



Singular Integral Operators in Generalized Morrey Spaces on Curves in the Complex Plane

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Abstract. We study the boundedness of the Cauchy singular integral operators on curves in complex plane in generalized Morrey spaces. We also consider the weighted case with radial weights. We apply these results to the study of Fredholm properties of singular integral operators in weighted generalized Morrey spaces.

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1. Introduction

In this paper, we deal with singular integral operators in generalized Morrey spaces. The well-known classical Morrey spaces were widely investigated during last decades; see for instance books [1, 21], survey paper [22], and references therein. We study the boundedness of a singular integral operator S_Γ in the space $L^{p,\varphi}(\Gamma, \rho)$, where Γ is a composite curve which is a union of a finite number of non-intersecting curves without self-intersection, satisfying arc-chord condition. The boundedness of singular integral operators in classical Morrey spaces on a single curve was studied in [23]. We also refer to the paper [20], where conditions for the weighted boundedness of a general class of multidimensional singular integral operators in generalized Morrey spaces are found. In [17] weighted results for singular integral operators were obtained in classical Morrey spaces, but with more general weights. To prove the boundedness of the singular integral operator S_Γ in the weighted generalized Morrey space $L^{p,\varphi}(\Gamma, \rho)$, first we have to prove the non-weighted boundedness of the maximal operator along such a curve in $L^{p,\varphi}(\Gamma)$. Then we derive the non-weighted boundedness of S_Γ via the Alvarez-Pérez-type point-wise estimate

$$M^\#(|S_\Gamma f|^s)(t) \leq C [Mf(t)]^s, \quad 0 < s < 1. \quad (1.1)$$

For two-weight estimates for the maximal operator in local Morrey spaces we refer to [26]. We apply the obtained results to the study of Fredholm properties of singular integral operators in weighted generalized Morrey spaces.

The theory of the Riemann boundary value problem and singular integral equations on curves in the complex plane, including Fredholm properties, is well known, see the books [5, 15, 16]. In particular, this theory was extensively developed in such spaces as Lebesgue, Orlicz and recently in variable exponent Lebesgue spaces and their weighted versions; see [7, 10]. For the case of composite curves we refer to [7].

We study the Fredholmness of the following singular integral operator:

$$\mathfrak{A}u := a(t)u(t) + b(t)(S_{\Gamma}u)(t), \quad (S_{\Gamma}u)(t) = \frac{1}{\pi i} \int_{\Gamma} \frac{u(\tau)}{\tau - t} d\tau, \quad t \in \Gamma, \quad (1.2)$$

in weighted generalized Morrey space $L^{p,\varphi}(\Gamma, \rho)$, where Γ is a set of non-intersecting oriented closed curves without self-intersection, satisfying archord condition. Γ may be such a single curve or union of such curves.

Fredholmness of such operators in classical weighted Morrey spaces was studied in [24]. The case of generalized Morrey spaces on an interval was studied in [13]. We apply the methods from these papers to extend the results obtained there to the case of generalized weighted Morrey spaces on composite curves.

The paper is organized as follows: in Sect. 2, we provide necessary definitions on generalized Morrey spaces, Zygmund classes of functions and Matuszewska–Orlicz indices. In Sect. 3, we describe some known facts we use. In Sect. 4, we present our new results on the boundedness and Fredholmness of singular integral operators in weighted generalized Morrey spaces on composite curves. First we prove the Fefferman–Stein inequality $\|Mf\|_{L^{p,\varphi}(X)} \leq C\|M^{\#}f\|_{L^{p,\varphi}(X)}$ for a metric space X to derive the non-weighted boundedness of S_{Γ} via (1.1). Then we prove the boundedness of S_{Γ} and Fredholmness of \mathfrak{A} in the weighted case.

2. Definitions

2.1. Generalized Morrey Spaces on Homogeneous Underlying Spaces

Let (X, d, μ) be a homogeneous metric measure space with quasi-distance d and measure μ . We restrict ourselves to the case where X has constant dimension: there exists a number $N > 0$ (not necessarily integer) such that

$$C_1 r^N \leq \mu B(x, r) \leq C_2 r^N, \quad (2.1)$$

where the constants C_1 and C_2 do not depend on $x \in X$ and $r > 0$. In this case, the generalized Morrey space $L^{p,\varphi}(X)$ may be defined by the norm:

$$\|f\|_{p,\varphi} = \sup_{x \in X, r > 0} \left\{ \frac{1}{\varphi(r)} \int_{B(x,r)} |f(y)|^p d\mu(y) \right\}^{\frac{1}{p}}, \quad (2.2)$$

where $1 \leq p < \infty$ and $0 \leq M(\varphi) < N$ and the standard notation $B(x, r) = \{y \in X : d(x, y) < r\}$ is used.

Everywhere in the sequel it is assumed that $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a measurable function satisfying the following assumptions:

1. $\varphi(r)$ is continuous in a neighborhood of the origin;
2. $\varphi(0) = 0$;
3. $\inf_{r>\delta} \varphi(r) > 0$ for every $\delta > 0$

and

$$\varphi(r) \geq cr^n \tag{2.3}$$

for $0 < r \leq l$, if $l < \infty$, and $0 < r \leq N$ with an arbitrary $N > 0$, if $l = \infty$, the constant c depends on N in the latter case. Condition (2.3) makes the space $L^{p,\varphi}(X)$ non-trivial (see [18, Corollary 3.4]).

2.2. Curves Satisfying Arc-Chord Condition

Let Γ be a bounded curve in the complex plane \mathbb{C} . We denote $\tau = t(\sigma)$, $t = t(s)$, where σ and s stand for the arc abscissas of the points τ and t , and $d\mu(\tau) = d\sigma$ will stand for the arc-measure on Γ . We also use the notation

$$\Gamma(t, r) = \{\tau \in \Gamma : |\tau - t| < r\} \quad \text{and} \quad \Gamma_*(t, r) = \{\tau \in \Gamma : |\sigma - s| < r\},$$

so that $\Gamma_*(t, r) \subseteq \Gamma(t, r)$, and denote $\ell = \mu\Gamma = \text{length of } \Gamma$.

Definition 2.1. A curve Γ is said to satisfy *the arc-chord condition at a point* $t_0 = t(s_0) \in \Gamma$, if there exists a constant $k > 0$, not depending on t , such that

$$|s - s_0| \leq k|t - t_0|, \quad t = t(s) \in \Gamma. \tag{2.4}$$

Finally, a curve Γ is said to satisfy *the (uniform) arc-chord condition*, if

$$|s - \sigma| \leq k|t - \tau|, \quad t = t(s), \tau = t(\sigma) \in \Gamma. \tag{2.5}$$

In the sequel Γ is always assumed to be a curve satisfying the arc-chord condition.

The generalized Morrey space $L^{p,\varphi}(\Gamma)$ is defined by the norm

$$\|f\|_{p,\varphi} = \sup_{t \in \Gamma, r > 0} \left\{ \frac{1}{\varphi(r)} \int_{\Gamma(t,r)} |f(\tau)|^p d\mu(\tau) \right\}^{\frac{1}{p}}. \tag{2.6}$$

For a non-negative weight function $\varrho(t)$ the weighted generalized Morrey space is introduced as

$$L^{p,\varphi}(\Gamma, \varrho) = \{f : \varrho f \in L^{p,\varphi}(\Gamma)\} \tag{2.7}$$

with

$$\|f\|_{p,\varphi;\varrho} := \|f\|_{L^{p,\varphi}(\Gamma,\varrho)} = \|\varrho f\|_{L^{p,\varphi}(\Gamma)}. \tag{2.8}$$

2.3. On Admissible Weight Functions

In the sequel, when studying the singular operator S_Γ along a curve Γ in weighted generalized Morrey space, we deal with radial type weights of the form

$$\varrho(t) = w(|t - t_0|), \quad t_0 \in \Gamma. \tag{2.9}$$

We introduce below the class of weight functions $w_k(x)$, $x \in [0, \ell]$, admitted for our goals. Although the functions w_k should be defined only on $[0, d]$, where $d = \text{diam}\Gamma = \sup_{t, \tau \in \Gamma} |t - \tau| < \ell$, everywhere below we consider them as defined on $[0, \ell]$.

Definition 2.2. By $W = W(\mathbb{R}_+)$ we denote the class of functions φ which are continuous and positive on \mathbb{R}_+ and such that there exists the finite limit $\lim_{x \rightarrow 0} \varphi(x)$. By $\overline{W} = \overline{W}(\mathbb{R}_+)$ we denote the class of functions $\varphi \in W$ such that $x^a \varphi(x)$ is almost increasing on \mathbb{R}_+ for some $a = a(\varphi) \in \mathbb{R}$. By $\underline{W} = \underline{W}(\mathbb{R}_+)$ we denote the class of functions $\varphi \in W$ such that there exists a number $b \in \mathbb{R}$ such that $\frac{\varphi(x)}{x^b}$ is almost decreasing.

Definition 2.3. Let $x, y \in (0, l]$, $x_+ = \max(x, y)$ and $x_- = \min(x, y)$. By V_\pm we denote the classes of functions $w \in W$ defined by the following conditions:

$$V_+ : \left| \frac{w(x) - w(y)}{x - y} \right| \leq C \frac{w(x_+)}{x_+}, \tag{2.10}$$

$$V_- : \left| \frac{w(x) - w(y)}{x - y} \right| \leq C \frac{w(x_-)}{x_+}. \tag{2.11}$$

Lemma 2.4. Functions $w \in V_+$ are almost increasing on $[0, l]$ and functions $w \in V_-$ are almost decreasing on $[0, l]$.

For the proof of this lemma we refer to [18].

Definition 2.5. We say that a function $\varphi \in \overline{W}$ belongs to the Zygmund class \mathbb{Z}^β , $\beta \in \mathbb{R}^1$, if

$$\int_0^r \frac{\varphi(t)}{t^{1+\beta}} dt \leq c \frac{\varphi(r)}{r^\beta}, \quad r \in (0, \infty), \tag{2.12}$$

and to the Zygmund class \mathbb{Z}_γ , $\gamma \in \mathbb{R}^1$, if

$$\int_r^l \frac{\varphi(t)}{t^{1+\gamma}} dt \leq c \frac{\varphi(r)}{r^\gamma}, \quad r \in (0, \infty). \tag{2.13}$$

It is known that the property of a function to be almost increasing or almost decreasing after the multiplication (division) by a power function is closely related to the notion of the so-called Matuszewska–Orlicz indices. We refer for instance to [9, 19] for the properties of the indices of such a type.

For a function $\varphi \in \mathbb{W}(\mathbb{R}_+) := \underline{W} \cap \overline{W}$ such indices at the origin are defined as follows:

$$m(\varphi) = \lim_{r \rightarrow 0} \frac{\ln \left(\limsup_{h \rightarrow 0} \frac{\varphi(rh)}{\varphi(h)} \right)}{\ln r}; \quad M(\varphi) = \lim_{r \rightarrow \infty} \frac{\ln \left(\limsup_{h \rightarrow 0} \frac{\varphi(rh)}{\varphi(h)} \right)}{\ln r}. \tag{2.14}$$

The indices $m(\varphi)$ are finite numbers when $\varphi \in \overline{W}(\mathbb{R}_+)$ and $M(\varphi)$ are finite numbers when $\varphi \in \underline{W}(\mathbb{R}_+)$. Besides this,

$$m(\varphi) = \sup \left\{ a : \frac{\varphi(t)}{t^a} \text{ is almost increasing on } (0, 1] \right\}, \tag{2.15}$$

and

$$M(\varphi) = \inf \left\{ a : \frac{\varphi(t)}{t^a} \text{ is almost decreasing on } (0, 1] \right\}. \tag{2.16}$$

We will also use the following known properties:

$$\varphi \in \mathbb{Z}^\beta \iff m(\varphi) > \beta \tag{2.17}$$

and

$$\varphi \in \mathbb{Z}_\gamma \iff M(\varphi) < \gamma. \tag{2.18}$$

For other properties of the indices of functions $\varphi \in \mathbb{W}(\mathbb{R}_+)$, we refer for instance to the paper [25, Section 6] and references therein.

3. Preliminaries

3.1. Maximal Operator

To prove the boundedness of the singular integral operator in generalized Morrey space, we need to consider the maximal operator in this space. The boundedness of the maximal operator

$$Mf(x) = \sup_{r>0} \int_{B(x,r)} |f(y)| \, d\mu(y)$$

in the space $L^{p(\cdot),\varphi(\cdot)}(X)$ is known, see [8], in the setting of quasi-metric measure spaces. In particular, we can use this result for curves with arc-chord condition.

In the proof of Lemma 4.2 for maximal operator, we will use the following lemma.

Lemma 3.1. [2, Lemma 3.2] *Let $g(r) : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a non-negative almost decreasing function. Then*

$$\sum_{k=0}^\infty g(2^{k+1}r) \leq C \int_r^\infty \frac{g(t)}{t} \, dt, \quad r > 0. \tag{3.1}$$

3.2. Point-Wise Estimate for the Weighted Singular Integral Operator on a Curve Which Satisfies the Arc-Chord Condition

Let Γ be a curve satisfying the arc-chord condition. We consider radial type weights of the form $\varrho(t) = w(|t - t_0|)$, $t_0 \in \Gamma$.

We define the operator K in the following way:

$$Kf(t) := \left(\varrho S \frac{1}{\varrho} - S \right) f(t) = \int_\Gamma K(t, \tau) f(\tau) \, d\mu(\tau),$$

where

$$K(t, \tau) := \frac{\varrho(t) - \varrho(\tau)}{\varrho(\tau)(\tau - t)} = \frac{w(|t - t_0|) - w(|\tau - t_0|)}{w(|\tau - t_0|)(\tau - t)}.$$

In [23, Theorem 6.1] the following point-wise estimate was proved:

$$|Kf(t)| \leq C \frac{w(s)}{s} \int_0^s \frac{|f_*(\sigma)| d\sigma}{w(\sigma)} + C \int_s^l \frac{|f_*(\sigma)|}{\sigma} d\sigma, \quad t = t(s), \tag{3.2}$$

when $w \in V_+$, and

$$|Kf(t)| \leq \frac{C}{s} \int_0^s |f_*(\sigma)| d\sigma + Cw(s) \int_s^l \frac{|f_*(\sigma)|}{\sigma w(\sigma)} d\sigma, \quad t = t(s), \tag{3.3}$$

when $w \in V_-$, where $f_*(\sigma) = f[t(s)]$.

It is easy to see that in the right-hand sides of (3.2)–(3.3) we have Hardy-type operators. Thus, according to the point-wise estimate (3.2)–(3.3) the boundedness of the weighted singular integral operator is reduced to the boundedness of the corresponding Hardy-type operators.

3.3. Hardy-Type Operators

We study the following Hardy-type operators:

$$H_w f(x) = \frac{w(x)}{x} \int_0^x \frac{f(\tau) d\tau}{w(\tau)}, \quad \mathcal{H}_w f(x) = w(x) \int_x^l \frac{f(\tau) d\tau}{\tau w(\tau)}. \tag{3.4}$$

For Hardy-type operators in various function spaces, we refer for instance to [3, 14], and the recent book [12], see also references therein. The boundedness of the operators (3.4) in generalized Morrey spaces was proved in [13]. We present here this result in a slightly modified form so as we need it here.

Theorem 3.2. [13, Theorem 3.4] *Let $1 < p < \infty$. Let the function $\varphi \in \mathbb{W}(\mathbb{R}_+)$, defining the generalized Morrey space, satisfy the conditions*

$$0 < m(\varphi) \leq M(\varphi) \leq 1, \quad \frac{\varphi(r)}{r} \text{ is almost decreasing.}$$

Then the Hardy-type operators H_w and \mathcal{H}_w with the weight $w \in \mathbb{W}(\mathbb{R}_+)$ are bounded in the generalized Morrey space $L^{p,\varphi}(\mathbb{R})$ if the conditions

$$m\left(\frac{\varphi}{w^p}\right) > 1 - p, \quad \text{and} \quad M\left(\frac{\varphi}{w^p}\right) < 1, \tag{3.5}$$

respectively, are fulfilled.

4. Main Results

In this section, we study singular integral operators in generalized Morrey spaces on composite curves, which satisfy the arc-chord condition. The boundedness of singular integral operators on the single such a curve in classical Morrey spaces was proved in [23].

4.1. Singular Integral Operator on Composite Curves

By a composite curve we mean a union $\Gamma = \bigcup_{k=1}^m \Gamma_k$ of a finite number of non-intersecting curves without self-intersection, satisfying arc-chord condition. Then the singular integral operator $S_\Gamma f(t)$ on such curves can be defined in the following way:

It is convenient to treat the function $f(t)$, $t \in \Gamma$, defined on Γ , as follows: Denote $f_k(t) = f(t)|_{t \in \Gamma_k}$. Then we treat $f(t)$ as

$$f(t) = (f_1(t), f_2(t), \dots, f_m(t)).$$

Then

$$S_\Gamma f(t) = ((Sf)_1, (Sf)_2, \dots, (Sf)_m), \tag{4.1}$$

where

$$(Sf)_k = \frac{1}{\pi i} \sum_{j=1}^m \int_{\Gamma_j} \frac{f_j(\tau)}{\tau - t} d\tau = \frac{1}{\pi i} \int_{\Gamma_k} \frac{f_k(\tau)}{\tau - t} d\tau + \frac{1}{\pi i} \sum_{j=1, j \neq k}^m S_{\Gamma_j} f_j(t), \quad t \in \Gamma_k.$$

We define, for $k = 1, 2, \dots, m$:

$$S_{\Gamma_k} f_k(t) := \frac{1}{\pi i} \int_{\Gamma_k} \frac{f_k(\tau)}{\tau - t} d\tau, \quad T_k f(t) := \frac{1}{\pi i} \sum_{j=1, j \neq k}^m S_{\Gamma_j} f_j(t), \quad t \in \Gamma_k. \tag{4.2}$$

Hence,

$$S_\Gamma f(t)|_{t \in \Gamma_k} = S_{\Gamma_k} f_k(t) + T_k f(t), \quad t \in \Gamma_k, \tag{4.3}$$

where T_k is an operator with bounded kernel.

We define the norm in the Morrey space on composite curves in a natural way, namely as

$$\|f\|_{L^{p,\varphi}(\Gamma)} := \sum_{k=1}^m \|f_k\|_{L^{p,\varphi_k}(\Gamma_k)},$$

where $\varphi(r) = (\varphi_1(r), \varphi_2(r), \dots, \varphi_m(r))$. From (4.3) there follows the obvious statement: if the singular integral operator is bounded in the generalized Morrey spaces on every separate curve, then it is also bounded in this space on the composite curve.

4.2. Boundedness of the Singular Integral Operator on Composite Curves: Non-weighted Case

The main result in this case reads.

Theorem 4.1. *Let $1 < p < \infty$, $\varphi_k(r) \geq cr$ and $0 \leq M(\varphi_k) < 1$, $k = 1, 2, \dots, m$. Then the singular integral operator S_Γ is bounded in the space $L^{p,\varphi}(\Gamma)$, where $\varphi(r) = (\varphi_1(r), \varphi_2(r), \dots, \varphi_m(r))$.*

To prove this theorem we need the following new lemma, which is a generalization of the Lemma 5.3, proved in [23] for classical Morrey spaces.

Let $M^\#$ be defined as follows:

$$M^\# f(x) := \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y) - f_{B(x,r)}| \, d\mu(y),$$

$$f_{B(x,r)} = \int_{B(x,r)} f(y) \, d\mu(y).$$

Lemma 4.2. *Let X be a metric measure space with $\mu(X) = \infty$. Under the condition (2.1), i.e., $C_1 r^N \leq \mu B(x,r) \leq C_2 r^N$, the following estimate holds:*

$$\|Mf\|_{p,\varphi} \leq C \|M^\# f\|_{p,\varphi}, \quad 1 < p < \infty, \quad 0 \leq M(\varphi) < N.$$

Proof. To prove this statement, we use the following point-wise estimate from [23, Lemma 5.3]:

$$\int_{B(x,r)} |Mf(y)|^p \, d\mu(y) \leq \int_{B(x,r)} |M^\# f(y)|^p \, d\mu(y) + \sum_{j=0}^\infty \frac{C}{(2^{j+1} + 1)^{N\varepsilon}} \int_{B(x,2^{j+1}r)} |M^\# f(y)|^p \, d\mu(y).$$

Note that,

$$\begin{aligned} & \sum_{j=0}^\infty \frac{C}{(2^{j+1} + 1)^{N\varepsilon}} \int_{B(x,2^{j+1}r)} |M^\# f(y)|^p \, d\mu(y) \\ &= \sum_{j=0}^\infty \frac{C\varphi(2^{j+1}r)}{(2^{j+1} + 1)^{N\varepsilon}} \cdot \frac{1}{\varphi(2^{j+1}r)} \int_{B(x,2^{j+1}r)} |M^\# f(y)|^p \, d\mu(y) \\ &\leq \|M^\# f(y)\|_{p,\varphi}^p \sum_{j=0}^\infty \frac{C\varphi(2^{j+1}r)}{(2^{j+1} + 1)^{N\varepsilon}} \leq \|M^\# f(y)\|_{p,\varphi}^p \cdot r^{N\varepsilon} \sum_{j=0}^\infty \frac{C\varphi(2^{j+1}r)}{(2^{j+1}r)^{N\varepsilon}} \\ &\text{(by Lemma 3.1)} \leq C \|M^\# f(y)\|_{p,\varphi}^p \cdot r^{N\varepsilon} \int_r^\infty \frac{\varphi(t)}{t^{N\varepsilon+1}} \, dt \leq C \|M^\# f(y)\|_{p,\varphi}^p \cdot \varphi(r), \end{aligned}$$

for $\varepsilon \in \left(\frac{M(\varphi)}{N}, 1\right)$.

Then

$$\begin{aligned} \frac{1}{\varphi(r)} \int_{B(x,r)} |Mf(y)|^p d\mu(y) &\leq \frac{1}{\varphi(r)} \int_{B(x,r)} |M^\#f(y)|^p d\mu(y) + C \|M^\#f(y)\|_{p,\varphi}^p \\ &\leq C \|M^\#f(y)\|_{p,\varphi}^p. \end{aligned}$$

Therefore

$$\|Mf(y)\|_{p,\varphi} \leq C \|M^\#f(y)\|_{p,\varphi}.$$

□

Proof of Theorem 4.1. Due to (4.3), i.e., $S_\Gamma f(t)|_{t \in \Gamma_k} = S_{\Gamma_k} f_k(t) + T_k f(t)$, $t \in \Gamma_k$, we need to prove the boundedness of S_{Γ_k} and T_k .

Using the property $\|f\|_{p,\varphi_k} = \|f^s\|_{\frac{p}{s},\varphi_k}$, $0 < s < 1$, of the norm we have

$$\|S_{\Gamma_k} f\|_{p,\varphi_k} = \|(S_{\Gamma_k} f)^s\|_{\frac{p}{s},\varphi_k} \leq \|M[(S_{\Gamma_k} f)^s]\|.$$

To prove the boundedness of S_{Γ_k} we apply Lemma 4.2 and the inequality (1.1), and obtain

$$\|S_{\Gamma_k} f\|_{p,\varphi_k} \leq C \|M^\# [S_{\Gamma_k} f]^s\|_{\frac{p}{s},\varphi_k} \leq C \|(Mf)^s\|_{\frac{p}{s},\varphi_k} = C \|Mf\|_{p,\varphi_k}. \tag{4.4}$$

To this end, we need the boundedness of the maximal operator in $L^{p,\varphi_k}(\Gamma_k)$. According to [8] such a boundedness holds under the condition:

$$\sup_{t>r} \frac{\inf_{t<s} \varphi(s)}{|\tau - t|^{\frac{1}{p}}} \leq C \frac{\varphi(r)}{|\tau - r|^{\frac{1}{p}}}, \tag{4.5}$$

where C does not depend on τ and r . It is easy to check that the conditions of our theorem imply the validity of the condition (4.5). Thus, the boundedness of S_{Γ_k} is proved under the conditions of our theorem.

Boundedness of T_k is evident, since T_k is the operator with bounded kernel. Indeed, $\frac{1}{\tau - t_k} =: K_j(\tau, t_k)$ is bounded, since $\tau \in \Gamma_j$ and $t_k \in \Gamma_k$, $k \neq j$. Therefore,

$$\begin{aligned} |T_k f| &= \left| \sum_{j=1, j \neq k}^m (S_{\Gamma_j} f_j)(t_k) \right| = \left| \sum_{j=1, j \neq k}^m \int_{\Gamma_j} \frac{f_j(\tau) d\tau}{\tau - t_k} \right| \\ &= \left| \sum_{j=1, j \neq k}^m \int_{\Gamma_j} K_j(\tau, t_k) f_j(\tau) d\tau \right| \leq \sum_{j=1, j \neq k}^m \int_{\Gamma_j} |K_j(\tau, t_k)| |f_j(\tau)| d\tau \\ &\leq \sum_{j=1, j \neq k}^m C_j \|f_j(\tau)\|_{L^1(\Gamma_j)} \leq C \sum_{j=1, j \neq k}^m \|f_j(\tau)\|_{L^p(\Gamma_j)} \\ &\leq C_2 \sum_{j=1, j \neq k}^m \|f_j(\tau)\|_{L^{p,\varphi_j}(\Gamma_j)} \leq C_2 \|f\|_{L^{p,\varphi}(\Gamma)}. \end{aligned}$$

Hence, in particular

$$\|T_k f\|_{L^{p,\varphi}(\Gamma)} \leq C \|f\|_{L^{p,\varphi}(\Gamma)}, \quad \text{since } 1 \in L^{p,\varphi}(\Gamma).$$

The proof is complete. □

4.3. Weighted Case

For simplicity we fix the weight at a single point t_0^k on each curve Γ_k . Therefore,

$$\varrho = (\varrho_1, \dots, \varrho_m), \quad \text{where } \varrho_k = w_k (|t - t_0^k|), \tag{4.6}$$

$$w_k \in \mathbb{W} \cap (V_+ \cup V_-), \quad t_0^k \in \Gamma_k, \quad k = 1, 2, \dots, m.$$

Theorem 4.3. *Let $\Gamma = \bigcup_{k=1}^m \Gamma_k$, where Γ_k is a curve satisfying the arc-chord condition. Let $1 < p < \infty$, $\varphi_k(r) \geq cr$,*

$$0 < m(\varphi_k) \leq M(\varphi_k) < 1, \quad k = 1, 2, \dots, m, \tag{4.7}$$

and

$$\frac{\varphi_k(r)}{r} \quad \text{be an almost decreasing function.} \tag{4.8}$$

Then the operator S_Γ is bounded in the generalized Morrey space $L^{p,\varphi}(\Gamma, \varrho)$, if

$$m \left(\frac{\varphi_k}{w_k^p} \right) > 1 - p, \quad M \left(\frac{\varphi_k}{w_k^p} \right) < 1, \tag{4.9}$$

where $\varphi = (\varphi_1, \varphi_2, \dots, \varphi_m)$ and $\varrho = (\varrho_1, \varrho_2, \dots, \varrho_m)$.

Proof. According to the representation (4.3) to prove the boundedness of the weighted singular integral operator on composite curves we have to prove the weighted boundedness of S_{Γ_k} and T_k . To prove the weighted boundedness of S_{Γ_k} in generalized Morrey space we use the point-wise estimates (3.2)–(3.3). According to these estimates, we need the boundedness of the non-weighted S_{Γ_k} and weighted boundedness of the corresponding Hardy-type operators.

The non-weighted boundedness of $S_{\Gamma_k} f_k$ is proved under the right-hand side condition in (4.7) in Theorem 4.1. The boundedness of the Hardy-type operators H_w and \mathcal{H}_w was proved in Theorem 3.2 under Conditions (4.7), (4.8) and (4.9).

The boundedness of $T_k f$ in the weighted space $L^{p,\varphi}(\Gamma, \varrho)$ can be proved by using the same arguments as in the non-weighted case, proved in Theorem 4.1. The right-hand side condition in (4.9) is sufficient for such boundedness. The proof is complete. □

4.4. On Some Basics Related to Fredholmness

In the sequel $\Gamma = \bigcup_{k=1}^m \Gamma_k$ is regarded as a union of a finite number of non-intersecting closed oriented curves Γ_k without self-intersection, satisfying arc-chord condition. The contour Γ_m has counterclockwise orientation and the contours Γ_k , $k \in \overline{1, m-1} = \{1, 2, \dots, m-1\}$ have clockwise orientation. Thus, the contours Γ_k , $k \in \overline{1, m-1}$, are inside the contour Γ_m . This orientation divides the complex plane into two areas D^+ and D^- , where D^+

is to the left of Γ , and $D^- = \mathbb{C} \setminus \overline{D^+} = \bigcup_{k=1}^m D_k^-$, where D_k^- , $k \in \overline{1, m}$, are to the right of the contours Γ_k , $k \in \overline{1, m}$. For simplicity we assume that the origin is in D^+ .

Let $X = X(\Gamma)$ be any Banach space of functions on Γ . We recall that a linear operator \mathfrak{A} in a Banach space X is called Fredholm if its kernel $\ker \mathfrak{A} := \{u \in X : \mathfrak{A}u = 0\}$ has a finite dimension $\alpha := \dim(\ker \mathfrak{A}) < \infty$, the range $R(\mathfrak{A}) := \{f \in X : f = \mathfrak{A}u, u \in X\}$ is closed in X and has a finite dimension $\beta := \dim(R(\mathfrak{A})) < \infty$. Following [6], the ordered pair (α, β) will be referred to as the d -characteristic of the operator \mathfrak{A} . The difference $Ind_X \mathfrak{A} := \alpha - \beta$ is called the index of \mathfrak{A} .

In this section we consider the operator

$$\mathfrak{A}u := a(t)u(t) + b(t)(S_\Gamma u)(t), \quad t \in \Gamma,$$

where $a, b \in C(\Gamma)$.

It is well known that the following conditions:

- (1) the singular integral operator S_Γ in $L^{p,\varphi}(\Gamma, \varrho)$ is bounded;
- (2) the commutator $gS_\Gamma - S_\Gamma g$ in $L^{p,\varphi}(\Gamma, \varrho)$, where $g \in C(\Gamma)$ is compact;
- (3) $S_\Gamma^2 = I$ in this setting

guarantee the Fredholmness of the singular integral operator $aI + bS_\Gamma$ with continuous coefficients a and b , under the condition $a(t) \neq \pm b(t)$, $t \in \Gamma$. See for instance [11, Theorem A], where Fredholmness of singular integral operators in the setting of an abstract Banach space of functions on curves was proved.

Thus, it remains to check the above conditions (1)–(3). The condition (1) was proved in Theorem 4.3. Condition (3) on composite curves is known, see [7]. As regards condition (2) in weighted generalized Morrey space it is covered by the following theorem.

Theorem 4.4. *Let $g \in C(\Gamma)$ and ϱ be a weight (4.6). Under the assumptions $1 < p < \infty$, $m(\varphi) < 1$ and condition (4.9), the commutator $S_\Gamma g - gS_\Gamma$ is compact in the weighted generalized Morrey space $L^{p,\varphi}(\Gamma, \varrho)$.*

Proof. From (4.3) it is easy to see that to prove compactness of commutator $S_\Gamma g - gS_\Gamma$, we have to prove the compactness of the commutators $S_{\Gamma_k} g - gS_{\Gamma_k}$ and $T_k g - gT_k$, $k = 1, 2, \dots, m$. We start with the commutator $S_{\Gamma_k} g - gS_{\Gamma_k}$. From the famous Mergelyan’s result, see for instance [4], p. 169, it is known that the continuous functions may be uniformly approximated by rational functions for an arbitrary Jordan curve Γ . Since the compactness of such commutator for rational functions is known, and the boundedness of our singular integral operators is proved in Theorem 4.3, the statement of this theorem for this commutator is proved.

Now we consider the commutator $T_k g - gT_k$. Since the function g is continuous and a product of a compact operator with a continuous function is compact, we need to prove the compactness of the operator

$$T_k f(t) := \frac{1}{\pi i} \sum_{j=1, j \neq k}^m (S_{\Gamma_j} f_j)(t) = \frac{1}{\pi i} \sum_{j=1, j \neq k}^m (T_{kj} f_j)(t), \quad t \in \Gamma_k,$$

defined in (4.2), where

$$T_{kj}f_j = \frac{1}{\pi i} \int_{\Gamma_j} \frac{f_j(\tau)}{\tau - t} d\tau, \quad t \in \Gamma_k, \quad j \neq k.$$

We denote

$$k(s, \sigma) := \frac{1}{\tau - t} = \frac{1}{\tau(\sigma) - t(s)}.$$

Thus,

$$T_{kj}f_j = \frac{1}{\pi i} \int_{\Gamma_j} k(s(t), \sigma(\tau))f_j(\tau) d\tau, \quad t \in \Gamma_k, \quad j \neq k. \tag{4.10}$$

First we consider the kernel $k(s, \sigma)$, where $t = t(s)$, $s \in [0, l_k]$ and $\tau = \tau(\sigma)$, $\sigma \in [0, l_j]$, $k, j = 1, 2, \dots, m$, $j \neq k$. It is easy to see that the function $K(s, \sigma)$ is continuous function on $\Pi := [0, l_k] \times [0, l_j]$. Since, by the Stone-Weierstrass theorem, any continuous function on a compact set Π in \mathbb{R}^n can be uniformly approximated by a polynomial, we have that for any $\varepsilon > 0$, $\exists p_n(s, \sigma) = \sum_{0 \leq \mu + \nu \leq n} C_{\mu, \nu} s^\mu \sigma^\nu$, such that

$$|R_n| = |k(s, \sigma) - p_n(s, \sigma)| < \varepsilon, \tag{4.11}$$

uniformly in both variables.

We denote

$$P_n f_j := \frac{1}{\pi i} \int_{\Gamma_j} p_n(s(t), \sigma(\tau)) d\tau = \frac{1}{\pi i} \sum_{0 \leq \mu + \nu \leq n} C_{\mu, \nu} s^\mu(t) \int_{\Gamma_j} \sigma^\nu(\tau) f_j(\tau) d\tau. \tag{4.12}$$

Then

$$T_{kj}f_j = P_n f_j + R_n f_j.$$

Since P_n is a finite dimensional operator, it is enough to prove its boundedness by the norm, to claim that P_n is compact.

Hence, for the operator P_n , defined by (4.12), we have

$$\begin{aligned} |P_n f_j| &\leq \sum_{0 \leq \mu + \nu \leq n} C_{\mu, \nu} |s^\mu(t)| \int_{\Gamma_j} |\sigma^\nu(\tau)| |f_j(\tau)| d\tau \\ &\leq C \int_{\Gamma_j} |f_j(\tau)| d\tau \leq C_1 \|f_j\|_{L^1(\Gamma_j)}, \quad \text{since } \sigma \text{ and } s \text{ are bounded.} \end{aligned}$$

Then, applying the norm we get

$$\|P_n f_j\|_{L^{p, \varphi_j}(\Gamma_j, \varrho_j)} \leq C \|f_j\|_{L^1(\Gamma_j)}, \quad \text{since } 1 \in L^{p, \varphi_j}(\Gamma_j, \varrho_j).$$

It is not hard to show that, under the assumptions of our theorem,

$$\|f_j\|_{L^1(\Gamma_j)} \leq C \|f_j\|_{L^{p, \varphi_j}(\Gamma_j, \varrho_j)}. \tag{4.13}$$

Therefore, by (4.13) we obtain that

$$\|P_n f_j\|_{L^{p,\varphi_j}(\Gamma_j, \varrho_j)} \leq C \|f_j\|_{L^{p,\varphi_j}(\Gamma_j, \varrho_j)}.$$

Thus, the operator P_n is compact in the space $L^{p,\varphi_j}(\Gamma_j, \varrho_j)$. Now we have to estimate

$$|R_n f_j| = |T_{kj} f_j - P_n f_j| \tag{4.14}$$

by the norm of our space. Passing to the norm in (4.14), by (4.11) and (4.13) we obtain

$$\begin{aligned} \|R_n f_j\|_{L^{p,\varphi_j}(\Gamma_j, \varrho_j)} &= \|T_{kj} f_j - P_n f_j\|_{L^{p,\varphi_j}(\Gamma_j, \varrho_j)} \\ &\leq C\varepsilon \|f_j\|_{L^1(\Gamma_j)} \leq \varepsilon \|f_j\|_{L^{p,\varphi_j}(\Gamma_j, \varrho_j)} \rightarrow 0. \end{aligned}$$

Thus, we can conclude that the operator $T_k f$ is compact in $L^{p,\varphi}(\Gamma, \varrho)$, since $T_k f$ is the limit of the sequence of compact operators. The proof is complete. \square

Now we can formulate the theorem on Fredholmness of the operator \mathfrak{A} , which indeed is proved by the above statements.

Theorem 4.5. *Let $1 < p < \infty, \varphi_k(r) \geq cr$. Let $\Gamma = \bigcup_{k=1}^m \Gamma_k$ be a set of non-intersecting oriented closed curves without self-intersection, satisfying arc-chord condition. Let $a(t), b(t) \in C(\Gamma)$ and $\varrho(t)$ be a weight (4.6), i.e., $\varrho = (\varrho_1, \dots, \varrho_m)$. Then the operator \mathfrak{A} is Fredholm in the space $L^{p,\varphi}(\Gamma, \varrho)$ if*

$$\inf_{t \in \Gamma} |a(t) \pm b(t)| \neq 0, \tag{4.15}$$

and

$$m \left(\frac{\varphi_k}{w_k^p} \right) > 1 - p, \quad M \left(\frac{\varphi_k}{w_k^p} \right) < 1, \quad k = 1, 2, \dots, m. \tag{4.16}$$

The index of the operator \mathfrak{A} in the space $L^{p,\varphi}(\Gamma, \varrho)$ is equal to the $L^{p,\varphi}$ -index of the function g :

$$Ind_{L^{p,\varphi}(\Gamma, \varrho)} \mathfrak{A} = -ind_{L^{p,\varphi}(\Gamma, \varrho)} g(t) := \varkappa$$

(and the d -characteristic is equal to $(\varkappa, 0)$, if $\varkappa \geq 0$ and $(0, |\varkappa|)$, if $\varkappa \leq 0$), where $g(t) = \frac{a(t)+b(t)}{a(t)-b(t)}$.

Indeed: (1) The boundedness of S_Γ in the weighted space $L^{p,\varphi}(\Gamma, \varrho)$ was proved in Theorem 4.3. (2) The compactness of the commutator $gS_\Gamma - S_\Gamma g$, where $g(t) = \frac{a(t)+b(t)}{a(t)-b(t)}$, under Condition (4.15) on $g(t)$, follows from Theorem 4.4. (3) For the fact that $S_\Gamma^2 = I$, we refer for instance to [7]. Then the operator \mathfrak{A} is Fredholm in the space $L^{p,\varphi}(\Gamma, \varrho)$, since all (1)–(3) conditions are satisfied.

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