = 0$,
(2) $\langle \lambda x + y, z \rangle = \lambda \langle x, z \rangle + \langle y, z \rangle$,
(3) $\langle ax, y \rangle = a \langle x, y \rangle$,
(4) $\langle x, y \rangle = \langle y, x \rangle^*$.

The mapping $\langle \cdot, \cdot \rangle$ is called an $\mathcal{A}$-valued inner product. The inner product induces a norm on $\mathcal{M}$: $\|x\| = \|\langle x, x \rangle\|^{1/2}$. $\mathcal{M}$ is called a *Hilbert $\mathcal{A}$-module* (or more exactly, a Hilbert C*-module over $\mathcal{A}$), if it is complete with respect to this norm.

We denote by $\langle \mathcal{M}, \mathcal{M} \rangle$ the closure of the linear span of all the elements of the form $\langle x, y \rangle, x, y \in \mathcal{M}$. $\mathcal{M}$ is called a *full* Hilbert $\mathcal{A}$-module if $\langle \mathcal{M}, \mathcal{M} \rangle = \mathcal{A}$.

For a full Hilbert $\mathcal{A}$-module $\mathcal{M}$, we have the following lemma:

Lemma 1.1. *Let $\mathcal{A}$ be a C*-algebra with unit e and $\mathcal{M}$ be a full Hilbert $\mathcal{A}$-module. There exists a sequence $\{x_i\}_{i=1}^n \subseteq \mathcal{M}$, such that $\sum_{i=1}^n \langle x_i, x_i \rangle = e$.*

A linear mapping T from $\mathcal{M}$ into itself is said to be $\mathcal{A}$-*linear* if $T(ax) = aT(x)$ for each $a \in \mathcal{A}$ and $x \in \mathcal{M}$. A bounded $\mathcal{A}$-linear mapping from $\mathcal{M}$ into itself is called an *operator* on $\mathcal{M}$. Denote by $\mathrm{End}_{\mathcal{A}}(\mathcal{M})$ all operators on $\mathcal{M}$. $\mathrm{End}_{\mathcal{A}}(\mathcal{M})$ is a Banach algebra.

A mapping T from $\mathcal{M}$ into itself is said to be *adjointable* if there exists a mapping T^* such that $\langle Tx, y \rangle = \langle x, T^*y \rangle$, for all $x, y \in \mathcal{M}$. Notice that each adjointable mapping must be an operator. Denote by $\mathrm{End}^*_{\mathcal{A}}(\mathcal{M})$ all adjointable operators on $\mathcal{M}$. $\mathrm{End}^*_{\mathcal{A}}(\mathcal{M})$ is a C*-algebra.

Similarly, a linear mapping f from $\mathcal{M}$ into $\mathcal{A}$ is said to be $\mathcal{A}$-linear if $f(ax) = af(x)$ for each $a \in \mathcal{A}$ and $x \in \mathcal{M}$. The set of all bounded $\mathcal{A}$-linear mappings from $\mathcal{M}$ to $\mathcal{A}$ is denoted by $\mathcal{M}'$.

For each x in $\mathcal{M}$, one can define a mapping $\hat{x}$ from $\mathcal{M}$ to $\mathcal{A}$ as follows: $\hat{x}(y) = \langle y, x \rangle$, for all $y \in \mathcal{M}$. Obviously, $\hat{x} \in \mathcal{M}'$.

For each x in $\mathcal{M}$ and f in $\mathcal{M}'$, one can define a mapping $\theta_{x,f}$ from $\mathcal{M}$ into itself as follows: $\theta_{x,f} y = f(y)x$, for all $y \in \mathcal{M}$. Obviously, $\theta_{x,f} \in \mathrm{End}_{\mathcal{A}}(\mathcal{M})$.

In particular, for each x, y in $\mathcal{M}$, we have $\theta_{x,\hat{y}} z = \hat{y}(z)x = \langle z, y \rangle x$, for all $z \in \mathcal{M}$.

For the operators of the above forms, we have the following lemmas:

Lemma 1.2. *Let $\mathcal{M}$ be a Hilbert C*-module over a C*-algebra $\mathcal{A}$. For all $a \in \mathcal{A}, x, y \in \mathcal{M}, f, g \in \mathcal{M}', A \in End_{\mathcal{A}}(\mathcal{M})$, we have*

(1) $\theta_{x,f} A = \theta_{x,f \circ A}$,
(2) $A\theta_{x,f} = \theta_{Ax,f}$,
(3) *if in addition, $\mathcal{A}$ is commutative, then* $\theta_{x,f}\theta_{y,g} = f(y)\theta_{x,g}$, $\theta_{ax,f} = a\theta_{x,f}$.

Lemma 1.3. *Let $\mathcal{M}$ be a Hilbert C*-module over a C*-algebra $\mathcal{A}$. For all $a \in \mathcal{A}, x, y, z, w \in \mathcal{M}, A \in End^*_{\mathcal{A}}(\mathcal{M})$, we have*

(1) $\theta_{x,\hat{y}} \in End^*_{\mathcal{A}}(\mathcal{M})$, *and* $\theta^*_{x,\hat{y}} = \theta_{y,\hat{x}}$,
(2) $\theta_{x,\hat{y}} A = \theta_{x,\hat{y} \circ A} = \theta_{x,\widehat{A^*y}}$,
(3) $A\theta_{x,\hat{y}} = \theta_{Ax,\hat{y}}$,
(4) *if in addition, $\mathcal{A}$ is commutative, then* $\theta_{x,\hat{y}}\theta_{z,\hat{w}} = \langle z, y \rangle \theta_{x,\hat{w}}$, $\theta_{ax,\hat{y}} = a\theta_{x,\hat{y}} = \theta_{x,\widehat{a^*y}}$.

For a commutative C*-algebra $\mathcal{A}$, for each a in $\mathcal{A}$, one can define a mapping T_a from $\mathcal{M}$ into itself as follows: $T_a x = ax$, for all $x \in \mathcal{M}$. Obviously, $T_a \in \mathrm{End}_{\mathcal{A}}(\mathcal{M})$. It is worthwhile to notice that if $\mathcal{A}$ is not commutative, then T_a is not $\mathcal{A}$-linear. In this case, T_a is not in $\mathrm{End}_{\mathcal{A}}(\mathcal{M})$.

Lemma 1.4. *Let $\mathcal{A}$ be a commutative C^*-algebra with unit e and $\mathcal{M}$ be a full Hilbert $\mathcal{A}$-module. Then $\mathcal{Z}(End_{\mathcal{A}}(\mathcal{M})) = \{T_a : \ a \in \mathcal{A}\}$.*

Proof. For each A in $\mathrm{End}_{\mathcal{A}}(\mathcal{M})$ and x in $\mathcal{M}$, since $AT_ax = A(ax) = aAx = T_aAx$, we have $AT_a = T_aA$. It is to say $T_a \in \mathcal{Z}(\mathrm{End}_{\mathcal{A}}(\mathcal{M}))$.

On the other hand, assume $A \in \mathcal{Z}(\mathrm{End}_{\mathcal{A}}(\mathcal{M}))$. By Lemma 1.1, there exists a sequence $\{x_i\}_{i=1}^n \subseteq \mathcal{M}$, such that $\sum_{i=1}^n \langle x_i, x_i\rangle = e$. Thus we have

$$\sum_{i=1}^n A\theta_{x,\hat{x_i}}x_i = \sum_{i=1}^n \langle x_i, x_i\rangle Ax = Ax,$$

and

$$\sum_{i=1}^n \theta_{x,\hat{x_i}}Ax_i = \sum_{i=1}^n \langle Ax_i, x_i\rangle x.$$

Let $\sum_{i=1}^n \langle Ax_i, x_i\rangle = a$. Then we have $A = T_a$. The proof is complete. $\square$

For an algebra $\mathcal{A}$, if for each a in $\mathcal{A}$, $a\mathcal{A}a = 0$ implies that $a = 0$, then it is said to be *semi-prime*.

Lemma 1.5. *Let $\mathcal{A}$ be a C^*-algebra and $\mathcal{M}$ be a Hilbert $\mathcal{A}$-module. Then $End_{\mathcal{A}}(\mathcal{M})$ is a semi-prime Banach algebra.*

Proof. Let A be in $\mathrm{End}_{\mathcal{A}}(\mathcal{M})$. Assume that $ABA = 0$ for each B in $\mathrm{End}_{\mathcal{A}}(\mathcal{M})$. In particular, for each $x \in \mathcal{M}$ and $f \in \mathcal{M}'$, we have

$$A\theta_{x,f}Ax = \theta_{Ax,f\circ A}x = f(Ax)Ax = 0.$$

By taking $y = Ax$ and $f = \hat{y}$, we have $\langle y, y\rangle y = 0$. It follows that

$$\langle\langle y, y\rangle y, \langle y, y\rangle y\rangle = \langle y, y\rangle^3 = 0.$$

Since $\langle y, y\rangle$ is a self-adjoint element, we have $\langle y, y\rangle = 0$, and $y = 0$. Hence $A = 0$, and $\mathrm{End}_{\mathcal{A}}(\mathcal{M})$ is semi-prime. The proof is complete. $\square$

2. Derivations on $\mathrm{End}_{\mathcal{A}}(\mathcal{M})$

In this section, we study derivations on $\mathrm{End}_{\mathcal{A}}(\mathcal{M})$. We begin with several lemmas.

Lemma 2.1. *Let $\mathcal{A}$ be a commutative unital C^*-algebra and $\mathcal{M}$ be a full Hilbert $\mathcal{A}$-module. Then each derivation on $End_{\mathcal{A}}(\mathcal{M})$ is $\mathcal{A}$-linear, i.e. $d(aA) = ad(A)$, for each $a \in \mathcal{A}$ and $A \in End_{\mathcal{A}}(\mathcal{M})$.*

Proof. Suppose d is a derivation on $\mathrm{End}_{\mathcal{A}}(\mathcal{M})$.

By Lemma 1.4, we have $\mathcal{Z}(\mathrm{End}_{\mathcal{A}}(\mathcal{M})) = \{T_a : \ a \in \mathcal{A}\}$. For each A in $\mathrm{End}_{\mathcal{A}}(\mathcal{M})$, By

$$d(T_aA) = d(T_a)A + T_ad(A)$$

and

$$d(AT_a) = Ad(T_a) + d(A)T_a,$$

we obtain $d(T_a)A = Ad(T_a)$. Hence $d(T_a) \in \mathcal{Z}(\mathrm{End}_{\mathcal{A}}(\mathcal{M}))$, and $d(\mathcal{Z}(\mathrm{End}_{\mathcal{A}}(\mathcal{M}))) \subseteq \mathcal{Z}(\mathrm{End}_{\mathcal{A}}(\mathcal{M}))$.

Since $\mathcal{Z}(\mathrm{End}_{\mathcal{A}}(\mathcal{M})) = \{T_a : \ a \in \mathcal{A}\}$ is a commutative C*-algebra, and every derivation on a commutative C*-algebra is zero, we have $d(T_a) = 0$.

It follows that

$$d(aA) = d(T_aA) = d(T_a)A + T_a d(A) = T_a d(A) = ad(A),$$

which means that d is $\mathcal{A}$-linear. The proof is complete. □

Lemma 2.2. *Let $\mathcal{A}$ be a commutative unital C*-algebra and $\mathcal{M}$ be a full Hilbert $\mathcal{A}$-module. Then each derivation on $\mathrm{End}_{\mathcal{A}}(\mathcal{M})$ is continuous.*

Proof. Suppose d is a derivation on $\mathrm{End}_{\mathcal{A}}(\mathcal{M})$. Assume that $\{T_n\}$ is a sequence converging to zero in $\mathrm{End}_{\mathcal{A}}(\mathcal{M})$, and $\{d(T_n)\}$ converges to T.

According to the closed graph theorem, to show d is continuous, it is sufficient to prove that $T = 0$.

By Lemma 2.1, we know d is $\mathcal{A}$-linear. For $x, y \in \mathcal{M}$, $f, g \in \mathcal{M}'$, we have

$$d(\theta_{x,f} T_n \theta_{y,g}) = d(f(T_n y)\theta_{x,g}) = f(T_n y) d(\theta_{x,g}) \to 0,$$

and

$$d(\theta_{x,f} T_n \theta_{y,g}) = d(\theta_{x,f}) T_n \theta_{y,g} + \theta_{x,f} d(T_n) \theta_{y,g} + \theta_{x,f} T_n d(\theta_{y,g}).$$

Since $\{T_n\}$ converges to zero and $\{d(T_n)\}$ converges to T, we have

$$d(\theta_{x,f} T_n \theta_{y,g}) \to \theta_{x,f} T \theta_{y,g} = f(Ty)\theta_{x,g}.$$

It follows that $f(Ty)\theta_{x,g} = 0$.

Let $a = f(Ty)$; then we have $a\theta_{x,g} = \theta_{ax,g} = 0$. For each $z \in \mathcal{M}$, we have

$$\theta_{ax,g} z = g(z)ax = 0. \tag{1}$$

By taking $g = \widehat{ax}$ and $z = ax$ in (1), we can obtain $ax = 0$, i.e.

$$f(Ty)x = 0. \tag{2}$$

By taking $f = \widehat{Ty}$ and $x = Ty$ in (2), we can obtain $Ty = 0$. i.e. $T = 0$. The proof is complete. □

Now we can prove our main theorem in this section.

Theorem 2.3. *Let $\mathcal{A}$ be a commutative C*-algebra with unit e and $\mathcal{M}$ be a full Hilbert $\mathcal{A}$-module. Then each derivation on $\mathrm{End}_{\mathcal{A}}(\mathcal{M})$ is an inner derivation.*

Proof. Suppose d is a derivation on $\mathrm{End}_{\mathcal{A}}(\mathcal{M})$ and $\{x_i\}_{i=1}^n$ is a sequence in $\mathcal{M}$ such that $\sum_{i=1}^n \langle x_i, x_i \rangle = e$.

Define a mapping T from $\mathcal{M}$ into itself by the following:

$$Tx = \sum_{i=1}^n d(\theta_{x,x_i}) x_i,$$

for all $x \in \mathcal{M}$.

By Lemmas 2.1 and 2.2, d is $\mathcal{A}$-linear and continuous; thus T is also $\mathcal{A}$-linear and continuous. That is to say $T \in \mathrm{End}_{\mathcal{A}}(\mathcal{M})$.

Now it is sufficient to show that $d(A) = TA - AT$, for each $A \in \mathrm{End}_{\mathcal{A}}(\mathcal{M})$.

For each $x \in \mathcal{M}$, we have

$$\begin{aligned} TAx &= \sum_{i=1}^{n} d(\theta_{Ax,x_i})x_i \\ &= \sum_{i=1}^{n} d(A\theta_{x,x_i})x_i \\ &= \sum_{i=1}^{n} d(A)\theta_{x,x_i}x_i + \sum_{i=1}^{n} Ad(\theta_{x,x_i})x_i \\ &= d(A)\sum_{i=1}^{n}\langle x_i, x_i\rangle x + A\sum_{i=1}^{n} d(\theta_{x,x_i})x_i \\ &= d(A)x + ATx. \end{aligned}$$

It implies that $d(A) = TA - AT$. Hence d is an inner derivation. The proof is complete. □

3. 2-Local Derivations on $\mathrm{End}_{\mathcal{A}}(\mathcal{M})$ and $\mathrm{End}^*_{\mathcal{A}}(\mathcal{M})$

In this section, we characterize 2-local derivations on $\mathrm{End}_{\mathcal{A}}(\mathcal{M})$ and $\mathrm{End}^*_{\mathcal{A}}(\mathcal{M})$. First, we show the following lemma:

Lemma 3.1. *Let $\mathcal{A}$ be a commutative unital C^*-algebra and $\mathcal{M}$ be a Hilbert $\mathcal{A}$-module. For $x_i \in \mathcal{M}$ and $f_i \in \mathcal{M}'$, if $\sum_{i=1}^n \theta_{x_i,f_i} = 0$, then $\sum_{i=1}^n f_i(x_i) = 0$.*

Proof. Let $a_{i,j} = f_j(x_i) \in \mathcal{A}$ and $\Lambda = (a_{i,j})_{n\times n} \in M_n(\mathcal{A})$. We have

$$\sum_{i=1}^{n} f_i(x_k)x_i = \sum_{i=1}^{n} \theta_{x_i,f_i}x_k = 0.$$

It follows that

$$\sum_{i=1}^{n} a_{k,i}a_{i,j} = \sum_{i=1}^{n} f_i(x_k)f_j(x_i) = f_j(\sum_{i=1}^{n} f_i(x_k)x_i) = 0,$$

which implies that $\Lambda^2 = 0$.

Since $\mathcal{A}$ is a commutative unital C*-algebra, it is well known that $\mathcal{A}$ is $*$-isomorphic to $C(\mathcal{S})$ for some compact Hausdorff space $\mathcal{S}$. Without loss of generality, we can assume $\mathcal{A} = C(\mathcal{S})$.

Then for each $t \in \mathcal{S}$, we have $a_{i,j}(t) \in \mathbb{C}$ and $\Lambda(t), \Lambda^2(t) \in M_n(\mathbb{C})$.

Recall that for a matrix A in $M_n(\mathbb{C})$, $A^2 = 0$ implies that $tr(A) = 0$, where $tr(A)$ denotes the trace of A, i.e. the sum of all the diagonal elements.

Hence $\Lambda^2(t) = 0$ implies that $tr(\Lambda(t)) = 0$. It follows that $tr(\Lambda) = 0$, that is to say $\sum_{i=1}^n f_i(x_i) = 0$. The proof is complete. □

Theorem 3.2. *Let $\mathcal{A}$ be a commutative unital C^*-algebra and $\mathcal{M}$ be a full Hilbert $\mathcal{A}$-module. Then each 2-local derivation on $End_{\mathcal{A}}(\mathcal{M})$ is a derivation.*

Proof. Denote by $\Gamma(\mathcal{M})$ the linear span of the set $\{\theta_{x,f} : x \in \mathcal{M}, f \in \mathcal{M}'\}$. By Lemma 1.2, $\Gamma(\mathcal{M})$ is a two-side ideal of $\mathrm{End}_{\mathcal{A}}(\mathcal{M})$.

For each $S = \sum_{i=1}^{n} \theta_{x_i,f_i} \in \Gamma(\mathcal{M})$, define $\phi(S) = \sum_{i=1}^{n} f_i(x_i)$.

One can verify that ϕ is well defined by Lemma 3.1. And obviously, ϕ is $\mathcal{A}$-linear. Moreover, for each $A \in \mathrm{End}_{\mathcal{A}}(\mathcal{M})$, we have

$$\phi(\theta_{x,f}A) = \phi(\theta_{x,f\circ A}) = f(Ax) = \phi(\theta_{Ax,f}) = \phi(A\theta_{x,f}).$$

It follows that $\phi(SA) = \phi(AS)$ for each $A \in \mathrm{End}_{\mathcal{A}}(\mathcal{M})$ and $S \in \Gamma(\mathcal{M})$.

Suppose δ is a 2-local derivation on $\mathrm{End}_{\mathcal{A}}(\mathcal{M})$. By the definition of 2-local derivation, there exists a derivation d on $\mathrm{End}_{\mathcal{A}}(\mathcal{M})$ such that $\delta(A) = d(A)$ and $\delta(S) = d(S)$. By Theorem 2.3, d is an inner derivation, i.e. there exists an element $T \in \mathrm{End}_{\mathcal{A}}(\mathcal{M})$ such that $d = D_T$.

Thus we have

$$\delta(A)S + A\delta(S) = d(A)S + Ad(S) = d(AS) = D_T(AS) = TAS - AST.$$

Since $\Gamma(\mathcal{M})$ is a two-side ideal of $\mathrm{End}_{\mathcal{A}}(\mathcal{M})$, we know that $AS \in \Gamma(\mathcal{M})$. Hence

$$\phi(\delta(A)S + A\delta(S)) = \phi(TAS - AST) = 0,$$

which follows that $\phi(\delta(A)S) = -\phi(A\delta(S))$.

Now, for each $A, B \in \mathrm{End}_{\mathcal{A}}(\mathcal{M})$ and $S \in \Gamma(\mathcal{M})$, we have

$$\begin{aligned}\phi(\delta(A+B)S) &= -\phi((A+B)\delta(S)) \\ &= -\phi(A\delta(S)) - \phi(B\delta(S)) \\ &= \phi(\delta(A)S) + \phi(\delta(B)S) \\ &= \phi((\delta(A) + \delta(B))S).\end{aligned}$$

Let $C = \delta(A+B) - \delta(A) - \delta(B)$; we obtain $\phi(CS) = 0$.

By taking $S = \theta_{x,f}$, we have

$$\phi(C\theta_{x,f}) = f(Cx) = 0 \Rightarrow \langle Cx, Cx\rangle = 0 \Rightarrow Cx = 0 \Rightarrow C = 0.$$

It means that $\delta(A+B) = \delta(A) + \delta(B)$. That is to say δ is an additive mapping. In addition, by the definition of 2-local derivation, it is easy to show that δ is homogeneous and $\delta(A^2) = A\delta(A) + \delta(A)A$ for each $A \in \mathrm{End}_{\mathcal{A}}(\mathcal{M})$. Hence δ is a Jordan derivation.

By Lemma 1.5, $\mathrm{End}_{\mathcal{A}}(\mathcal{M})$ is a semi-prime Banach algebra. According to the classical result that every Jordan derivation on a semi-prime Banach algebra is a derivation [5], we obtain that δ is a derivation. The proof is complete. □

Theorem 3.3. *Let $\mathcal{A}$ be a commutative unital C^*-algebra and $\mathcal{M}$ be a full Hilbert $\mathcal{A}$-module. Then each 2-local derivation on $\mathrm{End}^*{}_{\mathcal{A}}(\mathcal{M})$ is a derivation.*

Proof. Denote by $\Gamma^*(\mathcal{M})$ the linear span of the set $\{\theta_{x,\hat{y}} : x, y \in \mathcal{M}\}$. By Lemma 1.3, $\Gamma^*(\mathcal{M})$ is a two-side ideal of $\mathrm{End}^*_{\mathcal{A}}(\mathcal{M})$.

For each $S = \sum_{i=1}^{n} \theta_{x_i,\widehat{y_i}} \in \Gamma^*(\mathcal{M})$, define $\phi(S) = \sum_{i=1}^{n} \langle x_i, y_i\rangle$.

By Lemma 3.1, ϕ is well defined. For each $A \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{M})$, we have

$$\phi(\theta_{x,\widehat{y}}A) = \phi(\theta_{x,\widehat{A^*y}}) = \langle x, A^*y\rangle = \langle Ax, y\rangle = \phi(\theta_{Ax,\widehat{y}}) = \phi(A\theta_{x,\widehat{y}}).$$

It follows that $\phi(SA) = \phi(AS)$ for each $A \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{M})$ and $S \in \Gamma^*(\mathcal{M})$.

In [15], the authors prove that for a commutative unital C*-algebra $\mathcal{A}$ and a full Hilbert $\mathcal{A}$-module $\mathcal{M}$, each derivation on $\mathrm{End}^*_{\mathcal{A}}(\mathcal{M})$ is an inner derivation.

The rest of the proof is similar to Theorem 3.2, so we omit it. □

4. Local Derivations on $\mathrm{End}_{\mathcal{A}}(\mathcal{M})$

In this section, we discuss local derivations on $\mathrm{End}_{\mathcal{A}}(\mathcal{M})$. Through this section, we assume that $\mathcal{A}$ is a commutative C*-algebra with unit e, and $\mathcal{M}$ is a Hilbert $\mathcal{A}$-module, and moreover, there exist x_0 in $\mathcal{M}$ and f_0 in $\mathcal{M}^{'}$ such that $f_0(x_0) = e$. Denote the unit of $\mathrm{End}_{\mathcal{A}}(\mathcal{M})$ by I. Define $\mathcal{L} = span\{\theta_{x,f_0} : x \in \mathcal{M}\}$, and $\mathcal{R} = span\{\theta_{x_0,f} : f \in \mathcal{M}^{'}\}$.

Lemma 4.1. (1) *θ_{x_0,f_0} is an idempotent;*

(2) *each element in $\mathcal{L}$ is an $\mathcal{A}$-linear combination of some idempotents in $\mathcal{L}$, and each element in $\mathcal{R}$ is an $\mathcal{A}$-linear combination of some idempotents in $\mathcal{R}$;*

(3) *$\mathcal{L}$ is a left ideal of $\mathrm{End}_{\mathcal{A}}(\mathcal{M})$, and $\mathcal{R}$ is a right ideal of $\mathrm{End}_{\mathcal{A}}(\mathcal{M})$;*

(4) *$\mathcal{L}$ is a left separating set of $\mathrm{End}_{\mathcal{A}}(\mathcal{M})$, i.e. for each A in $\mathrm{End}_{\mathcal{A}}(\mathcal{M})$, $A\mathcal{L} = 0$ implies that $A = 0$, and $\mathcal{R}$ is a right separating set of $\mathrm{End}_{\mathcal{A}}(\mathcal{M})$, i.e. for each A in $\mathrm{End}_{\mathcal{A}}(\mathcal{M})$, $\mathcal{R}A = 0$ implies that $A = 0$*

Proof. (1) $\theta_{x_0,f_0}\theta_{x_0,f_0} = f_0(x_0)\theta_{x_0,f_0} = \theta_{x_0,f_0}$.

(2) For each $x \in \mathcal{M}$, there exists a non-zero complex number $\lambda \in \mathbb{C}$, such that $e - \lambda f_0(x)$ is invertible in $\mathcal{A}$. Denote $e - \lambda f_0(x)$ by a^{-1}; then we have

$$f_0(a(x_0 - \lambda x)) = af_0(x_0 - \lambda x) = a(e - \lambda f_0(x)) = aa^{-1} = e.$$

By (1), we know that $\theta_{a(x_0-\lambda x),f_0}$ is an idempotent.

Thus we have

$$\theta_{x,f_0} = \lambda^{-1}\theta_{x_0,f_0} - \lambda^{-1}a^{-1}\theta_{a(x_0-\lambda x),f_0}.$$

That is to say θ_{x,f_0} is an $\mathcal{A}$-linear combination of idempotents in $\mathcal{L}$.

Similarly, for each $f \in \mathcal{M}^{'}$, there exists a non-zero complex number $\lambda \in \mathbb{C}$, such that $e - \lambda f(x_0)$ is invertible in $\mathcal{A}$. Denote $e - \lambda f(x_0)$ by a^{-1}, then we have

$$(a(f_0 - \lambda f))(x_0) = a(e - \lambda f(x_0)) = aa^{-1} = e.$$

Again by (1), we know that $\theta_{x_0,a(f_0-\lambda f)}$ is an idempotent.

Thus we have

$$\theta_{x_0,f} = \lambda^{-1}\theta_{x_0,f_0} - \lambda^{-1}a^{-1}\theta_{x_0,a(f_0-\lambda f)}.$$

(3) For each $A \in \mathrm{End}_{\mathcal{A}}(\mathcal{M})$, since $A\theta_{x,f_0} = \theta_{Ax,f_0}$, we know that $\mathcal{L}$ is a left ideal of $\mathrm{End}_{\mathcal{A}}(\mathcal{M})$. Similarly, $\mathcal{R}$ is a right ideal of $\mathrm{End}_{\mathcal{A}}(\mathcal{M})$ since $\theta_{x_0,f}A = \theta_{x_0,f\circ A}$.

(4) Suppose $A \in \mathrm{End}_{\mathcal{A}}(\mathcal{M}), \theta_{x,f_0} \in \mathcal{L}, \theta_{x_0,f} \in \mathcal{R}$.

If $A\theta_{x,f_0} = 0$, then

$$0 = A\theta_{x,f_0}x_0 = \theta_{Ax,f_0}x_0 = f_0(x_0)Ax = Ax,$$

i.e. $A = 0$.

If $\theta_{x_0,f}A = 0$, then for each x in $\mathcal{M}$, we have $f(Ax)x_0 = \theta_{x_0,f}Ax = 0$. It follows that $f(Ax) = f(Ax)f_0(x_0) = f_0(f(Ax)x_0) = 0$. Since f is arbitrarily chosen, we can obtain $\langle Ax, Ax\rangle = 0$, which means that $Ax = 0$. Hence $A = 0$. The proof is complete. □

Let $\mathcal{J}$ be a left $\mathcal{A}$-module, and ϕ be a bilinear mapping from $\mathrm{End}_{\mathcal{A}}(\mathcal{M})\times \mathrm{End}_{\mathcal{A}}(\mathcal{M})$ into $\mathcal{J}$.

We say that ϕ is $\mathcal{A}$-*bilinear* if $\phi(aA, B) = \phi(A, aB) = a\phi(A, B)$ for each $A, B \in \mathrm{End}_{\mathcal{A}}(\mathcal{M})$ and $a \in \mathcal{A}$.

We say that ϕ preserves zero product if $AB = 0$ implies that $\phi(A, B) = 0$ for each $A, B \in \mathrm{End}_{\mathcal{A}}(\mathcal{M})$.

Lemma 4.2. *Let $\mathcal{J}$ be a left $\mathcal{A}$-module, and $\phi : End_{\mathcal{A}}(\mathcal{M}) \times End_{\mathcal{A}}(\mathcal{M}) \to \mathcal{J}$ be an $\mathcal{A}$-bilinear mapping preserving zero product. Then for each $A, B \in End_{\mathcal{A}}(\mathcal{M}), L \in \mathcal{L}$, and $R \in \mathcal{R}$, we have*

$$\phi(A, LB) = \phi(AL, B) = \phi(I, ALB) \tag{3}$$

and

$$\phi(AR, B) = \phi(A, RB) = \phi(ARB, I). \tag{4}$$

Proof. Suppose P is an idempotent in $\mathrm{End}_{\mathcal{A}}(\mathcal{M})$. Let $Q = I - P$.

Since ϕ preserves zero product, we have

$$\phi(A, PB) = \phi(AP + AQ, PB) = \phi(AP, PB) = \phi(AP, B - QB) = \phi(AP, B).$$

By Lemma 4.1(2), each element in $\mathcal{L}$ is an $\mathcal{A}$-linear combination of idempotents in $\mathcal{L}$. Considering ϕ is $\mathcal{A}$-bilinear, we obtain that $\phi(A, LB) = \phi(AL, B)$.

By Lemma 4.1(3), $\mathcal{L}$ is a left ideal, so $AL \in \mathcal{L}$. Hence $\phi(AL, B) = \phi(I, ALB)$.

Similarly, we can show the equation (4.2) is true. □

For an algebra $\mathcal{A}$ with unit e, a linear mapping δ on $\mathcal{A}$ is said to be a *generalized derivation* if $\delta(ab) = a\delta(b) + \delta(a)b - a\delta(e)b$, for all a, b in $\mathcal{A}$.

Theorem 4.3. *Suppose that $\mathcal{A}$ is a commutative C^*-algebra with unit e, and $\mathcal{M}$ is a Hilbert $\mathcal{A}$-module, and moreover, there exist x_0 in $\mathcal{M}$ and f_0 in $\mathcal{M}'$ such that $f_0(x_0) = e$. If δ is an $\mathcal{A}$-linear mapping from $End_{\mathcal{A}}(\mathcal{M})$ into itself such that: for each A, B, C in $End_{\mathcal{A}}(\mathcal{M})$, $AB = BC = 0$ implies that $A\delta(B)C = 0$, then δ is a generalized derivation. In particular, if $\delta(I) = 0$, where I is the unit of $End_{\mathcal{A}}(\mathcal{M})$, then δ is a derivation.*

Proof. Suppose A, B, X, Y, A_0, B_0 are arbitrary elements in $\mathrm{End}_{\mathcal{A}}(\mathcal{M})$, where $A_0B_0 = 0$, L and R are arbitrary elements in $\mathcal{L}$ and $\mathcal{R}$, respectively.

Define a bilinear mapping ϕ_1: $\phi_1(X, Y) = X\delta(YA_0)B_0$. Then ϕ_1 is an $\mathcal{A}$-bilinear mapping preserving zero product.

By Lemma 4.2, we have

$$\phi_1(R, A) = \phi_1(RA, I),$$

i.e.

$$R\delta(AA_0)B_0 = RA\delta(A_0)B_0.$$

Since $\mathcal{R}$ is a right separating set of $\mathrm{End}_{\mathcal{A}}(\mathcal{M})$, we have

$$\delta(AA_0)B_0 = A\delta(A_0)B_0.$$

Now define a bilinear mapping ϕ_2: $\phi_2(X,Y) = \delta(AX)Y - A\delta(X)Y$. Then ϕ_2 is also an $\mathcal{A}$-bilinear mapping preserving zero product.

Again by Lemma 4.2, we have

$$\phi_2(B,L) = \phi_2(I,BL),$$

i.e.

$$\delta(AB)L - A\delta(B)L = \delta(A)BL - A\delta(I)BL.$$

Since $\mathcal{L}$ is a left separating set of $\mathrm{End}_{\mathcal{A}}(\mathcal{M})$, we obtain that

$$\delta(AB) = A\delta(B) + \delta(A)B - A\delta(I)B.$$

That is to say δ is a generalized derivation. The proof is complete. □

Applying the above Theorem, we can get the following corollary immediately:

Corollary 4.4. *Suppose $\mathcal{A}$ is a commutative C*-algebra with unit e, $\mathcal{M}$ is a Hilbert $\mathcal{A}$-module, and moreover, there exist x_0 in $\mathcal{M}$ and f_0 in $\mathcal{M}'$ such that $f_0(x_0) = e$. Then each $\mathcal{A}$-linear local derivation δ on $\mathrm{End}_{\mathcal{A}}(\mathcal{M})$ is a derivation.*

Proof. For each A, B, C in $\mathrm{End}_{\mathcal{A}}(\mathcal{M})$, if $AB = BC = 0$, by the definition of local derivation, there exists a derivation δ_B such that $\delta_B(B) = \delta(B)$. Thus we have

$$A\delta(B)C = A\delta_B(B)C = \delta_B(ABC) - \delta_B(A)BC - AB\delta_B(C) = 0.$$

Let I be the unit of $\mathrm{End}_{\mathcal{A}}(\mathcal{M})$, by the definition of local derivation; there exists a derivation δ_I such that $\delta_I(I) = \delta(I) = 0$.

By Theorem 4.3, δ is a derivation. The proof is complete. □

Acknowledgements

This paper was partially supported by National Natural Science Foundation of China (Grant No. 11371136).

References

[1] Ayupov, S., Kudaybergenov, K.: 2-Local derivations on von Neumann algebras. Positivity **19**, 445–455 (2014)

[2] Ayupov, S., Kudaybergenov, K., Peralta, A.: A survey on local and 2-local derivations on C*- and von Neuman algebras. Top. Funct. Anal. Algebra Contemp. Math. **672**, 73–126 (2016)

[3] Christensen, E.: Derivations of nest algebras. Math. Ann. **229**, 155–161 (1977)

[4] Crist, R.: Local derivations on operator algebras. J. Funct. Anal. **135**, 72–92 (1996)

[5] Cusack, J.: Jordan derivations on rings. Proc. Am. Math. Soc. **53**, 321–324 (1975)

[6] Hadwin, D., Li, J.: Local derivations and local automorphisms on some algebras. J. Oper. Theory **60**, 29–44 (2008)

[7] He, J., Li, J., An, G., Huang, W.: Characterizations of 2-local derivations and local Lie derivations on some algebras. Sib. Math. J. **(to appear)** (2017)

[8] Johnson, B.: Local derivations on C*-algebras are derivations. Trans. Am. Math. Soc. **353**, 313–325 (2001)

[9] Kadison, R.: Derivations of operator algebras. Ann. Math. **83**, 280–293 (1966)

[10] Kadison, R.: Local derivations. J. Algebra **130**, 494–509 (1990)

[11] Kim, S., Kim, J.: Local automorphisms and derivations on $M_n(\mathbb{C})$. Proc. Am. Math. Soc. **132**, 1389–1392 (2004)

[12] Lance, E.: Hilbert C*-Modules: A Toolkit for Operator Algebraists. Cambridge University Press, Cambridge (1995)

[13] Larson, D., Sourour, A.: Local derivations and local automorphisms. Proc. Symp. Pure Math. **51**, 187–194 (1990)

[14] Li, J., Pan, Z.: Annihilator-preserving maps, multipliers and local derivations. Linear Algebra Appl. **432**, 5–13 (2010)

[15] Li, P., Han, D., Tang, W.: Derivations on the algebra of operators in Hilbert C*-modules. Acta Math. Sin. (Engl. Ser.) **28**, 1615–1622 (2012)

[16] Moghadam, M., Miri, M., Janfada, A.: A note on derivations on the algebra of operators in Hilbert C*-modules. Mediterr. J. Math. **13**, 1167–1175 (2016)

[17] Sakai, S.: Derivations of W*-algebras. Ann. Math. **83**, 273–279 (1966)

[18] Šemrl, P.: Local automorphisms and derivations on $B(H)$. Proc. Am. Math. Soc. **125**, 2677–2680 (1997)

[19] Zhang, J., Li, H.: 2-Loacl derivations on digraph algebras. Acta Math. Sin. (Chin. Ser.) **49**, 1401–1406 (2006)

Jun He, Jiankui Li and Danjun Zhao
Department of Mathematics
East China University of Science and Technology
Shanghai 200237
China
e-mail: jiankuili@yahoo.com

Received: June 1, 2017.

Revised: October 12, 2017.

Accepted: October 20, 2017.