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Oscillatory Behavior of Second-Order Nonlinear Differential Equations with a Nonpositive Neutral Term

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Abstract. We shall present new oscillation criteria of second-order nonlinear differential equations with a nonpositive neutral term of the form:

$$\left((a(t)\left(\left(x(t)-p(t)x(\sigma(t))'\right)^{\gamma}\right)' + q(t)x^{\beta}(\tau(t)) = 0,$$

with positive coefficients. The obtained results answer an open problem raised in Li et al. [Adv Differ Equ 35:7, 2015, Remark 4.3 (P2)]. Examples are given to illustrate the main results.

Mathematics Subject Classification. 34N05, 39A10.

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1. Introduction

This paper deals with oscillatory behavior of all solutions of the nonlinear second-order differential equations with a nonpositive neutral term of the form:

$$((a(t)((x(t)-p(t)x(\sigma(t))')^{\gamma})' + q(t)x^{\beta}(\tau(t)) = 0. \tag{1.1}$$

We assume that

(i) γ, β are the ratios of positive odd integers, $\gamma \geq \beta$;
(ii) a, p, q : $[t_0, \infty) \to \mathrm{R}^+$ are continuous functions, and $0 < p(t) < p_0 < 1$,
(iii) $\tau, \sigma : [t_0, \infty) \to R$ are continuous functions $\tau(t) \leq t, \sigma(t) \leq t, \tau'(t) > 0, \sigma'(t) > 0$ for $t \geq t_0$, and $\tau(t), \sigma(t) \to \infty$ as $t \to \infty$.
(iv) $h(t) = \sigma^{-1}(\tau(t)) \leq t, h'(t) \geq 0$ and $h(t) \to \infty$ as $t \to \infty$. We let

$$A(v,u) = \int_{\mathrm{u}}^{\mathrm{v}} \frac{1}{a^{1/\gamma}(s)} \mathrm{d}s, \quad v \geq u \geq t_0,$$

and assume that

$$A(t, t_0) \to \infty \text{ as } t \to \infty. \tag{1.2}$$

By a solution of Eq. (1.1), we mean a function x(t) with both quasi-derivatives $a(t)y'(t)$, and $(a(t)y'(t))'$ are continuous on $[T_{\mathrm{x}},\infty), T_{\mathrm{x}} \geq t_0$, which satisfies Eq. (1.1) on $[T_{\mathrm{x}},\infty)$, where $y(t) = x(t) - p(t)x(\sigma(t))$. We consider only those solutions $x(t)$ of (1.1) which satisfy $\sup\{|x(t)| : t \geq T\} > 0$ for all T $\geq T_{\mathrm{x}}\}$.

A solution of (1.1) is said to be oscillatory if it has arbitrarily large zeros, and otherwise, it is called nonoscillatory. Equation (1.1) is said to be oscillatory if all its solutions are oscillatory.

In recent years, there has been much research activity concerning the oscillation and nonoscillation of solutions of various differential equations. Meanwhile, there also have been numerous research for second-order neutral functional differential equations, due to the comprehensive use in natural science and theoretical study.

For the oscillation results of second-order neutral functional differential equations, we refer the reader to [1–6, 8–13] and the references cited therein. A commonly employed condition is as follows:

$$-1 \leq p(t) \leq 0$$

as well as the condition

$$-\infty < -p_0 \leq p(t) \leq 0.$$

In [12], several oscillation results were obtained for Eq. (1.1) in the special case $\gamma = 1$ and under the assumptions:

$$0 \leq p(t) \leq p_0 < 1, \quad \tau(t) = t - \tau_0 \text{ and } \sigma(t) = t - \sigma_0.$$

Further contribution for Eq. (1.1) and its particular cases were made in [5, 8] where authors established sufficient conditions ensuring that every solution $x(t)$ of Eq. (1.1) is either oscillatory or converge to zero as $t \to \infty$.

In [8, Remark 4.3 (P2)], the authors proposed the following open problem:"*Is it possible to suggest a different method to study Eq.* (1.1) *and obtain some sufficient conditions which ensure that all solutions of Eq.* (1.1) *are oscillatory.*"

The main objective of this paper is to give an affirmative answer to this problem. We shall present some new criteria for the oscillation of second-order nonlinear differential equations with a nonpositive neutral term of type (1.1). Examples are inserted to illustrate the main results.

2. Main Results

For $t \geq T$ for some $T \geq t_0$, we let

$$\mu(t) = a^{1/\gamma}(t)A(t,T) \text{ and } Q(t) = \int_{\mathrm{t}}^{\infty} q(s)\mathrm{d}s.$$

We begin with the following new result.

Theorem 2.1. *Let conditions (i)–(iv) and (1.2) hold. If there exists a positive continuously differentiable function $\rho(t)$ and $\rho'(t) \geq 0$, such that*

$$\limsup_{t\to\infty}\left[\rho(t)Q(t)+\int_{t_0}^{\mathrm{t}}\left[\rho(s)q(s)-\frac{\gamma^\gamma}{(1+\gamma)^{\gamma+1}}\right.\right.$$
$$\left.\left.\times\frac{a(\tau(s))}{(\beta\tau'(s)g(s))^\gamma}\left(\frac{(\rho'(s))^{\gamma+1}}{\rho^\gamma(s)}\right)\right]\mathrm{d}s\right]=\infty, \tag{2.1}$$

where

$$g(t)=\begin{cases}1 & \text{when } \beta=\gamma\\ c(A^{(\beta-\gamma)/\gamma}(t)) & \text{when } \beta<\gamma \text{ for some constant } c\ >0,\end{cases} \tag{2.2}$$

$$\limsup_{\mathrm{t}\to\infty}\int_{\mathrm{h(t)}}^{\mathrm{t}}A^\beta(h(t),h(s))q(s)\mathrm{d}s>1 \text{ when } \beta=\gamma \tag{2.3}$$

and

$$\limsup_{t\to\infty}\int_{\mathrm{h(t)}}^{\mathrm{t}}A^\beta(h(t),h(s))q(s)\mathrm{d}s>\ 0 \text{ when } \beta<\gamma, \tag{2.4}$$

then Eq. (1.1) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of Eq. (1.1), say $x(t)>0$, $x(\tau(t))>0, x(\sigma(t))>0$ for $t\geq t_1$ for some $t_1\geq t_0$. It follows from Eq. (1.1) that:

$$\left(a(t)\left(y'(t)\right)^\gamma\right)'\leq -q(t)x^\beta(\tau(t)) \tag{2.5}$$

where $y(t)=x(t)-p(t)x(\sigma(t))$. Hence, $a(t)\left(y'(t)\right)^\gamma$ is nonincreasing and of one sign. That is, there exists a $t_2\geq t_1$, such that $y'(t)>0$ or $y'(t)<0$ for $t\geq \mathrm{t}_2$. We claim that $y'(t)>0$ for $t\geq t_2$. To prove it, we assume that $y'(t)<0$ for $t\geq t_2$. Then

$$a(t)\left(y'(t)\right)^\gamma\leq -c<0 \text{ for } t\geq t_2,$$

where $\mathrm{c}=-a(t_2)\left(y'(t_2)\right)^\gamma>0$. Thus, we conclude that

$$y(t)\leq y(t_2)-c^{1/\gamma}\int_{t_2}^{\mathrm{t}}a^{-1/\gamma}(s)\mathrm{d}s.$$

By virtue of (1.2), $\lim_{t\to\infty}y(t)=-\infty$. □

Now, we consider the following two cases:

Case 1. If $x(t)$ is unbounded, then there exists a sequence $\{t_{\mathrm{k}}\}$, such that $\lim_{k\to\infty}t_{\mathrm{k}}=\infty$ and $\lim_{k\to\infty}x(t_{\mathrm{k}})=\infty$ where $x(t_{\mathrm{k}})=\max\{x(s):t_0\leq s\leq t_{\mathrm{k}}\}$. Since $\lim_{t\to\infty}\sigma(t)=\infty, \sigma(t_{\mathrm{k}})>t_0$ for all sufficiently large k. By $\tau(t)\leq t$, $x(\tau(t_{\mathrm{k}}))=\max\{x(s):t_0\leq s\leq\tau(t_{\mathrm{k}})\}\leq\max\{x(s):t_0\leq s\leq t_{\mathrm{k}}\}=x(t_{\mathrm{k}})$.
Therefore, for all large k,

$$y(t_{\mathrm{k}})=x(t_{\mathrm{k}})-p(t_{\mathrm{k}})x(\tau(t_{\mathrm{k}}))\geq\left(1-p(t_{\mathrm{k}})\right)x(t_{\mathrm{k}})\geq(1-p(t_{\mathrm{k}}))x(t_{\mathrm{k}})>0,$$

which contradicts the fact that $\lim_{t\to\infty}y(t)=-\infty$.

Case 2. If $x(t)$ is bounded, then $y(t)$ is also bounded, which contradicts $\lim_{t\to\infty} y(t) = -\infty$.
This completes the prove of the claim and concludes that $y'(t) > 0$ for $t \geq t_2$.

Next, we have two cases to consider:

$$\text{(I) y (t)} > 0 \text{ or, (II) y (t) } < 0 \text{ for } t \geq t_2.$$

First assume that (I) holds. In view of (2.5) and $x(t) \geq$ y (t), we have

$$\left(a(t)\left(y'(t)\right)^{\gamma}\right)' \leq -q(t)y^{\beta}(\tau(t)) \leq 0. \tag{2.6}$$

It follows that:

$$\begin{aligned} y(t) &= y(t_2) + \int_{t_2}^{\mathrm{t}} \frac{\left(a(s)\left(y'(s)\right)^{\gamma}\right)^{1/\gamma}}{a^{1/\gamma}(s)} \mathrm{d}s \\ &\geq a^{1/\gamma}(t)\left(y'(t)\right) \int_{t_2}^{\mathrm{t}} a^{-1/\gamma}(s)\mathrm{d}s \\ &:= \mu(t)y'(t). \end{aligned} \tag{2.7}$$

Integrating (2.6) from t to u, letting u $\to \infty$ and using the fact that y (t) is increasing, we have

$$\begin{aligned} a(t)\left(y'(t)\right)^{\gamma} &\geq \int_{\mathrm{t}}^{\infty} q(s)y^{\beta}(\tau(s))\mathrm{d}s \geq y^{\beta}(\tau(t))\left(\int_{\mathrm{t}}^{\infty} q(s)\mathrm{d}s\right) \\ &:= Q(t)y^{\beta}(\tau(t)). \end{aligned} \tag{2.8}$$

Suppose that y (t) > 0 for $t \geq t_2$. Define

$$w(t) = \rho(t)\frac{a(t)(y'(t))^{\gamma}}{y^{\beta}(\tau(t))} > 0 \text{ for } t \geq t_2. \tag{2.9}$$

Then, it follows that:

$$w(t) = \rho(t)\frac{a(t)(y'(t))^{\gamma}}{y^{\beta}(\tau(t))} \geq \rho(t)\left(\int_{\mathrm{t}}^{\infty} q(s)\mathrm{d}s\right). \tag{2.10}$$

Now,

$$\begin{aligned} w'(t) &= \left(\frac{\rho(t)}{y^{\beta}(\tau(t))}\right)'(a(t)(y'(t))^{\gamma} \\ &\quad + ((a(t)(y'(t))^{\gamma})'\left(\frac{\rho(t)}{y^{\beta}(\tau(t))}\right) \\ &\leq -\rho(t)q(t) + \left(\frac{\rho'(t)}{\rho(t)}\right)\mathrm{w(t)} \\ &\quad - \beta\rho(\mathrm{t})\frac{a(t)(y'(t))^{\beta}y'(\tau(t))\tau'(t)}{y^{\beta+1}(\tau(t))}. \end{aligned} \tag{2.11}$$

Since $(a(t)(y'(t))^{\gamma}$ is decreasing, we have

$$\frac{y'(\tau(t))}{y'(t)} \geq \left(\frac{a(t)}{a(\tau(t))}\right)^{1/\gamma}. \tag{2.12}$$

Using (2.12) in (2.11), we obtain

$$w'(t) \leq -\rho(t)q(t) + \left(\frac{\rho'(t)}{\rho(t)}\right) \mathrm{w(t)} - \frac{\beta\rho(\mathrm{t})\tau'(t)}{a^{1/\gamma}(\tau(t))} \left(\frac{w(t)}{\rho(\mathrm{t})}\right)^{(\gamma+1)/\gamma} \mathrm{y}^{(\beta-\gamma)/\gamma}(\mathrm{t}).$$

For the case $\beta = \gamma$, we see that $\mathrm{y}^{(\beta-\gamma)/\gamma}(\mathrm{t}) = 1$, while for the case $\beta < \gamma$ and since $a(t)(y'(t))^\gamma$ is decreasing, there exists a constant $\mathrm{c} > 0$, such that

$$a(t)(y'(t))^\gamma \leq c \text{ for } \mathrm{t} \geq t_2.$$

Integrating this inequality from t_2 to t, we have

$$y(t) \leq y(t_2) + A(t, t_2),$$

and thus,

$$\mathrm{y}^{(\beta-\gamma)/\gamma}(t) \geq c^{(\beta-\gamma)/\gamma} A^{(\beta-\gamma)/\gamma}(t, t_2) := c^* A^{(\beta-\gamma)/\gamma}(t, t_2),$$

where $c^* = c^{(\beta-\gamma)/\gamma}$. Using those two cases and the definition of g(t), we get

$$w'(t) \leq -\rho(t)q(t) + \left(\frac{\rho'(t)}{\rho(t)}\right) \mathrm{w(t)} \text{ - } \frac{\beta\tau'(t)}{a^{1/\gamma}(\tau(t))\rho^{1/\gamma}(t)} g(t) w^{(\gamma+1)/\gamma}(t). \quad (2.13)$$

Setting

$$B := \left(\frac{\rho'(t)}{\rho(t)}\right) \text{ and } C := \frac{\beta\tau'(t)}{a^{1/\gamma}(\tau(t))\rho^{1/\gamma}(t)} \ \mathrm{g(t)},$$

and using

$$\text{Bu-Cu}^{(1+\gamma)/\gamma} \leq \frac{\gamma^\gamma}{(1+\gamma)^{\gamma+1}} \left(\frac{B^{\gamma+1}}{C^\gamma}\right),$$

(see [7]), we have

$$w'(t) \leq -\rho(t)q(t) + \frac{\gamma^\gamma}{(1+\gamma)^{\gamma+1}} \frac{a(\sigma(t))}{(\beta\tau'(t)g(t))^\gamma} \left(\frac{(\rho'(t))^{\gamma+1}}{\rho^\gamma(t)}\right).$$

Integrating this inequality from t_2 to t, we get

$$w(t) \leq w(t_2) - \int_{t_2}^{\mathrm{t}} \left[\rho(s)q(s) - \frac{\gamma^\gamma}{(1+\gamma)^{\gamma+1}} \frac{a(\tau(s))}{(\beta\tau'(s)g(s))^\gamma} \times \left(\frac{(\rho'(s))^{\gamma+1}}{\rho^\gamma(s)}\right)\right] \mathrm{d}s.$$

Taking into account (2.8), we find

$$w(t_2) \geq \rho(t)Q(t) + \int_{t_2}^{\mathrm{t}} \left[\rho(\mathrm{s})\mathrm{q(s)} - \frac{\gamma^\gamma}{(1+\gamma)^{\gamma+1}} \frac{a(\tau(s))}{(\beta\tau'(s)g(s))^\gamma} \times \left(\frac{(\rho'(s))^{\gamma+1}}{\rho^\gamma(s)}\right)\right] \mathrm{d}s.$$

Taking the lim sup of both sides in the above inequality as $t \to \infty$, we obtain a contradiction to the condition (2.1)

Consider now case (II). If we put z (t) = - y (t) > 0 for $t \geq t_2$, then Eq. (1.1) gives

$$\left(a(t)\left(z'(t)\right)^{\gamma}\right)' \geq q(t)x^{\beta}(\tau(t)),$$

and

$$z(t) = -y(t) = p(t)x(\sigma(t)) - x(t) \leq p(t)x(\sigma(t)),$$

or,

$$x(\sigma(t)) \geq z(t) \text{ or } z(t) = x(\sigma^{-1}(t)).$$

Using this inequality in (1.1), we have

$$\left(a(t)\left(z'(t)\right)^{\gamma}\right)' \geq q(t)z^{\beta}(\sigma^{-1}(\tau(t))) = q(t)z^{\beta}(h(t)). \tag{2.14}$$

Clearly, we have $z'(t) < 0$. Now, for $t_2 \leq u \leq v$, we may write

$$\begin{aligned} z(u) - z(v) &= -\int_{\mathrm{u}}^{\mathrm{v}} (a^{-1/\gamma}(s)\left(a(s)(z'(s))^{\gamma}\right)^{1/\gamma} \mathrm{d}s \\ &\geq A(v,u)\left(-\left(a(s)(z'(s))^{\gamma}\right)^{1/\gamma}\right); \end{aligned}$$

for $t \geq s \geq t_2$, setting u=h(s) and v=h(t) in the above inequality, we get $z(h(s)) = A(h(t),h(s))\left(-\left(a(h(t))(z'(h(t)))^{\gamma}\right)^{1/\gamma}\right)$.

Integrating inequality (2.14) from $h(t) \geq t_2$ to t, we find

$$\begin{aligned} Z(t) &:= -a(h(t))(z'(h(t)))^{\gamma} \\ &\geq (-a(h(t)(z'(h(t))^{\gamma})^{\beta/\gamma} \int_{h(t)}^{\mathrm{t}} A^{\beta}(h(t),h(s))q(s)\mathrm{d}s \\ &= Z^{\beta/\gamma}(t) \int_{h(t)}^{\mathrm{t}} A^{\beta}(h(t),h(s))q(s)\mathrm{d}s \end{aligned}$$

and hence

$$Z^{1-\beta/\gamma}(t) \geq \int_{h(t)}^{\mathrm{t}} A^{\beta}(h(t),h(s))q(s)\mathrm{d}s.$$

Taking lim sup of both sides of this inequality as $t \to \infty$, we arrive at a contradiction to (2.3) when $\beta = \gamma$ and (2.4) when $\beta < \gamma$. This completes the proof.

Remark 2.1. We note that Theorem 2.1 holds when $Q(t) < \infty$ and the additional term $\rho(t)Q(t)$ in condition (2.1) may improve some of the well-known existing results appeared in the literature.

In the case when $Q(t)$ does not exists as $t \to \infty$, we see that condition (2.1) can be replaced by

$$\begin{aligned} \limsup_{t\to\infty} \int_{t_2}^{\mathrm{t}} \Bigg[\rho(s)q(s) - \frac{\gamma^{\gamma}}{(1+\gamma)^{\gamma+1}} \frac{a(\tau(s))}{(\beta\tau'(s)g(s))^{\gamma}} \\ \left(\frac{(\rho'(s))^{\gamma+1}}{\rho^{\gamma}(s)}\right)\Bigg] \mathrm{d}s = \infty \end{aligned} \tag{2.15}$$

and the conclusion of Theorem 2.1 holds.

For the nonneutral equations, i.e., Eq. (1.1) when $p(t) = 0$ and $q(t)$ is either nonnegative or nonpositive for all large t, Eq. (1.1) is reduced to the equation

$$\left((a(t)\left((x(t))'\right)^{\gamma}\right)' + \delta q(t)x^{\beta}(\tau(t)) = 0 \qquad (1, \delta),$$

where $\delta \pm 1$. From Theorem 2.1, we extract the following immediate results.

Corollary 2.1. *Let conditions (i)–(iii) and* (1.2) *hold. If there exists a positive continuously differentiable function $\rho(t)$ and $\rho'(t) \geq 0$, such that condition* (2.1) *holds, then equation (1,+1) is oscillatory.*

Proof. The proof is contained in the proof of Theorem 2.1-Case (I) and, hence, is omitted. □

We note that Corollary 2.1 is related to some of the results in [1] and the references cited therein. The details are left to the reader.

Corollary 2.2. *Let conditions (i)–(iv) and* (1.2) *hold. If condition* (2.3) *or* (2.4) *holds, then every bounded solution of equation (1,−1) is oscillatory.*

Proof. The proof is contained in the proof of Theorem 2.1-Case (II) and, hence, is omitted. □

The following examples are illustrative.

Example 2.1. Consider the neutral equation

$$\left(x(t) - \frac{1}{2}x\left(t - \frac{\pi}{2}\right)\right)'' + 8x(t - \pi) = 0. \tag{2.16}$$

Here, $\sigma(t) = t - \frac{\pi}{2}$ and $\sigma^{-1}(t) = t + \frac{\pi}{2}, \tau(t) = t - \pi$ and so, $\mathrm{h(t)} = t - \frac{\pi}{2}$. All conditions of Theorem 2.1 with condition (2.1) be replaced by condition (2.15) are satisfied, and hence, Eq. (2.16) is oscillatory. One such solution is $x(t) = \sin 4\mathrm{t}$. We may note that the results in [8] are failed to conclude that all solutions of Eq. (2.16) are oscillatory. In fact, the results in [8] can be applied to Eq. (2.16) to conclude that every solution $x(t)$ of Eq. (2.16) is oscillatory or, $\lim_{t\to\infty} x(t) = 0$.

Example 2.2. Consider the neutral equation

$$\left(\left(x(t) - \frac{1}{2}x(\sqrt{t}\right)^{3}\right)'' + \frac{m}{t^{5/4}}x(t^{1/4}) = 0, \tag{2.17}$$

where m is a positive constant. Here, $\sigma(t) = \sqrt{t}$ and $\sigma^{-1}(t) = t^2, \tau(t) = t^{1/4}$ and so, $\mathrm{h(t)} = \sqrt{t}$. All conditions of Theorem 2.1 are satisfied for suitable m and all large t, and hence, Eq. (2.17) is oscillatory. One can easily see that the results reported in [3–6,9–13] cannot be applied to (1.1) with $p(t) > 0$. Therefore, these results are not applicable to Eq. (2.17).

Next, we present the following interesting results.

Theorem 2.2. *Let the hypotheses of Theorem 2.1 hold with $\rho'(t) \le 0$ for $t \ge t_0$ and condition (2.1) be replaced by*

$$\limsup_{t\to\infty}\left[\rho(t)Q(t) + \int_{t_0}^{t} \rho(s)q(s)\mathrm{d}s\right] = \infty. \tag{2.18}$$

Then, Eq. (1.1) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of Eq. (1.1), say x(t) $>$ $0, x(\tau(t)) > 0, x(\sigma(t)) > 0$ for $t \ge t_1$ for some $t_1 \ge t_0$. Proceeding as in the proof of Theorem 2.1, we conclude that $y'(t) > 0$ for $t \ge t_2$ and we have two cases to consider: (I) y(t) > 0 or y(t) < 0 for $t \ge t_2$.
Case (I). Suppose that y(t) > 0 $t \ge t_2$. As in the proof of Theorem 2.1, we obtain (2.13). Thus

$$w'(t) \le -\rho(t)q(t).$$

Integrating this inequality and using (2.8), we arrived at the desired contradiction.

In the following theorem, we employ different approaches to replace condition (2.1) in Theorem 2.1. □

Theorem 2.3. *Let the hypotheses of Theorem 2.1 hold with $\gamma \le 1$, and condition (2.1) be replaced by*

$$\limsup_{t\to\infty}\left[\rho(t)Q(t) + \int_{t_0}^{t} \left(\rho(s)\, q(s) - \frac{a^{1/\gamma}(\tau(t))\,(\rho'(t))^2}{4\beta\tau'(t)g(t)\rho(t)Q^{(1/\gamma)-1}(t)}\right)\mathrm{d}s\right] = \infty. \tag{2.19}$$

Then, Eq. (1.1) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of Eq. (1.1), say x(t) $>$ $0, x(\tau(t)) > 0, x(\sigma(t)) > 0$ for $t \ge t_1$ for some $t_1 \ge t_0$. Proceeding as in the proof of Theorem 2.1, we conclude that $y'(t) > 0$ for $t \ge t_2$ and y(t) satisfies either (I) or (II) for $t \ge t_2$.

If (I) holds, then as in the proof of Theorem 2.1, we obtain (2.13). Thus

$$\begin{aligned} w'(t) &\le -\rho(t)q(t) + \left(\frac{\rho'(t)}{\rho(t)}\right)\mathrm{w(t)} - \frac{\beta\rho^{-1/\gamma}(\mathrm{t})\tau'(t)}{a^{1/\gamma}(\tau(t))}\,\mathrm{g(t)}\, w^{(\gamma+1)/\gamma}(\mathrm{t}) \\ &\le -\rho(t)q(t) + \left(\frac{\rho'(t)}{\rho(t)}\right)w(t) - \frac{\beta\tau'(t)g(t)}{a^{1/\gamma}(\tau(t))\rho(t)}Q^{(1/\gamma)-1}(t)w^2(t) \\ &:= -\rho(t)q(t) - \left(\sqrt{\frac{\beta\tau'(t)g(t)}{a^{1/\gamma}(\tau(t))\rho(t)}Q^{(1/\gamma)-1}(t)}\,w(t) \right. \\ &\quad \left. - \frac{\frac{\rho'(t)}{\rho(t)}}{2\sqrt{\frac{\beta\tau'(t)g(t)}{a^{1/\gamma}(\tau(t))\rho(t)}Q^{(1/\gamma)-1}(t)}}\right)^2 \end{aligned}$$

$$+\frac{a^{1/\gamma}(\tau(t))\left(\rho'(t)\right)^2}{4\beta\tau'(t)g(t)\rho(t)Q^{(1/\gamma)-1}(t)}$$
$$\leq -\rho(t)q(t)+\frac{a^{1/\gamma}(\tau(t))\left(\rho'(t)\right)^2}{4\beta\tau'(t)g(t)\rho(t)Q^{(1/\gamma)-1}(t)}.$$

The rest of the proof is similar to that of Theorem 2.1 and, hence, is omitted. □

Next, we present the following new and easily verifiable oscillation criteria for Eq. (1.1).

Theorem 2.4. *Let conditions (i)–(iv) and* (1.2) *hold.*

Assume that condition (2.3) *and*

$$\limsup_{t\to\infty} A^{\beta}(\tau(t),t_0)Q(t)>1, \tag{2.20}$$

hold when $\beta=\gamma$ *and condition* (2.4) *and*

$$\limsup_{t\to\infty} A^{\beta}(\tau(t),t_0)Q(t)>0, \tag{2.21}$$

hold when $\beta<\gamma$*, then Eq.* (1.1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of Eq. (1.1), say x(t) > $0, x(\tau(t))>0, x(\sigma(t))>0$ for $t\geq t_1$ for some $t_1\geq t_0$. Proceeding as in the proof of Theorem 2.1, we conclude that $y'(t)>0$ for $t\geq t_2$ and y(t) satisfies either (I) or (II) for $t\geq t_2$.

If (I) holds, then as in the proof of Theorem 2.1, we obtain (2.7) and (2.8).

Using the facts that $\tau(t)\leq t$ and $a(t)\left(y'(t)\right)^{\gamma}$ is decreasing, we find

$$\begin{aligned}
w(t) := a(t)\left(y'(t)\right)^{\gamma} &\geq Q(t)\mu^{\beta}(\tau(t))\left(y'(\tau(t))\right)^{\beta}\\
&= Q(t)\mu^{\beta}(\tau(t))\left(a^{-\beta/\gamma}(\tau(t))\right)\left(a(\tau(t))(y'(\tau(t)))^{\gamma}\right)^{\beta/\gamma}\\
&\geq Q(t)\mu^{\beta}(\tau(t))\left(a^{-\beta/\gamma}(\tau(t))\right)\left(a(t)(y'(t))^{\gamma}\right)^{\beta/\gamma}\\
&= Q(t)\mu^{\beta}(\tau(t))\left(a^{-\beta/\gamma}(\tau(t))\right)w^{\beta/\gamma}(t),
\end{aligned}$$

or,

$$\begin{aligned}
\mathrm{w}^{1-\beta/\gamma}(\mathrm{t}) &\geq Q(t)\mu^{\beta}(\tau(t))\left(a^{-\beta/\gamma}(\tau(t))\right)\\
&= Q(t)\left(\int_{t_2}^{\tau(t)} a^{-1/\gamma}(s)\right)^{\beta}\mathrm{d}s = A^{\beta}(\tau(t),t_2)Q(t).
\end{aligned}$$

Taking lim sup of both sides of this inequality as $t\to\infty$, we arrive at a contradiction to condition (2.20) when $\beta=\gamma$ and condition (2.21) when $\beta<\gamma$. The proof of case (II) is similar to that of Theorem 2.1 and, hence, is omitted. □

For Eq. (1.1) with advanced argument, we present the following result.

Theorem 2.5. *Let $\tau(t) \geq t$, conditions (i)–(iv) and (1.2) hold.*
Assume that the conditions

$$\limsup_{t\to\infty} A(t, t_0)Q^{1/\gamma}(t) > 1 \tag{2.22}$$

$$\limsup_{t\to\infty} \int_{h(t)}^{\mathrm{t}} \left(\frac{1}{a(u)} \int_{\mathrm{u}}^{\mathrm{t}} q(s)\mathrm{d}s\right)^{1/\gamma} \mathrm{d}u > 1, \tag{2.23}$$

hold when $\gamma = \beta$ and the conditions

$$\limsup_{t\to\infty} A(t, t_0)Q^{1/\gamma}(t) = \infty \tag{2.24}$$

$$\limsup_{t\to\infty} \int_{h(t)}^{\mathrm{t}} \left(\frac{1}{a(u)} \int_{\mathrm{u}}^{\mathrm{t}} q(s)\mathrm{d}s\right)^{1/\gamma} \mathrm{d}u > 0, \tag{2.25}$$

hold when $\beta \leq \gamma$, then Eq. (1.1) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of Eq. (1.1), say $\mathrm{x(t)} > 0, x(\tau(t)) > 0, x(\sigma(t)) > 0$ for $t \geq t_1$ for some $\mathrm{t}_1 \geq t_0$. Proceeding as in the proof of Theorem 2.1 and consider the two cases (I) and (II)
First, suppose case (I) holds. From (2.8), we have

$$(y'(t))^{\gamma} \geq \left(\frac{Q(t)}{a(t)}\right) y^{\beta}(\tau(t)),$$

or,

$$y'(t) \geq \left(\frac{Q(t)}{a(t)}\right)^{1/\gamma} y^{\beta/\gamma}(\tau(t)).$$

Using (2.7) in the above inequality, we get

$$\begin{aligned} y(t) \geq \mu(t)y'(t) &\geq \mu(t) \left(\frac{1}{a(t)} \int_{\mathrm{t}}^{\infty} q(s)\mathrm{d}s\right)^{1/\gamma} y^{\beta/\gamma}(\tau(t)) \\ &\geq A(t, t_2)Q^{1/\gamma}(t)y^{\beta/\gamma}(t), \end{aligned}$$

or,

$$y^{1-\beta/\gamma}(t) \geq A(t, t_2)Q^{1/\gamma}(t).$$

Taking lim sup of both sides of this inequality as $t \to \infty$, we arrive at a contradiction to (2.22) when $\beta = \gamma$ and (2.24) when $\beta \leq \gamma$. □

If (II) holds, then as in the proof of Theorem 2.1-Case (II), we obtain (2.14) Integrating this inequality from u to t

$$(a(t)(z'(t))^{\gamma} - (a(u)(z'(u))^{\gamma} \geq \int_{\mathrm{u}}^{\mathrm{t}} q(s)z^{\beta}(h(s))\mathrm{d}s$$

or,

$$\begin{aligned} -z'(u) &\geq \left(\frac{1}{a(u)} \int_{\mathrm{u}}^{\mathrm{t}} q(s)z^{\beta}(h(s))\mathrm{d}s\right)^{1/\gamma} \\ &\geq \left(\frac{1}{a(u)} \int_{\mathrm{u}}^{\mathrm{t}} q(s)\mathrm{d}s\right)^{1/\gamma} z^{\beta/\gamma}(h(t)). \end{aligned}$$

Integrating this inequality from h (t) $\geq t_2$ to t, we arrive at a contradiction to (2.22) when $\beta = \gamma$ or, (2.24) when $\beta \leq \gamma$.

Remark 2.2. We may note that corollaries similar to Corollaries 2.1 and 2.2 can be also drawn from Theorems 2.2–2.5. The details are left to the reader.

Remarks:

1. Our new results of this paper can be extended to higher order equations of the form
$$(a(t)\left(\left(x(t)-p(t)x(\sigma(t))^{(n-1)}\right)^{\gamma}\right)' \\ +q(t)x^{\beta}(\tau(t))=0, \text{ n is a positive integer.} \tag{2.26}$$
The details are left to the reader.
2. It will be of interest to study Eq. (1.1) with $\beta > \gamma$.
3. The work in this paper can be extended to second-order damped equation of the form
$$\left((a(t)\left((x(t)-p(t)x(\sigma(t))'\right)^{\gamma}\right)' \\ +b(t)\left((x(t)-p(t)x(\sigma(t))'\right)^{\gamma}+q(t)x^{\beta}(\tau(t))=0. \tag{2.27}$$
Here, we let $C(t) = \exp\int_{t_0}^{\mathrm{t}} \frac{-b(s)}{a(s)}\mathrm{d}s$. One can easily see that Eq. (2.26) is reduced to
$$\left(\frac{(a(t)}{C(t)}\left(z(t)'\right)^{\gamma}\right)'+q(t)x^{\beta}(\tau(t))=0,$$
where z(t)= $(x(t)-p(t)x(\sigma(t))$. To obtain similar results as these above, we impose the condition
$$A(v,u)=\int_{\mathrm{u}}^{\mathrm{v}}\left(\frac{C(s)}{a(s)}\right)^{-1/\gamma}\mathrm{d}s, \quad v\geq u\geq t_0 \text{ and } A(t,t_0)\to\infty \text{ as t}\to\infty.$$
The details are left to the reader.

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