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Lipschitz Conditions in Laguerre Hypergroup

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Abstract. The purpose of this paper is to prove analogous of Titchmarsh's theorems for the Laguerre transform. More precisely, we give a Lipschitz-type condition on f in $L^p(\mathbb{K})$ for which its Laguerre transform belongs to $L^\beta(\hat{\mathbb{K}})$ for some values of β, where $\mathbb{K} = [0, +\infty) \times \mathbb{R}$ and $\hat{\mathbb{K}}$ is its dual. In the particular case, when $p = 2$, we provide equivalence theorem : we get a characterization of the space $\mathrm{Lip}_\alpha(\gamma, 2)$ of Lipschitz class functions by means of asymptotic estimate growth of the norm of their Laguerre transform for $0 < \gamma < 1$. Furthermore, we introduce Laguerre–Dini–Lipschitz class $\mathrm{LDLip}_\alpha(\gamma, \delta, p)$ and we obtain analogous of Titchmarsh's theorems in this occurence.

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1. Introduction

Lipschitz condition states that

$$|f(x) - f(x')| \leq M\,|x - x'|^\alpha; \quad 0 < \alpha \leq 1. \tag{1.1}$$

It was first considered by Lipschitz in 1864 while studying the convergence of the Fourier series of a periodic function f. He proved that inequality (1.1) is sufficient to have that the Fourier series of f converges everywhere to the value of f. A strengthening criterion was introduced by Dini in 1872 whose conclusion states that the convergence is in addition uniform. If we denote $w(h, f) = \sup_{|x-x'|<h} |f(x) - f(x')|$ the modulus of continuity, Lipschitz condition can be written as

$$w(h, f) = O(h^\alpha), \quad 0 < \alpha \leq 1$$

and the Dini–Lipschitz condition states

$$\lim_{h\to 0} \ln \frac{1}{h}\, w(h, f) = 0 \quad \text{i.e} \quad w(h, f) = o\left(\ln(\frac{1}{h})^{-1}\right).$$

A continuous version was studied by Titchmarsh [13]. He proved that if f belongs to the Lipschitz class $\mathrm{Lip}(\alpha, p)$, then its Fourier transform $\hat{f}$ belongs to $L^{\beta}(\mathbb{R})$ for $\frac{p}{p+\alpha p-1} < \beta \leq \frac{p}{p-1}$. Note that a function $f \in L^p(\mathbb{R})$ is said to be in Lipschitz class if the integral modulus of continuity $w_p(h, f)$ verifies

$$w_p(h, f) = \|f(x+h) - f(x)\|_p = O(h^{\alpha}).$$

A second result [13, Theorem 85] states that f belongs to Lipschitz class $\mathrm{Lip}(\alpha, 2)$ if and only if $\int_{\lambda \geq r} \hat{f}(\lambda) \mathrm{d}\lambda = O(r^{-2\alpha})$ as $r \to +\infty$.

An extension of these theorems to functions of several variables on $\mathbb{R}^n$ and on the torus group T^n was studied by Younis [15,16]. Later, analogous results were given, where considering generalized Fourier transforms: Bessel, Dunkl, Jacobi,... One can cite [2,3,7].

Krovokin [9] considered Dini–Lipschitz class as the set of functions in $L^p(\mathbb{R})$, such that $w_p(h, f) = o(\big(\ln \frac{1}{h}\big)^{-1})$. Younis [17], showed that the result of Titchmarsh's theorem [13, Theorem 84] does not hold for the Dini–Lipschitz functions: it does not improve the Hausdorff–Young inequality and the conclusion is that $\hat{f}$ belongs to $L^{p'}(\mathbb{R})$. Therefore, he considered some conditions which are rather situated in between the Lipschitz and the Dini–Lipschitz conditions. These were inspired from Weiss and Zygmund [14]. It states that

$$w(h, f) = O\left(h^{\alpha} \ln \left(\frac{1}{h}\right)^{-\delta}\right), \quad \delta \geq 0.$$

He showed that Titchmarsh's theorems [13, Theorem 84, Theorem 85] could be extended.

In this paper, we are interested in the Laguerre hypergroup $\mathbb{K} = [0, +\infty) \times \mathbb{R}$ which can be seen as a deformation of the hypergroup of radial functions on the Heisenberg group [5]. Let $\alpha \geq 0$, $\mathbb{K}$ is provided with the convolution product $*_{\alpha}$ generalizing the convolution product of radial functions on the $(2n+1)$-dimensional Heisenberg group $\mathbb{H}^n$. It was seen that $(\mathbb{K}, *_{\alpha})$ is a commutative hypergroup in the sense of Jewett with the involution the homeomorphism $i(x, t) = (x, -t)$ and the Haar measure dm_{α}, given by

$$dm_{\alpha}(x, t) = \frac{x^{2\alpha+1}}{\pi \Gamma(\alpha+1)} \mathrm{d}x \, \mathrm{d}t. \tag{1.2}$$

The unit element of $(\mathbb{K}, *_{\alpha})$ is given by $e = (0, 0)$, since $\delta_{(x,t)} *_{\alpha} \delta_{(0,0)} = \delta_{(0,0)} *_{\alpha} \delta_{(x,t)} = \delta_{(x,t)}$. The convolution product of two bounded Radon measures μ_1 and μ_2 on $\mathbb{K}$ is defined by

$$\left\langle \mu_1 *_{\alpha} \mu_2, \, f \right\rangle = \int_{\mathbb{K} \times \mathbb{K}} T^{\alpha}_{x,t} f(y, s) \, \mathrm{d}\mu_1 \mathrm{d}\mu_2,$$

where $T^{\alpha}_{x,t}$, $(x, t) \in \mathbb{K}$ are the generalized translation operators on $\mathbb{K}$ given, for $\alpha = 0$, by

$$T^{\alpha}_{x,t} f(y, s) = \frac{1}{2\pi} \int_0^{2\pi} f\left(\sqrt{x^2 + y^2 + 2xy\cos\theta}, t + s + xy\sin\theta\right) \mathrm{d}\theta$$

and, for $\alpha > 0$, by

$$T^{\alpha}_{x,t}f(y,s) = \frac{\alpha}{\pi}\int_0^{2\pi}\int_0^1 f\left(\sqrt{x^2+y^2+2xyr\cos\theta}, t+s+xyr\sin\theta\right) r(1-r^2)^{\alpha-1}\mathrm{d}r\,\mathrm{d}\theta.$$

If μ_1 and μ_2 are Dirac measure at (x,t) and $(y,s) \in \mathbb{K}$, then

$$(\delta_{(x,t)} *_{\alpha} \delta_{(y,s)})(f) = T^{\alpha}_{x,t}f(y,s). \tag{1.3}$$

The outline of this paper is given as follows:

First in Sect. 2, we collect some useful results related to Laguerre hypergroup. Second, in Sect. 3, we prove analogous of Titchmarsh theorem [13, Theorem 84]: we prove that if $f \in L^p(\mathbb{K})$ satisfying $\| T^{\alpha}_{x,y}f - f\|_{p,m_\alpha} = O(x^{\gamma})$ for $1 < p \leq 2$ and $0 < \gamma \leq 1$, then $\mathcal{F}_L f$ belongs to $L^{\beta}(\mathbb{R}\times\mathbb{N})$, where $\dfrac{(\alpha+2)p}{(\alpha+2)(p-1)+\frac{\gamma p}{2}} < \beta \leq \dfrac{p}{p-1}$. Third, in Sect. 4, we introduce the Laguerre–Lipschitz class $\mathrm{Lip}_{\alpha}(\gamma,2)$ and we establish the analog of [13, Theorem 85]. Finally, in Sect. 5, we extend these results to Laguerre–Dini–Lipschitz-type class $\mathrm{LDLip}_{\alpha}(\gamma,\delta,p)$, $\delta \geq 0$, using similar technics as in Sects. 3 and 4.

Throughout this paper, p and p' are real numbers, such that $1 \leq p \leq 2$ and $\frac{1}{p}+\frac{1}{p'} = 1$ and C denote a positive constant which can differ from line to other.

2. Preliminaries

Recall $\varphi_{\lambda,m}(x,t)$ the Laguerre Kernel given by

$$\forall (x,t) \in \mathbb{K}, \quad \varphi_{\lambda,m}(x,t) = e^{i\lambda t}\mathcal{L}^{(\alpha)}_m(|\lambda|\,x^2), \tag{2.1}$$

where $\mathcal{L}^{(\alpha)}_m$ is the Laguerre function defined on $\mathbb{R}_+$ by

$$\mathcal{L}^{(\alpha)}_m(x) = e^{-\frac{x}{2}}\frac{L^{\alpha}_m(x)}{L^{\alpha}_m(0)}$$

and L^{α}_m is the Laguerre polynomial of degree m and order α, given by

$$L^{\alpha}_m(x) = \sum_{k=0}^{m}(-1)^k\frac{\Gamma(m+\alpha+1)}{\Gamma(k+\alpha+1)}\frac{1}{k!(m-k)!}x^k. \tag{2.2}$$

For $(\lambda,m) \in \mathbb{R}\times\mathbb{N}$, $\varphi_{\lambda,m}$ is the unique solution of the initial problem:

$$\begin{cases} D_1 u = i\,\lambda\, u, \\ D_2\, u = -4|\lambda|(m+\dfrac{\alpha+1}{2})u \\ u(0,0) = 1, \ \dfrac{\partial u}{\partial x}(0,t) = 0 \quad \text{for all } t \in \mathbb{R}, \end{cases}$$

where for all $(x,t) \in \mathbb{K}$ and $\alpha \geq 0$

$$\begin{cases} D_1 = \dfrac{\partial}{\partial t} \\ D_2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{2\alpha+1}{x}\dfrac{\partial}{\partial x} + x^2\dfrac{\partial^2}{\partial t^2} \end{cases}$$

When $\alpha = n-1$, $n \in \mathbb{N}\backslash\{0\}$, the operator D_2 is the radial part of the sub-Laplacian on the Heisenberg group $\mathbb{H}_n$ and $\varphi_{\lambda,m}(\|z\|,t)$ are zonal spherical functions of the Gelfand pairs $(G, \mathcal{U}(\mathbb{C}^n))$, where G is the semi-direct product of $\mathcal{U}(\mathbb{C}^n))$ by $\mathbb{H}^n$ (see [5]). For the general case $\alpha \geq 0$, the functions $\varphi_{\lambda,m}$ $(\lambda, m) \in \mathbb{R}\times\mathbb{N}$ are characters of the Laguerre hypergroup $(\mathbb{K}, *_\alpha)$, i.e, elements of the dual space $\hat{\mathbb{K}}$ the space of all bounded functions $\chi : \mathbb{K} \to \mathbb{C}$, such that $\tilde{\chi}(x,t) = \overline{\chi(x,-t)} = \chi(x,t)$, where $(x,t) \in \mathbb{K}$. Indeed, this space is given by

$$\{\varphi_{\lambda,m}\,;\ (\lambda,m) \in \mathbb{R}^* \times \mathbb{N}\} \cup \{\varphi_\rho\,;\ \rho \geq 0\},$$

where $\varphi_\rho = j_\alpha(\rho x)$; j_α is the normalized Bessel function of order α. This space $\hat{\mathbb{K}}$ can be topologically identified to the so–called Heisenberg fan [5]:

$$\bigcup_{m\in\mathbb{N}} \left\{(\lambda,\mu) \in \mathbb{R}^2; \mu = |\lambda|(2m+\alpha+1)\right\} \bigcup \left\{(0,\mu) \in \mathbb{R}^2; \mu \geq 0\right\}.$$

The subset $\{(0,\mu) \in \mathbb{R}^2; \mu \geq 0\}$ is usually disregarded, since it has zero Plancherel measure.

For all $(\lambda,m) \in \mathbb{R}\times\mathbb{N}$, the kernel $\varphi_{\lambda,m}$ verifies the following product formula

$$\varphi_{\lambda,m}(x,t)\varphi_{\lambda,m}(y,s) = T^\alpha_{x,t}\varphi_{\lambda,m}(y,s), \quad (x,t),(y,s) \in \mathbb{K} \tag{2.3}$$

and has the property

$$\forall(\lambda,m) \in \mathbb{R}\times\mathbb{N}, \qquad \sup_{(x,t)\in\mathbb{K}} |\varphi_{\lambda,m}(x,t)| = 1. \tag{2.4}$$

Denote $L^p(\mathbb{K}) = L^p(\mathbb{K}, dm_\alpha)$ the space of measurable functions $f : \mathbb{K} \longrightarrow \mathbb{C}$, such that

$$\|f\|_{p,m_\alpha} = \left(\int_{\mathbb{K}} |f(x,t)|^p dm_\alpha(x,t)\right)^{\frac{1}{p}} < +\infty.$$

The Fourier–Laguerre transform of a function in $L^1(\mathbb{K})$ is given by

$$\mathcal{F}_L f(\lambda,m) = \int_{\mathbb{K}} f(x,t)\,\varphi_{-\lambda,m}(x,t)dm_\alpha(x,t).$$

From [11], it is well known that Fourier–Laguerre transform can be inverted to

$$\mathcal{F}_L^{-1} f(x,t) = \int_{\mathbb{R}\times\mathbb{N}} f(\lambda,m)\,\varphi_{\lambda,m}(x,t)d\gamma_\alpha(\lambda,m),$$

where

$$d\gamma_\alpha(\lambda,m) = L^\alpha_m(0)\delta_m \otimes |\lambda|^{\alpha+1}\mathrm{d}\lambda.$$

In the following, we can write $d\gamma_\alpha$ to designate $d\gamma_\alpha(\lambda,m)$ if necessary. Nessibi and Trimèche proved in [11] the Plancherel formula:

$$\|\mathcal{F}_L f\|_{2,\gamma_\alpha} = \|f\|_{2,m_\alpha}.$$

Furthermore, we have

$$\|\mathcal{F}_L f\|_{\infty,\gamma_\alpha} \leq \|f\|_{1,m_\alpha}.$$

Then, applying the Riez–Thorin interpolation theorem [6], we can extend the definition of $\mathcal{F}_L f$ to $L^p(\mathbb{K})$ for $1 \le p \le 2$ and we have the following Hausdorff–Young inequality:

$$\|\mathcal{F}_L f\|_{p',\gamma_\alpha} \le C\,\|f\|_{p,m_\alpha}. \tag{2.5}$$

It is well known from [10,11] that the translation $T^\alpha_{x,t}$ is linear operator from $L^p(\mathbb{K})$ onto itself and we have

$$\|T^\alpha_{(x,t)} f\|_{p,m_\alpha} \le C\,\|f\|_{p,m_\alpha}$$

and it verifies, as a consequence of the product formula (2.3), the relation

$$\mathcal{F}_L(T^\alpha_{x,t} f) = \varphi_{\lambda,m}(x,t)\mathcal{F}_L f(\lambda,m). \tag{2.6}$$

These results would be useful in the following sections.

3. Lipschitz Conditions in Laguerre Hypergroup

We denote, for all $(x,t) \in \mathbb{K}$, $|x,t| = |(x,t)|_{\mathbb{K}} = (x^4 + 4t^2)^{\frac{1}{4}}$ the homogeneous norm on $\mathbb{K}$ and for all $(\lambda,m) \in \mathbb{R}\times\mathbb{N}$, $|\lambda,m|$ the quasinorm on $\mathbb{R}\times\mathbb{N}$ defined by $|\lambda,m| = |(\lambda,m)|_{\mathbb{R}\times\mathbb{N}} = 4|\lambda|\kappa_m$, where $\kappa_m = m + \frac{\alpha+1}{2}$ (cf. [12]). Lets denote $\mathbb{B}_r$ the ball centered on 0 and of radius r, defined by, $\mathbb{B}_r = \{(\lambda,m) \in \mathbb{R}\times\mathbb{N};\ |\lambda,m| < r\}$ and $\mathbb{B}^c_r = (\mathbb{R}\times\mathbb{N})\backslash\mathbb{B}_r$.

Theorem 3.1. *Let f be a function in $L^p(\mathbb{K})$, such that $\|\,T^\alpha_{x,y} f - f\|_{p,m_\alpha} = O(x^\gamma)$ for $1 < p \le 2$ and $0 < \gamma \le 1$. Then, $\mathcal{F}_L f$ belongs to $L^\beta(\mathbb{R}\times\mathbb{N})$, where*

$$\frac{(\alpha+2)p}{(\alpha+2)(p-1) + \frac{\gamma p}{2}} < \beta \le \frac{p}{p-1}.$$

Proof. For fixed $(x,t) \in \mathbb{K}$, we have, using relations (2.6) and (2.5)

$$\int_{\mathbb{R}\times\mathbb{N}} |\varphi_{\lambda,m}(x,t) - 1|^{p'}\, |\mathcal{F}_L f(\lambda,m)|^{p'} d\gamma_\alpha(\lambda,m) = O(x^{\gamma p'}).$$

On the other hand, relations (2.1) and (2.2) yield to

$$\lim_{|\lambda,m|x^2\to 0} \left(\frac{||\varphi_{\lambda,m}(x,t)| - 1|}{|\lambda,m|x^2}\right) = \frac{1}{4(\alpha+1)} > 0. \tag{3.1}$$

Consequently, there exists a constant C, such that if $|\lambda,m|x^2 < \eta$, then

$$||\varphi_{\lambda,m}(x,t)| - 1| \ge C\,|\lambda,m|x^2.$$

Therefore

$$\begin{aligned}\int_{\mathbb{B}_{\frac{\eta}{x^2}}} |\lambda,m|^{p'} |\mathcal{F}_L f(\lambda,m)|^{p'} d\gamma_\alpha &\le x^{-2p'} \int_{\mathbb{B}_{\frac{\eta}{x^2}}} |\varphi_{\lambda,m}(x,t) - 1|^{p'}\, |\mathcal{F}_L f(\lambda,m)|^{p'} d\gamma_\alpha \\ &\le C x^{(\gamma-2)p'}.\end{aligned}$$

Now, let $\beta \le p'$. From Hölder inequality, one gets

$$\int_{\mathbb{B}_X} |\lambda,m|^{\beta} |\mathcal{F}_L f(\lambda,m)|^{\beta} d\gamma_\alpha \le \left(\int_{\mathbb{B}_X} |\lambda,m|^{p'} |\mathcal{F}_L f(\lambda,m)|^{p'} d\gamma_\alpha\right)^{\frac{\beta}{p'}} \left(\int_{\mathbb{B}_X} 1 d\gamma_\alpha\right)^{1-\frac{\beta}{p'}}.$$

Therefore

$$\int_{\mathbb{B}_X} |\lambda, m|^\beta |\mathcal{F}_L f(\lambda, m)|^\beta d\gamma_\alpha(\lambda, m) = O\left(X^{\frac{(2-\gamma)p'}{2}\frac{\beta}{p'}+(\alpha+2)\left(1-\frac{\beta}{p'}\right)}\right). \quad (3.2)$$

Recall that $\mathbb{B}_1^c = (\mathbb{R}\times\mathbb{N})\backslash\mathbb{B}_1$. To get the theorem, it is enough to prove that $\int_{\mathbb{B}_1^c\cap\mathbb{B}_X} |\mathcal{F}_L f(\lambda, m)|^\beta d\gamma_\alpha(\lambda, m)$ is bounded when $X \to +\infty$. Therefore, we can write

$$\int_{\mathbb{B}_1^c\cap\mathbb{B}_X} |\mathcal{F}_L f(\lambda, m)|^\beta d\gamma_\alpha(\lambda, m) = \sum_{m=0}^{+\infty} L_m^\alpha(0)\, I,$$

where I depend on m and X and has the expression

$$I = \int_{\frac{1}{4m+2\alpha+2}}^{\frac{X}{4m+2\alpha+2}} \left(|\mathcal{F}_L f(\lambda, m)|^\beta + |\mathcal{F}_L f(-\lambda, m)|^\beta\right)\lambda^{\alpha+1}\mathrm{d}\lambda.$$

Consider

$$\Phi_m(X) = \int_{\frac{1}{4m+2\alpha+2}}^{\frac{X}{4m+2\alpha+2}} |(\lambda, m)|^\beta \left(|\mathcal{F}_L f(\lambda, m)|^\beta + |\mathcal{F}_L f(-\lambda, m)|^\beta\right)\lambda^{\alpha+1}\mathrm{d}\lambda.$$

Thus

$$I = \int_{\frac{1}{4m+2\alpha+2}}^{\frac{X}{4m+2\alpha+2}} (4m+2\alpha+2)|(\lambda, m)|^{-\beta}\Phi_m'(|(\lambda, m)|)\mathrm{d}\lambda.$$

Making a change of variables and an integration by parts, we get

$$I = \Phi_m(X)X^{-\beta} + \beta\int_1^X t^{-\beta-1}\Phi_m(t)\mathrm{d}t.$$

Consequently

$$\int_{\mathbb{B}_1^c\cap\mathbb{B}_X} |\mathcal{F}_L f(\lambda, m)|^\beta d\gamma_\alpha(\lambda, m) = X^{-\beta}\psi(X) + \beta\int_1^X t^{-\beta-1}\psi(t)\mathrm{d}t, \quad (3.3)$$

where

$$\psi(X) = \sum_{m=0}^{+\infty} L_m^\alpha(0)\Phi_m(X) = \int_{\mathbb{B}_1^c\cap\mathbb{B}_X} |(\lambda, m)|^\beta |\mathcal{F}_L f(\lambda, m)|^\beta d\gamma_\alpha(\lambda, m).$$

From relation (3.2), we have

$$\int_{\mathbb{B}_1^c\cap\mathbb{B}_X} |\mathcal{F}_L f(\lambda, m)|^\beta d\gamma_\alpha(\lambda, m) = O\left(X^{-\beta+\frac{2-\gamma}{2}\beta+(\alpha+2)\left(1-\frac{\beta}{p'}\right)}\right) + O\left(\int_1^X t^{-\beta-1} t^{\frac{2-\gamma}{2}\beta+(\alpha+2)(1-\frac{\beta}{p'})}\mathrm{d}t\right).$$

This is bounded as $X \to +\infty$ if $-\beta(\frac{\gamma}{2} + \frac{\alpha+2}{p'}) + (\alpha+2) < 0$ that gives $\beta > \dfrac{(\alpha+2)p}{(\alpha+2)(p-1)+\frac{\gamma p}{2}}$. □

4. An Equivalence Theorem for Laguerre–Lipschitz Class Functions

In this paragraph, we consider $0 < \gamma < 1$ and $p = 2$. We try to put the previous theorem into form in which it is reversible. Hence, we give a characterization of the space $\mathrm{Lip}_\alpha(\gamma, 2)$ of Laguerre–Lipschitz class functions by means of asymptotic estimate growth of the norm of their Laguerre transform.

The behavior in 0 of the characters $\varphi_{\lambda,m}(x,t)$ could be deduced from relations (2.1) and (2.2) as follows:

$$|\varphi_{\lambda,m}(x,t) - 1|^2 = |\lambda\, t|^2 + \frac{|\lambda, m|^2 x^4}{4^2(\alpha+1)^2} + o(|\lambda|^2 |x,t|^4). \tag{4.1}$$

Exploiting this result, one can find the two following propositions.

Proposition 4.1. *Let* $f \in L^2(\mathbb{K})$ *and* $0 < \gamma < 1$. *Assume that* $\int_{\mathbb{B}_r^c} |\mathcal{F}_L f(\lambda, m)|^2 \, d\gamma_\alpha(\lambda, m) = O(r^{-\gamma})$ *as* $r \to +\infty$, *then* f *verifies*

$$\|T^\alpha_{x,t} f - f\|_{2,m_\alpha} = O(|x,t|^\gamma).$$

Proof. Denote $r = \frac{\eta}{|x,t|^2}$. According to Plancherel formula, one has

$$\|T^\alpha_{x,t} f - f\|^2_{2,m_\alpha} = I_1 + I_2$$

where

$$I_1 = \int_{\mathbb{B}_r} |\varphi_{\lambda,m}(x,t) - 1|^2 |\mathcal{F}_L f(\lambda, m)|^2 d\gamma_\alpha(\lambda, m)$$

and

$$I_2 = \int_{\mathbb{B}_r^c} |\varphi_{\lambda,m}(x,t) - 1|^2 |\mathcal{F}_L f(\lambda, m)|^2 d\gamma_\alpha(\lambda, m).$$

Using relation (2.4), we find that

$$I_2 \le 4 \int_{\mathbb{B}_r^c} |\mathcal{F}_L f(\lambda, m)|^2 d\gamma_\alpha = O(r^{-\gamma}) = O(|x,t|^{2\gamma}).$$

Denote $g(X) = \displaystyle\int_X^\infty (|\mathcal{F}_L f(\lambda, m)|^2 + |\mathcal{F}_L f(-\lambda, m)|^2)\, \lambda^{\alpha+1} \mathrm{d}\lambda$, then

$$g'(\lambda) = -(|\mathcal{F}_L f(\lambda, m)|^2 + |\mathcal{F}_L f(-\lambda, m)|^2)\, \lambda^{\alpha+1}.$$

Using relation (4.1), there exist $C > 0$ and $\eta > 0$ such that for all $(x,t) \in \mathbb{K}$,

$$|\lambda|\, |x,t|^2 < \eta \quad \Longrightarrow \quad |\varphi_{\lambda,m}(x,t) - 1|^2 \le C |\lambda, m|^2 |x,t|^4$$

which gives

$$I_1 \le C|x,t|^4 \sum_{m=0}^\infty L_m^\alpha(0) J_m, \quad \textit{where } J_m = \int_0^{\frac{\eta}{4\kappa_m |x,t|^2}} (4\kappa_m)^2 \lambda^2 (-g'(\lambda)) \mathrm{d}\lambda.$$

By integration by parts, we have

$$J_m = -\frac{\eta^2}{|x,t|^4} g\left(\frac{\eta}{4\kappa_m |x,t|^2}\right) + (4\kappa_m)^2 \int_0^{\frac{\eta}{4\kappa_m |x,t|^2}} 2\lambda g(\lambda) \mathrm{d}\lambda.$$

Remark that

$$\sum_{m=0}^{\infty} L_m^\alpha(0) g\left(\frac{R}{4\kappa_m}\right) = \int_{\mathbb{B}_R^c} |\mathcal{F}_L f(\lambda,m)|^2 d\gamma_\alpha = O(R^{-\gamma}).$$

Making a change of variable in the second part of J_m, one gets

$$\begin{aligned} I_1 &\le C\,|x,t|^{2\gamma} + C\,|x,t|^4 \int_0^{\frac{\eta}{|x,t|^2}} u \sum_{m=0}^{\infty} L_m^\alpha(0) g\left(\frac{u}{4\kappa_m}\right) \mathrm{d}u \\ &= O(|x,t|^{2\gamma}) + |x,t|^4 O\left(\int_0^{\frac{\eta}{|x,t|^2}} u^{-\gamma+1}\mathrm{d}u\right) \\ &= O(|x,t|^{2\gamma}) \end{aligned}$$

which proves the wanted result. □

Proposition 4.2. *Let* $0<\gamma<1$ *and* $f \in L^2(\mathbb{K})$, *such that* $\|T_{x,t}^\alpha f - f\|_{2,m_\alpha} = O(|x,t|^\gamma)$. *Then*

$$\int_{|\lambda|>r} |\mathcal{F}_L f(\lambda,m)|^2\, d\gamma_\alpha = O(r^{-\gamma}) \quad \text{as } r \to +\infty.$$

Proof. From relation (4.1), one deduce that there exist $C>0$ and $\eta>0$, such that for all $(x,t)\in\mathbb{K}$

$$|\lambda|\,|x,t|^2 < \eta \quad \Longrightarrow \quad |\varphi_{\lambda,m}(x,t)-1|^2 \ge C|\lambda|^2|x,t|^4.$$

By Plancherel formula, we have

$$\begin{aligned} \int_{\frac{\eta}{2h^2}<|\lambda|<\frac{\eta}{h^2}} |\mathcal{F}_L f(\lambda,m)|^2 d\gamma_\alpha &\le C \int_{\frac{\eta}{2h^2}<|\lambda|<\frac{\eta}{h^2}} |\lambda|^2|x,t|^4\,|\mathcal{F}_L f(\lambda,m)|^2 d\gamma_\alpha \\ &\le \|\mathcal{F}_L((\varphi_{\lambda,m}(x,t)-1)f)\|_{2,\gamma_\alpha}^2 = O(|x,t|^{2\gamma}). \end{aligned}$$

If we denote $kr = \frac{\eta}{2|x,t|^2}$, then

$$\int_{kr}^{2kr} |\mathcal{F}_L f(\lambda,m)|^2 d\gamma_\alpha \le C(kr)^{-\gamma}.$$

Consequently

$$\begin{aligned} \int_{|\lambda|>r} |\mathcal{F}_L f(\lambda,m)|^2\, d\gamma_\alpha &= \sum_{k=0}^{\infty} \int_{2^k r}^{2^{k+1}r} |\mathcal{F}_L f(\lambda,m)|^2 d\gamma_\alpha \\ &\le C\sum_{k=1}^{\infty} (2^k r)^{-\gamma} = C\frac{r^{-\gamma}}{1-2^{-\gamma}} = O(r^{-\gamma}). \end{aligned}$$

□

The conclusion of Proposition 4.2 is not sufficient to get an equivalence theorem, since $\{(\lambda,m); |\lambda|>r\} \subset \{(\lambda,m); |\lambda,m|>r\}$. Therefore, we give here another property verified by Laguerre kernel $\varphi_{\lambda,m}$.

Lemma 4.3. $\forall\, x>0$ *and* $t\in\mathbb{R}$ *,we have*

$$\lim_{|\lambda,m|\to+\infty} \varphi_{\lambda,m}(x,t) = 0.$$

Proof. From (1.2) and (1.3), $\delta_{x,t} *_\alpha \delta_{x,t}$ is absolutely continuous with respect to the Haar measure dm_α; then, from [8, p. 41], we have that the Fourier–Laguerre transform of $\delta_{(x,t)}$ on the dual space $\hat{\mathbb{K}}$ of the hypergroup $(\mathbb{K}, *_\alpha)$ is a C_0 function. This implies that

$$\lim_{|\lambda,m|\to+\infty} \varphi_{\lambda,m}(x,t) = 0.$$

□

Definition 4.4. Let $0 \le \gamma \le 1$. A function f is said to be in Laguerre–Lipschitz class of order γ and we denote $f \in Lip_\alpha(\gamma, 2)$, if f belongs to $L^2(\mathbb{K})$ and verifies, for all $(x,t) \in \mathbb{K}$

$$w(h,f) = \|T^\alpha_{\Delta_h(x,t)} f - f\|_{2,m\alpha} = O(h^\gamma),$$

where $\Delta_h(x,t)$ is the dilated of $(x,t) \in \mathbb{K}$ given by $\Delta_r(x,t) = (rx, r^2t)$.

Now, we are able to establish the equivalence theorem.

Theorem 4.5. *Let $f \in L^2(\mathbb{K})$. Then, the two statements*

(i) *f is in Laguerre–Lipschitz class* $\mathrm{Lip}_\alpha(\gamma, 2)$, $0 < \gamma < 1$.
(ii) $\int_{\mathbb{B}^c_r} |\mathcal{F}_L f(\lambda,m)|^2 \, d\gamma_\alpha(\lambda,m) = O(r^{-\gamma})$ *as* $r \to +\infty$,

are equivalent.

Proof. Let $h > 0$, from relation (2.1), we have

$$\varphi_{h^2\lambda,m}(x,t) = \varphi_{\lambda,m}(\Delta_h(x,t))$$

Therefore, using Lemma 4.3, we get

$$\lim_{h^2|\lambda,m|\to+\infty} |\varphi_{\lambda,m}(\Delta_h(x,t)) - 1| = 1.$$

Hence, there exist $C > 0$ and $A > 0$, such that

$$|\lambda, m| > \frac{A}{h^2} \quad \Longrightarrow \quad |\varphi_{\lambda,m}(\Delta_h(x,t)) - 1|^2 \ge C.$$

Let $f \in L^2(\mathbb{K})$ verifying (i)

$$\begin{aligned}\int_{\mathbb{B}^c_{\frac{A}{h^2}}} |\mathcal{F}_L f(\lambda,m)|^2 d\gamma_\alpha &\le C \int_{\mathbb{B}^c_{\frac{A}{h^2}}} |\varphi_{\lambda,m}(\Delta_h(x,t)) - 1|^2 |\mathcal{F}_L f(\lambda,m)|^2 d\gamma_\alpha \\ &\le C\|T^\alpha_{\Delta_h(x,t)} f - f\|^2_{2,m\alpha} = O(h^{2\gamma}).\end{aligned}$$

Consequently, (ii) holds.

Lets prove that (ii) ⇒ (i). From Proposition 4.1, one has

$$\|T^\alpha_{x,t} f - f\|_{2,m\alpha} = O(|x,t|^\gamma) \quad as\ |x,t| \to 0.$$

Thus, for $(x,t) \in (0,+\infty) \times \mathbb{R}$

$$\|T^\alpha_{\Delta_h(x,t)} f - f\|_{2,m\alpha} = O(h^\gamma).$$

□

5. Laguerre Dini–Lipschitz Conditions

The reader can find analogous results of this section in the references [1,4,17].

Definition 5.1. Let $0 < \gamma < 1$ and $\delta \geq 0$, we define the Laguerre–Dini–Lipschitz class and we denote $LDLip_\alpha(\gamma, \delta, p)$ the set of functions f belonging to $L^p(\mathbb{K})$ satisfying

$$\forall (x,t) \in \mathbb{K}, \quad w_p(h,f) = \|T^\alpha_{\Delta_h(x,t)} f - f\|_{p,m_\alpha} = O\left(h^\gamma \ln(\frac{1}{h})^{-\delta}\right).$$

$\Delta_h(x,t)$ the dilated of (x,t) is given in Definition 4.4.

Since the same technics previously are available, then we remove details in the proofs of the theorems below.

Theorem 5.2. *Let $f \in L^2(\mathbb{K})$. Then, the following statements are equivalent.*

(i) *f belongs to Laguerre–Dini–Lipschitz class $LDLip_\alpha(\gamma, \delta, 2)$, $0 < \gamma < 1$, $\delta \geq 0$.*

(ii) $\int_{\mathbb{B}^c_r} |\mathcal{F}_L f(\lambda, m)|^2 \, d\gamma_\alpha(\lambda, m) = O(r^{-\gamma} \ln(r)^{-2\delta})$ *as $r \to +\infty$.*

Proof. By proceeding similarly to Theorem 4.5, we have

$$\int_{\mathbb{B}^c_{\frac{A}{h^2}}} |\mathcal{F}_L f(\lambda, m)|^2 d\gamma_\alpha = O(w_2^2(h,f)).$$

Thus, $\int_{\mathbb{B}^c_r} |\mathcal{F}_L f(\lambda, m)|^2 \, d\gamma_\alpha(\lambda, m) = O(r^{-\gamma} \ln(r)^{-2\delta}))$ as $r \to +\infty$.

The converse can be done in the same way as in Proposition 4.1. Consider the same notation $\|T^\alpha_{\Delta_h(x,t)} f - f\|^2_{m_\alpha} = I_1 + I_2$. Then, we get

$$I_2 = O\left(h^{2\gamma} \ln(\frac{1}{h})^{-2\delta}\right)$$

and

$$\begin{aligned} I_1 &= O\left(h^{2\gamma} \ln\left(\frac{1}{h}\right)^{-2\delta}\right) + h^4 \int_0^{\frac{\eta}{h^2|x,t|^2}} u\, O\left(u^{-\gamma} \ln(u^{-2\delta})\right) \mathrm{d}u \\ &= O\left(h^{2\gamma} \ln\left(\frac{1}{h}\right)^{-2\delta}\right) \end{aligned}$$

which completes the proof. □

Theorem 5.3. *If $\gamma > 2$, $\delta \geq 0$ and $f \in LDLip(\gamma, \delta, 2)$, then $f = 0$ a.e.*

Proof. We have for all $(x,t) \in \mathbb{K}$, $\|T^\alpha_{\Delta_h(x,t)} f - f\|_{2,m_\alpha} = O(h^\gamma \ln(\frac{1}{h})^{-\delta})$. Thus

$$\int_{\mathbb{R}\times\mathbb{N}} \left| \frac{|\varphi_{h^2,m}(x,t)| - 1}{h^2 |\lambda, m|\, x^2} \right|^2 |\lambda, m|^2 |\mathcal{F}_L f(\lambda, m)|^2 d\gamma_\alpha \leq C h^{2\gamma - 4} \ln\left(\frac{1}{h}\right)^{-2\delta}.$$

Since $\gamma > 2$, then $\lim_{h\to 0} h^{2\gamma-4} \ln(\frac{1}{h})^{-2\delta} = 0$. Hence, from relation (3.1), one gets

$$\||\lambda, m|\mathcal{F}_L f(\lambda, m)\|_{2,\gamma_\alpha} = 0.$$

Thereby for all $(\lambda, m) \in \mathbb{R} \times \mathbb{N}$, $\quad |\lambda, m|\mathcal{F}_L f(\lambda, m) = 0$. The injectivity of the Fourier–Laguerre transform yields to the wanted result. □

Remark 5.4. The same conclusion holds if we consider a function such that $w_2(h, f) = o(h^2)$ and also if we take f a function in $\mathrm{LDLip}(\gamma, \delta, p)$, for $1 < p < 2$ and $\gamma > p'$ (by using Hausdorff–Young inequality).

Theorem 5.5. *Let* $0 < \gamma < 1$ *and* $f \in LDLip(\gamma, \delta, p)$, $1 < p < 2$. *Then,* $\mathcal{F}_L f$ *belongs to* L^β *for* $\dfrac{(\alpha+2)p}{(\alpha+2)(p-1)+\frac{\gamma p}{2}} < \beta \leq \dfrac{p}{p-1}$.

Proof. As in Theorem 4.5, we have, for fixed $x > 0$ and $t \in \mathbb{R}$

$$\|\varphi_{\lambda,m}(\Delta_h(x,t)-1)\mathcal{F}_L f\|_{p',\gamma_\alpha} = O\left(h^\gamma \ln(\frac{1}{h})^{-\delta}\right).$$

Therefore, for $\beta \leq p'$

$$\psi(X) = O\left(X^{\frac{(2-\gamma)\beta}{2}+(\alpha+2)\left(1-\frac{\beta}{p'}\right)} \ln(X)^{-\delta\beta}\right).$$

This allows us to deduce, by relation (3.3), that

$$\int_{\mathbb{B}_1^c\cap\mathbb{B}_X} |\mathcal{F}_L f(\lambda,m)|^\beta d\gamma_\alpha = O\left(X^{-\beta+\frac{2-\gamma}{2}\beta+(\alpha+2)\left(1-\frac{\beta}{p'}\right)} \ln(X)^{-\delta\beta}\right) + O\left(\int_1^X t^{-\beta-1} t^{\frac{2-\gamma}{2}\beta+(\alpha+2)\left(1-\frac{\beta}{p'}\right)} \ln(t)^{-\delta\beta}\mathrm{d}t\right).$$

If $\beta > \dfrac{(\alpha+2)p}{(\alpha+2)(p-1)+\frac{\gamma p}{2}}$, then this integral is bounded when X tends to infinity. □

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