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A New Collocation Algorithm for Solving Even-Order Boundary Value Problems via a Novel Matrix Method

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Abstract. This paper is dedicated to presenting and analyzing a numerical algorithm for the solution of even-order boundary value problems. The proposed solutions are spectral and they depend on introducing a new matrix of derivatives of certain shifted Legendre polynomial basis, along with the application of the collocation method. The nonzero elements of the introduced matrix are expressed in terms of the well-known harmonic numbers. Numerical examples provide favorable comparisons with other existing methods and ascertain the efficiency and applicability of the proposed algorithm.

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1. Introduction

We consider the even-order boundary value problems:

$$u^{(2n)}\left(x\right)=f\left(x,\mathbf{u}(x)\right),\quad x\in\left[a,b\right], \tag{1}$$

$$u^{(r)}(a)=\alpha_r,\quad u^{(r)}(b)=\beta_r,\quad r=0,1,\ldots,n-1, \tag{2}$$

where $\mathbf{u}(x)=\left(u(x),u'(x),\ldots,u^{(q)}(x)\right), 0\le q\le 2n-1$ and α_r,β_r are finite real constants.

We assume that $f:[a,b]\times\mathbb{R}^{q+1}\to\mathbb{R}$ is continuous at least in the interior of the domain of interest and it is Lipschitzian in $\mathbf{u}$, which means that there exist nonnegative constants L_k, $k=0,\ldots,q$, such that, for any $\mathbf{u}_1,\mathbf{u}_2$ in $\mathbb{R}^{q+1}$, the following inequality holds:

$$\left|f\left(x,\mathbf{u}_1\right)-f\left(x,\mathbf{u}_2\right)\right|\le\sum_{k=0}^{q}L_k\left|u_1^{(k)}(x)-u_2^{(k)}(x)\right|.$$

Moreover, we assume that the conditions for the existence and uniqueness of solution of Problems (1)–(2) in a certain appropriate domain of $[a,b]\times\mathbb{R}^{q+1}$

are satisfied [4] and that the solution $u(x)$ is differentiable with continuity up to what is necessary.

Even-order boundary value problems (BVPs) arise in a variety of different areas of applied mathematical, engineering and applied physics, in the study of astrophysics, hydrodynamic and hydro magnetic stability, fluid dynamics, astronomy, beam and long-wave theory (see [10,16]).

For example, eighth-order differential equations occur in torsional vibration of uniform beams [7] and in the motion of a circular cylindrical shell subjected to a load that is not symmetric about the axis of the shell [33]. They also occur in the study of the instability setting in an infinite horizontal layer of fluid, which is heated from below and is subject to the action of rotation. When this instability is ordinary convection, it can be modeled by a sixth-order ordinary differential equation; when the instability sets in as overstability, it can be modeled by an eighth-order ordinary differential equation [36].

High even-order BVPs can be involved also when a uniform magnetic field is applied across the fluid in the same direction as gravity. In this case ordinary convection and overstability yield, respectively, a 10th-order and a 12th-order differential equation [36]. Twizell et al. [38] developed numerical methods for eigenvalue problems of 8th-,10th- and 12th- orders arising in thermal instability.

In chemistry, differential equations are used in chemical kinetics, quantum mechanics and transport phenomena such as diffusion.

In most cases, this type of problems cannot be solved analytically; thus only approximate solutions can be expected. For this reason many numerical methods have been proposed in the literature. Some of these methods are finite difference and finite-element methods [11,18], perturbation and homotopy perturbation methods (see [26] and references therein), differential transform methods [28], variational iteration methods [32], collocation methods [13–15], spectral methods [12,23], etc.

Finite difference and finite-element methods are based on local representations of functions, usually by low-order polynomials. However, when a finite difference method has high order or when it is applied to a high-order differential equation, a large number of points are required, and the assigned boundary conditions may be insufficient. On the other hand, low-order finite-difference formulations are often inaccurate, particularly on coarse grids.

In the past decades, spectral methods [9,25] have emerged as a valid alternative to those methods. The basic idea of spectral methods is to use a set of basis functions $\psi_i, i = 0, \ldots, N$, also called trial or expansion approximating functions (very smooth and global such as polynomials), to represent the solution to a problem as a truncated series $\sum_{i=0}^{N} a_i \psi_i(x)$ where a_i are the unknown coefficients.

Finite-element methods are similar in philosophy to spectral algorithms. The major difference is that the trial functions for spectral methods are usually polynomials of high degree, and this assures higher accuracy with respect to finite-element methods. The main advantage of spectral methods is that for problems with smooth solutions they converge exponentially fast compared to algebraic convergence rates for finite difference and finite-element methods and with a degree of accuracy that local methods cannot match. A spectral method is characterized by a specific way to determine the coefficients. For instance, in *collocation* methods the numerical approximation $u_N(x)$ of the solution $u(x)$ to a problem is required to satisfy exactly the given differential equation in a discrete set of points, called collocation nodes, of the interval (a, b). The choice of appropriate spectral representation depends on the type of the differential equation and on the kind of boundary conditions involved in the problem. In recent years, spectral methods are extensively used to solve two-point boundary value problems of high orders. For example, in the series of papers [3,19,20], the authors obtained spectral solutions of high-even and high-odd order BVPs based on the application of Galerkin and Petrov–Galerkin methods. The algorithms in these articles were suitable for handling linear BVPs in one and two dimensions.

Operational matrices of derivatives of various orthogonal polynomials are utilized for handling several types of differential equations. For example, Abd-Elhameed [2] and Doha et al. [22] used Galerkin and tau matrices of derivatives to solve the singular Lane-Emden differential equations. Moreover, Abd-Elhameed in [1] has developed and used a novel harmonic numbers matrix of derivatives to solve linear and nonlinear sixth-order BVPs. Recently, Napoli and Abd-Elhameed in [31] used certain matrices of derivatives for the solution of initial value problems.

In this paper, we aim to solve even-order BVPs using a new collocation algorithm via introducing a novel matrix of derivatives of a certain combination of Legendre polynomials. This matrix generalizes the matrix of derivatives introduced in [1] for the solution of sixth-order BVPs. In [1], the author employed a matrix of derivatives along with the two spectral methods, namely Petrov–Galerkin and collocation methods for handling linear and nonlinear equations. In fact, the so-called Petrov–Galerkin method is utilized for solving linear equations while the collocation method is utilized for solving nonlinear equations. In the present paper, we use a unified collocation algorithm for handling linear and nonlinear even-order BVPs of type (1)–(2). The basis functions are special orthogonal polynomials related to Legendre polynomials.

The outline of the paper is the following: in Sect. 2 we review necessary background and definitions; in Sect. 3 we introduce a new matrix of derivatives of certain combination of Legendre polynomials; in Sect. 4 the proposed algorithm for solving general even-order boundary value problems is described in detail; Sect. 5 is devoted to the study of the convergence of the proposed method. Finally, in Sect. 6, we investigate numerically the proposed algorithm by solving some illustrative examples of some even-order boundary value problems. The results are compared with the approximate solutions obtained by other existing methods. A reliable good accuracy is obtained in all the considered cases.

2. Preliminaries and Used Formulae

In this section we recall some well-known definitions and properties which will be useful hereafter.

Definition 1. The n-th *harmonic number* H_n is the sum of the reciprocals of the first n natural numbers:

$$H_n = \sum_{i=1}^{n} \frac{1}{i}, \quad n = 1, 2, \ldots, \tag{3}$$

and $H_0 = 0$.

Based on their definition, the harmonic numbers satisfy the following recurrence relation:

$$H_n = H_{n-1} + \frac{1}{n}, \quad n \geq 1,$$

and the three-term recurrence relation:

$$(2n-1)H_{n-1} - (n-1)H_{n-2} = n\, H_n, \quad n \geq 2. \tag{4}$$

Definition 2. The *Legendre polynomials* $P_n(x)$, $n \geq 0$ can be defined over the interval $[-1, 1]$ by the recursive formulas [9]

$$\begin{cases} P_0(x) = 1, P_1(x) = x, \\ (n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x), \quad n \geq 1. \end{cases} \tag{5}$$

An important property of the Legendre polynomials is that they are orthogonal with respect to the L^2 inner product on the interval $[-1, 1]$:

$$\int_{-1}^{1} P_m(x)P_n(x)\, \mathrm{d}x = \frac{2}{2n+1}\delta_{mn},$$

where δ_{mn} is the well-known Kronecker delta function.

The shifted Legendre polynomials P_n^* are defined on $[a, b]$:

$$P_n^*(x) = P_n\left(\frac{2x - b - a}{b - a}\right),$$

and they are orthogonal on $[a, b]$ in the sense that

$$\int_a^b P_n^*(x)\, P_m^*(x)\, \mathrm{d}x = \frac{b-a}{2n+1}\delta_{mn}. \tag{6}$$

Moreover, they are a complete sequence in $L^2[a, b]$, for any compact interval $[a, b]$ [30].

Let us denote by $J_i^{(k)}(x)$ the k times repeated integration of $P_i^*(x)$:

$$J_i^{(k)}(x) = \underbrace{\int\int\cdots\int}_{k \text{ times}} P_i^*(x) \underbrace{\mathrm{d}x\,\mathrm{d}x\cdots\mathrm{d}x}_{k \text{ times}}.$$

The following theorem holds [1]:

Theorem 1.

$$J_i^{(k)}(x) = \frac{(b-a)^k}{2^{2k}} \sum_{m=0}^{k} \binom{k}{m} (-1)^m \frac{\left(i+k-2m+\frac{1}{2}\right)\Gamma(i-m+\frac{1}{2})}{\Gamma(i+k-m+\frac{3}{2})}$$
$$P^*_{i+k-2m}(x) + \pi_{k-1}(x),$$

where

$$\pi_{k-1}(x) = \begin{cases} a \text{ polynomial of degree } k-1 \text{ at most} & \text{if } i < k, \\ 0 & \text{if } i \geq k. \end{cases}$$

3. A New Matrix of Derivatives

In this section, we will construct a novel matrix of derivatives for handling even-order BVPs. For this purpose, consider the set of basis functions defined as

$$\psi_{i,n}(x) = (x-a)^n (b-x)^n P_i^*(x), \quad i = 0, 1, 2, \dots \tag{7}$$

From (5), it is clear that the three-term relation is satisfied:

$$\psi_{i,n}(x) = \frac{2i-1}{i} \frac{2x-b-a}{b-a} \psi_{i-1,n}(x) - \frac{i-1}{i} \psi_{i-2,n}(x), \quad i \geq 2. \tag{8}$$

Observe that $\psi_{i,n}(x)$ are orthogonal polynomials in $L_w^2(a,b)$ with respect to the weight function $w(x) = [(x-a)(b-x)]^{-2n}$. In fact, relation (6) yields the orthogonality relation for $\{\psi_{i,n}(x)\}_{i\geq 0}$:

$$\int_a^b \frac{\psi_{i,n}(x)\psi_{j,n}(x)}{(x-a)^{2n}(b-x)^{2n}} \mathrm{d}x = \frac{b-a}{2i+1}\, \delta_{ij}.$$

It can also be easily verified that $\psi_{i,n}(x)$ forms a complete orthogonal system in $L_w^2(a,b)$.

The following theorem expresses the derivative of the polynomials $\psi_{i,n}(x)$ in terms of the polynomials $\psi_{k,n}(x)$, $k = 0, \dots, i-1$.

Theorem 2. *Let $\psi_{i,n}(x)$, $i = 0, 1, 2, \dots$ be the polynomials defined in (7). The following relation holds:*

$$\psi'_{i,n}(x) = \frac{2}{b-a} \sum_{\substack{k=0 \\ (i+k)odd}}^{i-1} (2k+1)(1+2nH_i-2nH_k)\psi_{k,n}(x) + \eta_{i,n}(x), \tag{9}$$

where H_i are the harmonic numbers defined in (3), and

$$\eta_{i,n}(x) = n(x-a)^{n-1}(b-x)^{n-1} \begin{cases} a+b-2x, & i \text{ even}, \\ a-b, & i \text{ odd}. \end{cases}$$

Proof. Let us suppose $[a,b] = [-1,1]$. In this case $\psi_{i,n}(x) = (1-x^2)^n P_i(x)$ and

$$\eta_{i,n}(x) = -2n(1-x^2)^{n-1} \begin{cases} x, & i \text{ even}, \\ 1, & i \text{ odd}. \end{cases}$$

By induction on i, we will show that

$$\psi'_{i,n}(x) = \sum_{\substack{k=0 \\ (i+k)odd}}^{i-1} (2k+1)(1+2nH_i-2nH_k)\psi_{k,n}(x) + \eta_{i,n}(x). \tag{10}$$

It is easy to prove that for $i = 1$ the (10) is true. Let us suppose that relation (10) is valid for i and $(i-1)$. Now we use the same technique as in [1]: by differentiating (8) and by using (4) and the induction hypothesis, relation (10) can be obtained after performing some lengthy manipulations.

Finally, relation (9) can be obtained after some calculations, by replacing x in (10) by $\dfrac{2x-b-a}{b-a}$. □

Let us denote $\boldsymbol{\psi}(x) = [\psi_{0,n}(x), \psi_{1,n}(x), \ldots, \psi_{N,n}(x)]$.

Corollary 1. *The first derivative of* $\boldsymbol{\psi}(x)$ *can be written as*

$$\boldsymbol{\psi}'(x) = \boldsymbol{\psi}(x)\, S + \boldsymbol{\eta}_n(x), \tag{11}$$

where $\boldsymbol{\eta}_n(x) = (\eta_{0,n}(x), \eta_{1,n}(x), \ldots, \eta_{n,N}(x))$, *and* $S = (s_{i,j})$ *is an* $(N+1) \times (N+1)$ *upper triangular matrix defined as*

$$s_{i,j} = \begin{cases} \dfrac{2}{b-a}(2j+1)(1+2nH_i-2nH_j), \; i<j, & (i+j) \; odd, \\ 0, & otherwise. \end{cases}$$

Remark 1. It is worthy here to mention that relation (11) for $n = 0$ reduces to

$$\boldsymbol{\psi}'(x) = \boldsymbol{\psi}(x)\, S. \tag{12}$$

In this case, S is the operational matrix of derivatives of the shifted Legendre polynomials defined on $[a, b]$ which is derived and employed in a variety of papers. Bolek [8] derived this operational matrix for $[a, b] \equiv [-1, 1]$.

Corollary 2. *The* r*-th derivative of* $\boldsymbol{\psi}(x)$, $r = 0, \ldots, 2n+N$, *is*

$$\boldsymbol{\psi}^{(r)}(x) = \begin{cases} \boldsymbol{\psi}(x)S^r + \displaystyle\sum_{j=0}^{r-1} S^{r-j-1}\boldsymbol{\eta}_n^{(j)}(x), & r = 1\ldots, N, \\ \displaystyle\sum_{j=0}^{r-1} S^{r-j-1}\boldsymbol{\eta}_n^{(j)}(x), & r > N, \end{cases} \tag{13}$$

with $S^0 = I$.

4. A Collocation Algorithm for Handling the Even-Order BVPs

This section is devoted to describing the suggested collocation algorithm for solving even-order two-point BVPs. The harmonic numbers matrix of derivatives introduced in Sect. 3 will be employed for this purpose.

4.1. Homogeneous Boundary Conditions

Suppose that the boundary conditions are homogeneous, that is

$$u^{(r)}(a) = u^{(r)}(b) = 0, \quad r = 0, \ldots, n-1. \tag{14}$$

Let $H_w^n(a,b)$ be the weighted Sobolev space

$$H_w^n(a,b) = \left\{ v \in L_w^2(a,b) : \text{ for } 0 \le k \le n,\ v^{(k)} \in L_w^2(a,b) \right\},$$

where the last derivative is in the sense of distributions (see [9]).

Let us define the following subspace of $H_w^n(a,b)$:

$$H_{0,w}^n(a,b) = \left\{ v \in H_w^n(a,b) : \text{ for } 0 \le j \le n-1,\ v^{(j)}(a) = v^{(j)}(b) = 0 \right\}.$$

Any function $u(x) \in H_{0,w}^n(a,b)$ can be expanded in terms of the orthogonal functions $\psi_{i,n}(x)$

$$u(x) = \sum_{i=0}^{\infty} a_i \psi_{i,n}(x), \tag{15}$$

with

$$a_i = \frac{2i+1}{b-a} \int_a^b \frac{u(x)\ \psi_{i,n}(x)}{(x-a)^{2n}(b-x)^{2n}}\, \mathrm{d}x. \tag{16}$$

Let $u(x)$ be the solution of (1), (14) and let us suppose $u(x) \in H_{0,w}^n(a,b)$. If N is large enough, we can consider an approximate solution to $u(x)$ in the form

$$u(x) \approx u_N(x) = \sum_{i=0}^{N} a_i\, \psi_{i,n}(x). \tag{17}$$

Observe that $u_N(x)$ satisfies the boundary conditions (14).

If we put

$$A^T = [a_0, a_1, \ldots, a_N],$$

then $u_N(x) = \boldsymbol{\psi}(x)A$ and the differential equation (1) can be written as

$$\boldsymbol{\psi}^{(2n)}(x)A = f\left(x, \boldsymbol{\psi}(x)A, \boldsymbol{\psi}'(x)A, \ldots, \boldsymbol{\psi}^{(q)}(x)A\right). \tag{18}$$

Let $\{x_j\}_{j=0}^N$ be $(N+1)$ distinct points in $[a,b]$. In order to find numerical approximations to the solution of Problems (1)–(14), we enforce (18) to be satisfied exactly at $x_j, j = 0, \ldots, N$:

$$\boldsymbol{\psi}^{(2n)}(x_j)A = f\left(x_j, \boldsymbol{\psi}(x_j)A, \boldsymbol{\psi}'(x_j)A, \ldots, \boldsymbol{\psi}^{(q)}(x_j)A\right), \quad j = 0, \ldots, N. \tag{19}$$

Equation (19) constitutes a system of $(N+1)$ equations in the unknowns $a_0, \ldots, a_N$. It can be written as

$$\Omega A = F(A), \tag{20}$$

where

$$\Omega = \begin{bmatrix} \psi_{0,n}^{(2n)}(x_0) & \cdots & \psi_{N,n}^{(2n)}(x_0) \\ \vdots & & \vdots \\ \psi_{0,n}^{(2n)}(x_N) & \cdots & \psi_{N,n}^{(2n)}(x_N) \end{bmatrix},$$

and

$$F(A) = \begin{bmatrix} f\left(x_0, \boldsymbol{\psi}(x_0)A, \boldsymbol{\psi}'(x_0)A, \ldots, \boldsymbol{\psi}^{(q)}(x_0)A\right) \\ \vdots \\ f\left(x_N, \boldsymbol{\psi}(x_N)A, \boldsymbol{\psi}'(x_N)A, \ldots, \boldsymbol{\psi}^{(q)}(x_N)A\right) \end{bmatrix}.$$

Since the polynomials $\psi_{i,n}^{(2n)}(x)$, $i = 0, \ldots, N$, are orthogonal, they are linearly independent and form a basis for the space of polynomials of degree at most N. Hence, the $(N+1)$ vectors

$$\left[\psi_{0,n}^{(2n)}(x_0), \ldots, \psi_{0,n}^{(2n)}(x_N)\right], \left[\psi_{1,n}^{(2n)}(x_0), \ldots, \psi_{1,n}^{(2n)}(x_N)\right], \ldots, \\ \left[\psi_{N,n}^{(2n)}(x_0), \ldots, \psi_{N,n}^{(2n)}(x_N)\right]$$

are linearly independent so that Ω is a nonsingular matrix.

Observe that if $\omega_{ij} = \psi_{j-1,n}^{(2n)}(x_{i-1})$, $i, j = 1, \ldots, N+1$, are the entries of Ω, then $\omega_{i,1} = (-1)^n(2n)!$ for $i = 1, \ldots, N+1$. Also, if the nodes are symmetric with respect to $(a+b)/2$, then

$$\omega_{i,j} = (-1)^{j+1}\omega_{N-i+2,j}, \quad i = 1, \ldots, N+1, \; j = 2, \ldots, N+1.$$

Observe that Ω depends on n, N and on $\{x_j\}_{j=0}^{N}$.

In order to study system (20), we analyze the behavior of $\left\|\Omega^{-1}\right\|_\infty$ for different values of n, N and different sets of points in $[a, b] = [0, 1]$ (the zeros of Chebyshev polynomials of first kind (Cheb I), the zeros of Chebyshev polynomials of second kind (Cheb II), the zeros of Legendre polynomials (Legen), equidistant points in $[a, b]$ (EqPts)). A deeper theoretical study will be done in the future.

Tables 1, 2 and 3 show the numerical results.

Theorem 3. *Suppose that f is Lipschitzian in $\mathbf{u}$, with nonnegative constants L_k. If $T = \left\|\Omega^{-1}\right\|_\infty \sum_{k=0}^{q} L_k < 1$, then the system (20) has a unique solution which can be calculated by an iterative method*

$$A^{(\mu+1)} = G\left(A^{(\mu)}\right), \quad \mu = 1, 2, \ldots \tag{21}$$

with $A^{(0)}$ fixed and

Table 1. Different values of $\left\|\Omega^{-1}\right\|_\infty$ for $n = 4$

$n = 4$	$\left\|\Omega^{-1}\right\|_\infty$				
	$N = 2$	$N = 4$	$N = 10$	$N = 15$	$N = 20$
Cheb I	2.48×10^{-5}	3.45×10^{-5}	2.54×10^{-5}	2.48×10^{-5}	2.48×10^{-5}
Cheb II	2.48×10^{-5}	2.73×10^{-5}	2.48×10^{-5}	2.48×10^{-5}	2.48×10^{-5}
Legen	2.48×10^{-5}	2.53×10^{-5}	2.48×10^{-5}	2.48×10^{-5}	2.48×10^{-5}
Eqpts	2.48×10^{-5}	2.49×10^{-5}	2.48×10^{-5}	2.48×10^{-5}	2.48×10^{-5}

Table 2. Different values of $\|\Omega^{-1}\|_\infty$ for $n = 5$

$n = 5$	$\|\Omega^{-1}\|_\infty$				
	$N = 2$	$N = 4$	$N = 10$	$N = 15$	$N = 20$
Cheb I	2.76×10^{-7}	3.72×10^{-7}	2.89×10^{-7}	2.76×10^{-7}	2.76×10^{-7}
Cheb II	2.76×10^{-7}	3.01×10^{-7}	2.75×10^{-7}	2.76×10^{-7}	2.75×10^{-7}
Legen	2.76×10^{-7}	2.84×10^{-7}	2.76×10^{-7}	2.76×10^{-7}	2.76×10^{-7}
Eqpts	2.76×10^{-7}	2.78×10^{-7}	2.76×10^{-7}	2.76×10^{-7}	2.76×10^{-7}

Table 3. Different values of $\|\Omega^{-1}\|_\infty$ for $n = 6$

$n = 6$	$\|\Omega^{-1}\|_\infty$				
	$N = 2$	$N = 4$	$N = 10$	$N = 15$	$N = 20$
Cheb I	2.08×10^{-9}	2.75×10^{-9}	2.26×10^{-9}	2.09×10^{-9}	2.09×10^{-9}
Cheb II	2.08×10^{-9}	2.27×10^{-9}	2.10×10^{-9}	2.09×10^{-9}	2.09×10^{-9}
Legen	2.09×10^{-9}	2.16×10^{-9}	2.09×10^{-9}	2.09×10^{-9}	2.09×10^{-9}
Eqpts	2.08×10^{-9}	2.11×10^{-9}	2.09×10^{-9}	2.09×10^{-9}	2.09×10^{-9}

$$G(A) = \Omega^{-1}F(A).$$

Moreover,

$$\left\|A - A^{(\mu)}\right\|_\infty \leq \frac{T^\mu}{1-T}\left\|A^{(1)} - A^{(0)}\right\|_\infty . \tag{22}$$

Proof. If Ω is invertible, then the system in (20) can be written in the form

$$A = \Omega^{-1}F\left(A\right).$$

Putting $G(A) = \Omega^{-1}F(A)$, we have $A = G\left(A\right)$. Then, $\forall$ fixed N, if $X_1, X_2 \in \mathbb{R}^{N+1}$,

$$G\left(X_1\right) - G\left(X_2\right) = \Omega^{-1}\left[F\left(X_1\right) - F\left(X_2\right)\right]$$

and

$$\left\|G\left(X_1\right) - G\left(X_2\right)\right\|_\infty \leq \left\|\Omega^{-1}\right\|_\infty \sum_{k=0}^{q} L_k \left\|X_1 - X_2\right\|_\infty .$$

Hence, if $T < 1$, G is contractive. The proof goes on with usual techniques. □

Remark 2. From Tables 1, 2 and 3, the hypothesis $\left\|\Omega^{-1}\right\|_\infty \sum_{k=0}^{q} L_k < 1$ of Theorem 3 can be satisfied at least for $n = 4, 5, 6$, for N less than or equal to 20 and when the nodes $\{x_j\}_{j=0}^{N}$ are the Chebyshev and Legendre zeros or equidistant points in $[a, b]$.

Remark 3. Under the hypothesis of Theorem 3, if ν is the number of iterations required by method (21) for a fixed tolerance, let $A^{(\nu)} = \left[a_0^{(\nu)}, a_1^{(\nu)}, \ldots, a_N^{(\nu)}\right]$ be the solution of system (20). Let us denote by $u_{\nu,N}(x)$ the numerical solution of Problems (1)–(2)

$$u_{\nu,N}(x) = \sum_{i=0}^{N} a_i^{(\nu)} \psi_{i,n}(x).$$

Then, for all $x \in (a, b)$,

$$u_N(x) - u_{\nu,N}(x) = \sum_{i=0}^{N} \left(a_i - a_i^{(\nu)}\right) \psi_{i,n}(x) = \boldsymbol{\psi}(x) \left(A - A^{(\nu)}\right),$$

and

$$|u_N(x) - u_{\nu,N}(x)| \leq \frac{T^\nu}{1-T} \left\|A^{(1)} - A^{(0)}\right\|_\infty (b-a)^{2n},$$

since $|P_j^*(x)| \leq 1$, for $x \in [a, b]$ (see [9]).

Remark 4. If f is linear, that is $f(x, \mathbf{u}(x)) = \sum_{i=0}^{2n-1} h_i(x) u^{(i)}(x) + g(x)$, then system (19) is linear too

$$\sum_{i=0}^{2n} h_i(x_j) \boldsymbol{\psi}^{(i)}(x_j) A = -g(x_j), \quad j = 0, \ldots, N,$$

with $h_{2n}(x) = -1$. In matrix form:

$$HA = G,$$

with $G = [g(x_0), \ldots, g(x_N)]^T$ and

$$H = \begin{pmatrix} u_0(x_0) & \cdots & u_N(x_0) \\ \vdots & \ddots & \vdots \\ u_0(x_N) & \cdots & u_N(x_N) \end{pmatrix},$$

where $u_i(x_k) = \sum_{j=0}^{2n} h_j(x_k) \psi_i^{(j)}(x_k)$.

4.2. Nonhomogeneous Boundary Conditions

If the boundary conditions are nonhomogeneous as in (2), then the problem can be easily transformed into an equivalent problem subject to homogeneous boundary conditions [21] by the transformation:

$$U(x) = u(x) + \sum_{i=0}^{2n-1} c_i x^i,$$

where $c_i, i = 0, \ldots, 2n-1$ are constants to be determined such that

$$U^{(r)}(a) = U^{(r)}(b) = 0 \quad r = 0, \ldots, n-1.$$

To this aim, we solve the system

$$\begin{cases} \sum_{i=r}^{2n-1} \dfrac{i!}{(i-r)!} a^{i-r} c_i = -\alpha_r, & r = 0, \ldots, n-1, \\ \sum_{i=r}^{2n-1} \dfrac{i!}{(i-r)!} b^{i-r} c_i = -\beta_r, & r = 0, \ldots, n-1, \end{cases}$$

which can be written alternatively in the matrix form

$$M\,\mathbf{c} = \boldsymbol{\gamma},$$

where $\boldsymbol{c} = (c_0, \ldots, c_{2n-1})^T$, $\boldsymbol{\gamma} = (\alpha_0, \ldots, \alpha_{n-1}, \beta_0, \ldots, \beta_{n-1})^T$ and the elements m_{ij}, $i, j = 1, \ldots 2n$ of M are defined as follows:

for i even

$$m_{ij} = \begin{cases} 0, & 1 \le j < \frac{i}{2}, \\ \dfrac{(j-1)!}{(j-\frac{i}{2})!}\, b^{j-\frac{i}{2}}, & \frac{i}{2} \le j \le 2n, \end{cases}$$

and for i odd,

$$m_{ij} = \begin{cases} 0, & 1 \le j \le \frac{i-1}{2}, \\ \dfrac{(j-1)!}{(j-\frac{i+1}{2})!} a^{j-\frac{i+1}{2}}, & \frac{i-1}{2} < j \le 2n. \end{cases}$$

Let $Q(x) = \sum_{i=0}^{2n-1} c_i x^i$. Now we solve the problem

$$\begin{cases} U^{(2n)}(x) = f\left(x, (\mathbf{U} - \mathbf{Q})(x)\right), & x \in [a, b], \\ U^{(r)}(a) = U^{(r)}(b) = 0, & r = 0, \ldots, n-1, \end{cases} \tag{23}$$

where

$$(\mathbf{U} - \mathbf{Q})(x) = \left(u(x) - Q(x), u'(x) - Q'(x), \ldots, u^{(q)}(x) - Q^{(q)}(x)\right), \quad 0 \le q \le 2n-1.$$

5. Convergence of the Method

In this section, we will investigate the convergence analysis of the presented method.

Suppose that

$$u(x) = (x-a)^n (b-x)^n g(x), \tag{24}$$

with $g(x) \in C^n[a, b]$ and $|g^{(n)}(x)| \le M$, for all $x \in [a, b]$, and M a positive integer.

First we will derive an upper bound for the considered shifted Legendre expansion (15).

Theorem 4. *The expansion coefficients a_i in (15) satisfy the inequality*

$$|a_j| \le \frac{M(b-a)^n}{j^{n-1}}, \quad j \ge n, \tag{25}$$

and the series (15) converges uniformly to $u(x)$ in the L^2_w norm.

Proof. From (25) and (7), if $u(x)$ is defined as in (24), we have

$$a_j = \frac{2j+1}{b-a}\int_a^b \frac{u(x)P_j^*(x)}{(x-a)^n(b-x)^n}\mathrm{d}x = \frac{2j+1}{b-a}\int_a^b g(x)P_j^*(x)\,\mathrm{d}x. \tag{26}$$

Now we integrate the right-hand side of (26) by parts n times. For $i \geq n$, if we make use of Theorem 1, we get

$$a_j = (-1)^n \frac{2j+1}{b-a}\int_a^b g^{(n)}(x) I^{(n)}(x)\mathrm{d}x,$$

where

$$I^{(n)}(x) = \frac{(b-a)^n}{2^{2n}}\sum_{\ell=0}^{n}\binom{n}{\ell}(-1)^\ell \frac{\left(j+n-2\ell+\frac{1}{2}\right)\Gamma\left(j-\ell+\frac{1}{2}\right)}{\Gamma\left(j+n-\ell+\frac{3}{2}\right)} P^*_{j+n-2\ell}(x).$$

Since $|P_j^*(x)| \leq 1$, for $x \in [a,b]$ [9], it is easy to show that the following inequality holds:

$$|I^{(n)}(x)| \leq \frac{(b-a)^n}{2^{2n}}\sum_{\ell=0}^{n}\binom{n}{\ell}\left|\frac{\left(j+n-2\ell+\frac{1}{2}\right)\Gamma\left(j-\ell+\frac{1}{2}\right)}{\Gamma\left(j+n-\ell+\frac{3}{2}\right)}\right|.$$

Hence

$$\begin{aligned} |a_j| &\leq M\frac{(j+\frac{1}{2})(b-a)^n}{2^{2n-1}}\sum_{\ell=0}^{n}\binom{n}{\ell}\frac{j+n-2\ell+\frac{1}{2}}{(j-\ell+\frac{1}{2}+n)(j-\ell+\frac{1}{2}+n-1)\cdots(j-\ell+\frac{1}{2})} \\ &= M\frac{(b-a)^n}{2^{2n-1}}\sum_{\ell=0}^{n}\binom{n}{\ell}\frac{1}{\prod_{\substack{s=0\\ s\neq\ell}}^{n-1}(j-\ell+n-s+\frac{1}{2})} \\ &\leq M\frac{(b-a)^n}{2^{2n-1}}\sum_{\ell=0}^{n}\binom{n}{\ell}\frac{1}{(j-\ell+\frac{1}{2})^{n-1}} \\ &\leq M\frac{(b-a)^n}{2^{n}}\sum_{\ell=0}^{n}\binom{n}{\ell}\frac{1}{(2j-2\ell+1)^{n-1}} < \frac{L(b-a)^n}{j^{n-1}}. \end{aligned}$$

The uniform convergence of the series follows from (25) and the completeness of system $\{\psi_{i,n}(x)\}_{i\geq 0}$. □

Theorem 5. *For all $n \geq 0$, the following upper bound holds:*

$$\|u-u_N\|_w \leq \frac{M(b-a)^{n+\frac{1}{2}}}{2\sqrt{n-1}N^{n-1}}. \tag{27}$$

Proof.

$$\|u-u_N\|_w^2 = \sum_{j=N+1}^{\infty}\frac{b-a}{2j+1}\,a_j^2.$$

From Theorem 4

$$\|u-u_N\|_w^2 < (b-a)\sum_{j=N+1}^{\infty}\frac{M^2(b-a)^{2n}}{(2j+1)j^{2n-1}} < \frac{M^2(b-a)^{2n+1}}{2}\sum_{j=N+1}^{\infty}\frac{1}{j^{2n-1}}.$$

From the integral test for the convergence of the series, we know that the series $\sum_{i=1}^{\infty} \frac{1}{i^{2n-1}}$ is convergent for $n > 1$ and for the remainder $R_n = \sum_{i=N+1}^{\infty} \frac{1}{i^{2n-1}}$, the following inequality holds:

$$R_N \leq \int_N^{\infty} \frac{1}{x^{2n-1}} \mathrm{d}x.$$

Hence

$$\|u - u_N\|_w^2 \leq \frac{M^2(b-a)^{2n+1}}{4(n-1)N^{2n-2}},$$

and

$$\|u - u_N\|_w \leq \frac{M(b-a)^{n+\frac{1}{2}}}{2\sqrt{n-1}N^{n-1}}$$

and the thesis follows. □

In order to study the convergence of the method, we first prove the following Lemma.

Lemma 1. *Under the hypothesis of Theorem 3, for all N the error $\|u_N - u_{\nu,N}\|_w$ converges to zero as $\nu \to \infty$.*

Proof. From Remark 3, since $\lim_{\nu\to\infty} a_i^{(\nu)} = a_i$, $i = 0, \ldots, N$, for all N we get

$$\lim_{\nu\to\infty} \|u_N - u_{\nu,N}\|_w = 0.$$

□

Theorem 6. *Under the hypothesis of Theorem 3,*

$$\lim_{\substack{\nu\to\infty \\ N\to\infty}} \|u - u_{\nu,N}\|_w = 0,$$

where u and $u_{\nu,N}$ are, respectively, the exact and the numerical solution of Problems (1)–(2).

Proof. If $u_N(x)$ is defined as in (17), we get

$$\|u - u_{\nu,N}\|_w \leq \|u - u_N\|_w + \|u_N - u_{\nu,N}\|_w.$$

The result follows from Theorem 5 and Lemma 1. □

6. Numerical Examples

Now we present some numerical results obtained by applying the proposed method, which we call *LegBVP* method, to find numerical approximations of the solutions of some test problems. As the true solutions are known, we considered the error function $e(x) = |u(x) - u_{\nu,N}(x)|$ where $u_{\nu,N}(x)$ is the approximate solution. As collocation points we used the zeros of Chebyshev polynomials of second kind.

All computations are carried out using `Mathematica`, ver. 10. Particularly, the nonlinear system (18) is solved by the `NSolve` routine. For systems of algebraic equations, `NSolve` computes a numerical Gröbner basis using an efficient monomial ordering, then uses eigensystem methods to extract numerical roots.

In most cases analogous results are obtained by using equidistant points or the zeros of Chebyshev polynomials of the first kind.

Example 1. The computation of diffusion and reaction inside a small porous catalyst sphere (a pellet) is an important problem in chemical engineering. The goal is to predict the overall reaction rate, or the mass transfer into and out of the pellet. Suppose that the spherical pellet of radius r_p is isothermal and that within the pellet a substance (platinum) is distributed in order to catalyze the dehydrogenation of cyclohexane. The conservation of mass in a spherical region $0.1 \leq r \leq r_p$ is modeled by the boundary value problem [17]:

$$\begin{cases} D\left[\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}c}{\mathrm{d}r}\right)\right] = kg(c), \\ c(0.1) = 10\frac{\sinh(0.1\phi)}{c_0\sinh(\phi)}, \quad c(1) = c_0, \end{cases} \tag{28}$$

where c is the concentration of cyclohexane, D is the diffusion constant, k is the reaction rate constant, $\phi = r_p\sqrt{\frac{k}{D}}$, while the reaction rate function $g(c)$ defines the dependence on concentration.

By substituting $r/r_p \to R$, $c(r)/c(r_p) \to C(R)$, Problem (28) can be written in the following dimensionless form:

$$\begin{cases} \frac{\mathrm{d}^2C}{\mathrm{d}R^2} + \frac{2}{R}\frac{\mathrm{d}C}{\mathrm{d}R} = \phi^2 g(C), \\ C(0.1) = 10\frac{\sinh(0.1\phi)}{\sinh(\phi)}, \quad C(1) = 1, \end{cases} \tag{29}$$

where

$$C = \frac{\text{concentration of cyclohexane}}{\text{concentration of cyclohexane at the surface of the sphere}}.$$

Assume that the diameter of the pellet is 5 mm and the temperature 700 K. At 700 K the realistic parameter values are $k = 4\,\mathrm{s}^{-1}$, $D = 0.05\,\mathrm{cP}^2/\mathrm{s}$ so that $\phi \approx 2.236$.

If $g(C) = C$, the analytic solution is

$$C(R) = \frac{\sinh(\phi R)}{\sinh(\phi)}.$$

The maximum absolute errors for different values of N by *LegBVP* method are compared with the ones obtained by using the Matlab tool `Chebfun` [24]. They are displayed in Table 4.

Example 2 [5,27,34]. Consider the following eighth-order boundary value problem:

$$\begin{cases} u^{(8)}(x) + xu(x) = -(48 + 15x + x^3)e^x, \quad x \in [0,1], \\ u(0) = 0, \quad u(1) = 0, \\ u'(0) = 1, \quad u'(1) = -e, \\ u''(0) = 0, \quad u''(1) = -4e, \\ u'''(0) = -3, \ u'''(1) = -9e. \end{cases} \tag{30}$$

Table 4. Maximum absolute error in Problem (29)

Error in *LegBVP*		`Chebfun`
$N = 10$	$N = 16$	
7.4066×10^{-14}	3.0531×10^{-16}	6.7525×10^{-15}

Table 5. Absolute errors and comparisons for Problem (30)

x	Method in [27]	Method in [34]	Method in [5]	*LegBVP* $N = 4$	*LegBVP* $N = 8$
0.1	3.73×10^{-9}	5.62×10^{-10}	1.63×10^{-10}	2.17×10^{-13}	6.23×10^{-18}
0.2	6.61×10^{-9}	4.88×10^{-9}	1.63×10^{-9}	1.93×10^{-12}	1.94×10^{-18}
0.3	2.33×10^{-8}	1.37×10^{-8}	4.90×10^{-9}	4.13×10^{-12}	1.08×10^{-17}
0.4	5.17×10^{-8}	2.29×10^{-8}	8.46×10^{-9}	3.04×10^{-12}	1.40×10^{-17}
0.5	9.76×10^{-8}	2.71×10^{-8}	1.01×10^{-8}	1.98×10^{-12}	2.34×10^{-17}
0.6	1.78×10^{-6}	2.38×10^{-8}	8.68×10^{-9}	6.37×10^{-12}	1.03×10^{-17}
0.7	6.03×10^{-12}	1.49×10^{-8}	5.15×10^{-9}	2.53×10^{-12}	1.99×10^{-17}
0.8	1.83×10^{-8}	5.54×10^{-8}	1.76×10^{-9}	3.1×10^{-4}	6.06×10^{-17}

The analytic solution is $u\,(x) = x(1-x)e^{x}$.

The absolute errors obtained by using the proposed algorithm *LegBVP* for $N = 4$ and $N = 8$ are compared with those obtained by applying the three methods:

- Adomian decomposition method in [27].
- Non-polynomial spline technique in [34].
- Reproducing kernel space in [5].

As we can see from Table 5, we obtain smaller error by using *LegBVP* method than by applying the above mentioned methods even when N is small.

Example 3. Consider the nonlinear tenth-order boundary value problem [29]:

$$\begin{cases} u^{(10)}(x) - u'''(x) = 2e^{x}u^{2}(x), \quad x \in [0,1], \\ u^{(i)}(0) = (-1)^{i}, \quad u^{(i)}(1) = \frac{(-1)^{i}}{e}, \quad i = 0, \ldots, 4. \end{cases} \tag{31}$$

The exact solution of the above problem is $u\,(x) = e^{-x}$.

The proposed method compares favorably with the quintic B-spline collocation method presented in [29], as Table 6 shows.

Table 6. Absolute errors and comparisons for Problem (31)

x	Method in [29]	*LegBVP* $N=2$
0.1	4.82×10^{-6}	4.69×10^{-16}
0.2	2.44×10^{-6}	7.01×10^{-15}
0.3	1.48×10^{-5}	1.99×10^{-14}
0.4	1.78×10^{-5}	2.37×10^{-14}
0.5	1.23×10^{-5}	1.02×10^{-14}
0.6	4.78×10^{-6}	5.99×10^{-15}
0.7	4.59×10^{-6}	7.74×10^{-15}
0.8	5.72×10^{-6}	4.52×10^{-15}
0.9	2.23×10^{-6}	1.99×10^{-14}

Table 7. Absolute errors and comparisons for Problem (32)

x	Method in [28]	*LegBVP* $N=2$
0.1	7.51×10^{-14}	1.21×10^{-16}
0.2	2.77×10^{-12}	2.91×10^{-15}
0.3	1.73×10^{-13}	8.95×10^{-15}
0.4	5.02×10^{-11}	5.80×10^{-15}
0.5	9.34×10^{-11}	1.06×10^{-14}
0.6	1.28×10^{-10}	1.99×10^{-14}
0.7	1.39×10^{-10}	4.23×10^{-15}
0.8	1.23×10^{-10}	3.56×10^{-14}
0.9	7.45×10^{-11}	1.10×10^{-13}
1.0	1.95×10^{-11}	1.42×10^{-14}

Example 4. Consider the linear twelfth-order boundary value problem [28, 35, 37]

$$\begin{cases} u^{(12)}(x) + xu(x) = -(120 + 23x + x^3)e^x, \quad x \in [0,1], \\ u(0) = 0, \qquad u(1) = 0, \\ u'(0) = 1, \qquad u'(1) = -e, \\ u''(0) = 0, \qquad u''(1) = -4e, \\ u'''(0) = -3, \quad u'''(1) = -9e, \\ u^{(4)}(0) = -8, \quad u^{(4)}(1) = -16e, \\ u^{(5)}(0) = -15, \; u^{(5)}(1) = -25e. \end{cases} \tag{32}$$

The exact solution $u\,(x) = x(1-x)e^x$.

In Table 7, we compare the results obtained by the proposed method for $N = 2$, with those obtained by the application of the differential transform method developed in [28]. The table shows that our results are more accurate

Table 8. Comparison between different methods for Problem (32)

Best error	Method in [37]	Method in [35]	Method in [37]	*LegBVP* $N = 2$
E	2.7×10^{-3}	7.38×10^{-9}	1.39×10^{-10}	1.10×10^{-13}

Table 9. Absolute errors and comparisons for Problem (5)

x	Method in [28]	Method in [6]	*LegBVP* $N = 4$
0.1	4.11×10^{-15}	7.93×10^{-16}	0.00
0.2	1.30×10^{-13}	2.21×10^{-14}	3.33×10^{-16}
0.3	6.75×10^{-13}	1.11×10^{-13}	1.30×10^{-14}
0.4	1.53×10^{-12}	2.46×10^{-13}	6.59×10^{-14}
0.5	1.98×10^{-12}	3.12×10^{-13}	1.48×10^{-13}
0.6	1.57×10^{-12}	2.43×10^{-13}	1.89×10^{-13}
0.7	7.17×10^{-13}	1.15×10^{-13}	1.45×10^{-13}
0.8	1.42×10^{-13}	1.40×10^{-14}	5.53×10^{-14}
0.9	4.16×10^{-15}	1.97×10^{-14}	2.26×10^{-14}
1.0	1.22×10^{-15}	2.26×10^{-13}	9.46×10^{-14}

than those in [28]. In addition, in Table 8, we display the best absolute errors obtained by different methods.

Example 5. Consider the nonlinear twelfth-order boundary value problem [6, 28]

$$\begin{cases} u^{(12)}(x) = 2e^{x}u^{2}(x) + u^{(3)}(x), \quad x \in [0, 1], \\ u^{(k)}(0) = (-1)^{k}, \quad u^{(k)}(1) = (-1)^{k}e^{-1}, \quad k = 0, \ldots, 5. \end{cases} \tag{33}$$

The analytic solution of the above problem is $u\,(x) = e^{-x}$.

The proposed method compares favorably with the methods presented in [28] and [6], as Table 9 shows.

7. Conclusions

In this paper we have obtained new numerical solutions for general even-order two-point boundary value problems. The basic idea behind the suggested solutions is to introduce matrix of derivatives as a certain combination of Legendre polynomials along with the application of the collocation spectral method. Given a set of $(N+1)$ distinct points in (a, b), the true solution $u(x)$ of the problem is approximated by a polynomial $u_N(x)$ of degree $(2n + N)$. $u_N(x)$ is the truncated series expansion of $u(x)$ in terms of orthogonal polynomials related to Legendre polynomials. The coefficients of the expansion are determined by requiring that the differential equation is satisfied exactly at the $(N + 1)$ given points. Convergence and error analysis of the expansion has been carefully studied. Numerical examples support theoretical results and show that high accuracy in the approximation is achieved with small values of N.

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