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Positive Solution of Extremal Pucci's Equations with Singular and Sublinear Nonlinearity

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Abstract. In this paper, we establish the existence of a positive solution to

$$\begin{cases} -\mathcal{M}^+_{\lambda,\Lambda}(D^2u) = \frac{\mu k(x) f(u)}{u^\alpha} - \eta h(x) u^q & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$, $n \geq 1$. Under certain conditions on k, f and h, using viscosity sub- and super solution method with the aid of comparison principle, we establish the existence of a unique positive viscosity solution. This work extends and complements the earlier works on semilinear and singular elliptic equations with sublinear nonlinearity.

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1. Introduction

The goal of this paper is to establish the existence of a positive solution to

$$\begin{cases} -\mathcal{M}^+_{\lambda,\Lambda}(D^2u) = \frac{\mu k(x) f(u)}{u^\alpha} - \eta h(x) u^q & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$, $n \geq 1$ is a smooth bounded domain, $0 < \alpha,\ q < 1$ and μ, η are nonnegative parameters and $\mathcal{M}^+_{\lambda,\Lambda}$ is the extremal Pucci's operator, defined below. We will specify the conditions on k, f and h later. For a given λ, Λ satisfying $0 < \lambda \leq \Lambda < \infty$, extremal Pucci's operator is defined as follows:

$$\mathcal{M}^+_{\lambda,\Lambda}(M) = \Lambda \sum_{e_i > 0} e_i + \lambda \sum_{e_i < 0} e_i,$$

where e_is are the eigenvalues of M and $M \in S_{n\times n}$, where $S_{n\times n}$ is the set of all $n \times n$ real symmetric matrices. In case when $\lambda = \Lambda = 1$, it is easy to see that

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$$\mathcal{M}^{+}_{\lambda,\Lambda}(D^2u) = \Delta u.$$

There have been a good amount of research works for singular semilinear elliptic equations of type

$$\begin{cases} -\Delta u = \frac{\mu k(x) f(u)}{u^{\alpha}} - \eta h(x) u^q & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $\alpha > 0$. These problems arise in several branches of engineering and sciences and also have interesting applications such as steady state of thin films [1,2], modeling of MEMS devices [21]. There is a vast literature on this subject, but we just name those article which are closely related to this article. Crandall et al. [12] studied (1.2) for the existence of a solution in case when $\mu f(u) = 1$ and $\eta = 0$. Lazer and Mackenna [19] also dealt with the case $\mu f(u) = 1$ and $\eta = 0$ in (1.2) and established the existence and regularity results for the solution at the boundary.

Recently, Yijing and Wu [29] established the existence of a solution to (1.2), where $q > 1$ and $\eta h(x) = -\gamma > 0$ and $\mu f(u) = 1$. Their precise theorem says that $\exists\ \gamma^* > 0$ such that (1.2) has a solution for all $\gamma \in (0, \gamma^*)$ and no solution for $\gamma > \gamma^*$ see also [10,11]. Very recently, Yijing [30] established the existence of a solution to (1.2) in case when $\mu f(u) = 1$ and $\eta = -1$ and $\alpha > 1$. There are several other interesting papers in this direction which deals with similar type of singular equations as (1.2), see for instance [5,6,18,20,27,28] and the references therein.

We remark that recently fully nonlinear elliptic equations with singularity has been considered in [16], where the authors showed the existence, uniqueness and regularity of the solutions up to the boundary. When $q = 1$ in (1.1), then this kind of equations are studied in [16] and for the choice of $k(x) = 0$ in (1.1), this equation reduces to the equation considered in [22]. There have been also a good amount of interest on the existence and regularity questions to equations involving extremal Pucci's operator, see [14,15], where authors obtained Hölder estimate for the gradient of positive viscosity solutions of a class of fully nonlinear elliptic equations, including the extremal Pucci's equations without singular nonlinearity and see also Theorem 2 [16], where authors obtained the regularity for the solutions of (1.1) up to the boundary when $\eta = 0$. For the existence of eigenvalue, maximum principle and related regularity questions to fully nonlinear homogeneous operators and for the Dirichlet problem for singular fully nonlinear operators, we refer to [3,4]. For the recent developments on this area, we refer to survey paper [26]. Now, in the context of above research works, it is natural to ask whether can we obtain the existence of a positive solution to an equation which involves extremal Pucci's operator and singular and sublinear nonlinearity? The aim of this paper is to answer this question. More precisely, we establish the existence of a positive viscosity solution of (1.1). Due to nondivergence nature of the operator and singular and sublinear nonlinearity occurring in (1.1), it is of much interest to study (1.1), where the standard tools are not applicable. We use the viscosity sub- and super solution method and comparison principle to establish the existence of a unique viscosity solution to (1.1) under certain conditions on k, f and h and using standard arguments, we show the nonexistence of the solution to (1.1). We consider the cases $f(0) > 0$, $f(0) = 0$, and $f(0) < 0$, separately for the existence of a positive viscosity solution. There is a lot of research works for the existence of positive solutions to semilinear elliptic equations (1.2) in these cases but as far as our understanding goes, there is not much work available for equations involving extremal Pucci's operator with singular and sublinear nonlinearity.

This work complements and extends earlier research works on singular elliptic equations due to the following:

(i) In contrast to [30], this work establishes the existence of a solution to (1.1) which involves extremal Pucci's operator and where $\eta > 0$ and $0 < \alpha < 1$. This work complements the work of [30] even for fully nonlinear equations in the sense of viscosity solution.
(ii) This work extends the work of [9–11,19,29] and many others to extremal Pucci's equation in the framework of viscosity solution at least in case when $0 < \alpha < 1$, $0 < q < 1$.

We organize this paper as follows. Section 2 deals with the preliminaries and the auxiliary results which are needed in the proof of our main results. In Sect. 3, we state and prove the existence of a unique viscosity solution to (1.1). We also show the monotonicity of the solution with respect to the parameter η. Section 4 deals with the case $f(0) = 0$. In Sect. 5, we establish the existence of a positive solution in case $f(0) < 0$. We establish a nonexistence result in Sect. 6. Finally, two examples illustrating the main results are constructed in Sect. 7.

2. Preliminaries and Auxiliary Results

We begin this section with the basic definitions and auxiliary results which have been used in this paper.

Definition 2.1. A function $u \in C(\Omega)$ is an L^n-viscosity subsolution (supersolution) of (1.1) in Ω if for all $\phi \in W^{2,n}_{loc}(\Omega)$ and a point x at which $u - \phi$ has local maximum (minimum) we have

$$\begin{cases} \operatorname{ess\,lim\,inf}_{y\to x}(-\mathcal{M}^+_{\lambda,\Lambda}(D^2\phi) - \frac{\mu k(y) f(u)}{u^\alpha} + \eta h(y) u^q) \quad \le 0, \\ (\operatorname{ess\,lim\,sup}_{y\to x}(-\mathcal{M}^+_{\lambda,\Lambda}(D^2\phi) - \frac{\mu k(y) f(u)}{u^\alpha} + \eta h(y) u^q) \ge 0). \end{cases} \tag{2.1}$$

When a function satisfies an equation or inequality in the L^n-viscosity sense, then we will simply say that equation or inequality holds in the viscosity sense.

From the definition of Pucci's extremal operator, it is easy to see that

$$\mathcal{M}^+_{\lambda,\Lambda}(-M) = -\mathcal{M}^-_{\lambda,\Lambda}(M) \tag{2.2}$$

and

$$\mathcal{M}^{+}_{\lambda,\Lambda}(M) + \mathcal{M}^{-}_{\lambda,\Lambda}(N) \le \mathcal{M}^{+}_{\lambda,\Lambda}(M+N) \le \mathcal{M}^{+}_{\lambda,\Lambda}(M) + \mathcal{M}^{+}_{\lambda,\Lambda}(N), \quad \forall M, N \in S_{n\times n}. \tag{2.3}$$

Lemma 2.2. (Lemma 2.2 [22]) *Let Ω be a regular domain and let $u \in W^{2,n}_{loc}(\Omega) \cap C(\bar{\Omega})$ be a nonnegative solution to*

$$\mathcal{M}^{-}_{\lambda,\Lambda}(D^2u) + c(x)u \le 0 \quad in \;\; \Omega, \tag{2.4}$$

with $c(x) \in L^{\infty}(\Omega)$. Then either u vanishes identically in Ω or $u(x) > 0$ for all $x \in \Omega$. Moreover, in the latter case for any $x_0 \in \partial\Omega$ such that $u(x_0) = 0$, then

$$\limsup_{t\to 0} \frac{u(x_0 - t\nu) - u(x_0)}{t} < 0,$$

where ν is the outer normal to $\partial\Omega$.

Theorem 2.3. (Theorem 5 [16]) *Let $f_k \to f$ in $L^n(\Omega)$. Suppose that u_k is L^n-viscosity solution of*

$$\begin{cases} -\mathcal{M}^{+}_{\lambda,\Lambda}(D^2u_k) = f_k & in \;\; \Omega \\ u_k = 0 & on \;\; \partial\Omega, \end{cases} \tag{2.5}$$

and the sequence u_k is bounded in $L^{\infty}(\Omega)$. Then a subsequence of $\{u_k\}$ converges uniformly to a function u which satisfies

$$\begin{cases} -\mathcal{M}^{+}_{\lambda,\Lambda}(D^2u) = f & in \;\; \Omega \\ u = 0 & on \;\; \partial\Omega. \end{cases} \tag{2.6}$$

The following eigenvalue problem is considered in [23], see also [7].

Theorem 2.4. [Theorem 1.1 [23]] *Let us consider the following boundary value problem*

$$\begin{cases} -\mathcal{M}^{+}_{\lambda,\Lambda}(D^2u) = \mu u & in \;\; \Omega \\ u > 0 & in \;\; \Omega \\ u = 0 & on \;\; \partial\Omega. \end{cases} \tag{2.7}$$

Then there exists $\phi \in W^{2,p}_{loc}(\Omega) \cap C(\bar{\Omega})$ for all $p < \infty$ and $\mu_1^+ > 0$ such that (ϕ, μ_1^+) is a solution of (2.7). Furthermore, any other solution of (2.7) is of the form $(\mu_1^+, k\phi)$ for $k > 0$.

The following theorem is a consequence of the characterization of the eigenvalue; see [23] for the complete details.

Theorem 2.5. *If $u \in W^{2,n}_{loc}(\Omega) \cap C(\bar{\Omega})$ is a solution of*

$$\begin{cases} \mathcal{M}^{+}_{\lambda,\Lambda}(D^2u) + \eta u \ge 0 & in \;\; \Omega \\ u \le 0 & on \;\; \partial\Omega, \end{cases} \tag{2.8}$$

for some $\eta < \mu_1^+$, then $u \le 0$ in Ω.

3. Existence of a Positive Solution

Let us consider Equation (1.1), where we assume that k, f and h are nonnegative continuous functions unless otherwise stated and μ, η are nonnegative parameters. Since the problem under consideration is fully nonlinear and singular in nature, we regularize this problem and use the notion of the viscosity solution to prove the existence of a viscosity solution by passing the limit in the sequence of solutions to the regularized problem corresponding to (1.1). Let us consider the following regularized problem corresponding to (1.1):

$$(S_\delta)\begin{cases} -\mathcal{M}^+_{\lambda,\Lambda}(D^2u) = \mu\frac{k(x)f(u)}{(u+\delta)^\alpha} - \eta h(x)u^q & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

for each $\delta > 0$. We denote the solution of S_δ by u_δ and if $\delta = \frac{1}{n}$, then the solution will be denoted by u_n and equation by S_n instead of $u_{\frac{1}{n}}$ and $S_{\frac{1}{n}}$, respectively. We start by proving the following comparison lemma, which is valid for all $\delta \geq 0$, which will be used later in the paper.

Lemma 3.1. *Suppose h, k are nonnegative continuous functions on Ω and f is a nonnegative nonincreasing continuous function. Let $u, v \in C(\bar{\Omega})$ satisfy the following inequation in the viscosity sense*

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2u) + \mu\frac{k(x)f(u)}{(u+\delta)^\alpha} - \eta h(x)u^q \leq 0 \leq \mathcal{M}^+_{\lambda,\Lambda}(D^2v) + \mu\frac{k(x)f(v)}{(v+\delta)^\alpha} - \eta h(x)v^q \quad in \;\; \Omega$$

and $u \geq v$ on $\partial\Omega$. Further, if either $u \in W^{2,n}_{loc}$ or $v \in W^{2,n}_{loc}$ then $u \geq v$ in $\bar{\Omega}$.

Proof. We prove this lemma by method of contradiction. Let us consider

$$A = \{x \in \Omega \; : \; v(x) > u(x)\}.$$

If $A \neq \phi$, then A is an open set and also

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2u) + \mu\frac{k(x)f(u)}{(u+\delta)^\alpha} - \eta h(x)u^q - \mathcal{M}^+_{\lambda,\Lambda}(D^2v) - \mu\frac{k(x)f(v)}{(v+\delta)^\alpha} + \eta h(x)v^q \leq 0 \quad \text{in } \; A.$$

That is,

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2u) - \mathcal{M}^+_{\lambda,\Lambda}(D^2v) + \mu k(x)\left[\frac{f(u)}{(u+\delta)^\delta} - \frac{f(v)}{(v+\delta)^\alpha}\right] + \eta h(x)(v^q - u^q) \leq 0 \quad \text{in } \; A.$$

As f is a nonnegative nonincreasing function, the second and third terms are nonnegative on A and therefore we get the following

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2u) - \mathcal{M}^+_{\lambda,\Lambda}(D^2v) \leq 0 \quad \text{in } \; A,$$

which implies

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2(u-v)) \leq 0 \quad \text{in } \; A.$$

Now, as $u = v$ on ∂A, so by the maximum principle, see (Corollary 3.7(2) [8]), we get $u \geq v$ in A, which contradicts the definition of A and hence the lemma is proved.

Lemma 3.1 implies that positive solutions u_n of S_n form an increasing sequence because u_n and u_{n+1} satisfy

$$\begin{aligned} &\mathcal{M}^+_{\lambda,\Lambda}(D^2 u_{n+1}) + \mu\tfrac{k(x)f(u_{n+1})}{(u_{n+1}+\frac{1}{n+1})^\alpha} - \eta h(x)u^q_{n+1} = 0 \leq \mathcal{M}^+_{\lambda,\Lambda}(D^2 u_n) \\ &\quad + \mu\tfrac{k(x)f(u_n)}{(u_n+\frac{1}{n+1})^\alpha} - \eta h(x)u^q_n. \end{aligned} \tag{3.1}$$

The following theorem establishes the existence of a viscosity solution to the regularized problem.

Theorem 3.2. *Suppose that h and k are bounded nonnegative continuous functions such that $\inf_\Omega k > 0$ and f is a nonnegative, nonincreasing continuous function with $f(0) > 0$. Then for any positive $0 < \delta < 1$, there exists a unique solution u_δ of S_δ. Furthermore, there are positive constants m, M independent of δ such that*

$$m\phi \leq u_\delta \leq M\phi + \phi^\beta, \tag{3.2}$$

where $0 < \beta < 1$ and ϕ is defined in Theorem 2.4.

Proof. Since $f(0) > 0$, so by the continuity of f, we can find an $\epsilon > 0$ such that $f(s) > 0$ for $s \in [0, \epsilon]$. Let us define a constant $c = \inf_{s\in[0,\epsilon]} f(s) > 0$. Choose $m > 0$ sufficiently small such that

$$m^q \leq \frac{\mu \inf_\Omega k(x)c}{(m\|\phi\|_{L^\infty(\Omega)} + 1)^\alpha(\mu_1^+\|\phi\|_{L^\infty(\Omega)} + \eta\|h\|_{L^\infty(\Omega)}\|\phi^q\|_{L^\infty(\Omega)})},$$

and $m\|\phi\|_{L^\infty(\Omega)} \leq \epsilon$, where ϕ is defined in Theorem 2.4, which is always possible as $c\mu \inf_\Omega k > 0$. This value of m satisfies

$$m\mu_1^+\phi(x) \leq \frac{\inf_\Omega k(x)c\mu}{(m\phi(x)+\delta)^\alpha} - \sup_\Omega h(x)m^q\phi^q(x) \quad \text{for } \ x \in \Omega.$$

That is,

$$-\mathcal{M}^+_{\lambda,\Lambda}(D^2 m\phi) \leq \frac{k(x)\mu f(m\phi)}{(m\phi+\delta)^\alpha} - h(x)(m\phi)^q \quad \text{in } \ \Omega.$$

Thus $u = m\phi$ is a subsolution of the problem S_δ.

Let us consider $v = M\phi + \phi^\beta$, where $\beta < \max\{\frac{2}{(\alpha+1)}, 1\}$, and again ϕ is the eigenfunction from Theorem 2.4. After differentiating twice, we get

$$D^2 v = MD^2\phi + \beta(\beta-1)\phi^{\beta-2}D\phi \otimes D\phi + \beta\phi^{\beta-1}D^2\phi,$$

where $x \otimes x$ is an $n \times n$ matrix with (i, j)-entry $x_i x_j$. Let us calculate

$$\begin{cases} \mathcal{M}^+_{\lambda,\Lambda}(D^2 v) + \frac{\mu k(x)f(v)}{(v+\delta)^\alpha} - \eta h(x)v^q = \mathcal{M}^+_{\lambda,\Lambda}(MD^2\phi + \beta(\beta-1)\phi^{\beta-2}D\phi \otimes D\phi \\ \quad +\beta\phi^{\beta-1}D^2\phi) + \frac{\mu k(x)f(M\phi+\phi^\beta)}{(M\phi+\phi^\beta+\delta)^\alpha} - \eta h(x)(M\phi+\phi^\beta)^q. \end{cases} \tag{3.3}$$

Now, using Inequalities (2.2),(2.3) to (3.3) and noting that ϕ is an eigenfunction, we obtain

$$\begin{cases} \mathcal{M}^+_{\lambda,\Lambda}(D^2v)+\frac{\mu k(x)f(v)}{(v+\delta)^\alpha}-\eta h(x)v^q \leq \mathcal{M}^+_{\lambda,\Lambda}(MD^2\phi)+\mathcal{M}^+_{\lambda,\Lambda}(\beta(\beta-1)\phi^{\beta-2}D\phi\otimes D\phi) \\ \qquad +\mathcal{M}^+_{\lambda,\Lambda}(\beta\phi^{\beta-1}D^2\phi)+\frac{\mu k(x)f(M\phi+\phi^\beta)}{(\phi^\beta)^\alpha} \\ \qquad -\eta h(x)(M\phi+\phi^\beta)^q \\ \qquad = -M\mu_1^+\phi-\beta(1-\beta)\phi^{\beta-2}\lambda|D\phi|^2-\beta\mu_1^+\phi^\beta \\ \qquad +\frac{\mu k(x)f(M\phi+\phi^\beta)}{(\phi^\beta)^\alpha}-\eta h(x)(M\phi+\phi^\beta)^q. \end{cases} \tag{3.4}$$

Here in the last inequality, we have used that

$$\mathcal{M}^+_{\lambda,\Lambda}(\beta(\beta-1)\phi^{\beta-2}D\phi\otimes D\phi)=-\beta(1-\beta)\lambda|D\phi|^2\phi^{\beta-2},$$

which is a consequence of (2.2), $\beta-1<0$ and the fact that $D\phi\otimes D\phi$ has $|D\phi|^2$ as the only nontrivial eigenvalue. From (3.3), we get

$$\begin{cases} \mathcal{M}^+_{\lambda,\Lambda}(D^2v)+\frac{\mu k(x)f(v)}{(v+\delta)^\alpha}-\eta h(x)v^q \leq -M\mu_1^+\phi+\Big[\mu k(x)f(M\phi+\phi^\beta)\phi^{-\alpha\beta+2-\beta} \\ \qquad -\beta(1-\beta)\lambda|D\phi|^2\Big]\phi^{\beta-2}-\eta h(x)(M\phi+\phi^\beta)^q \\ \qquad \leq -M\mu_1^+\phi+\Big[\mu k(x)f(0)\phi^{-\alpha\beta+2-\beta} \\ \qquad -\beta(1-\beta)\lambda|D\phi|^2\Big]\phi^{\beta-2}-\eta h(x)(M\phi+\phi^\beta)^q. \end{cases} \tag{3.5}$$

The last inequality follows because f is nonincreasing function and $M\phi+\phi^\beta\geq 0$. Further, since $\phi=0$ on $\partial\Omega$, and $\phi\in C^1(\bar{\Omega})$ so by Lemma 2.2, there exist a neighbourhood, say N of $\partial\Omega$ and a constant $L>0$ such that

$$|D\phi|\geq L>0,$$

in N. So we can find some constant $C>0$ such that

$$\beta(1-\beta)\phi^{\beta-2}\lambda|D\phi|^2\geq C\phi^{\beta-2} \quad \text{in } N. \tag{3.6}$$

Now, since $2-\beta>\alpha\beta$ and $\phi=0$ on $\partial\Omega$, so from (3.5) and (3.6), we can find $M>0$ such that

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2v)+\frac{\mu k(x)f(v)}{(v+\delta)^\alpha}-\eta h(x)v^q\leq 0 \quad \text{in } N. \tag{3.7}$$

On the other hand, in $\Omega\backslash N$, by choosing $M>0$ large enough in (3.5), we find that v satisfies (3.7) also in $\Omega\backslash N$. Thus v satisfies Equation (3.7) in Ω and therefore v is a supersolution of S_δ. Now by using subsolution u and supersolution v, constructed above and applying monotone iteration method, we find a solutions u_δ of S_δ which vanishes on $\partial\Omega$. Furthermore, we can choose M large enough such that (3.2) is also satisfied. Also, by using regularity results of [24](see Theorem 3.1 and also Theorem 2.1, Theorem 4.1 [25]), we find that $u_\delta\in W^{2,n}_{loc}(\Omega)$.

Theorem 3.3. *Under the assumptions of Theorem 3.2, there exists a unique solution u of (1.1).*

Proof. Let us choose a sequence $\delta_n = \frac{1}{n}$ and corresponding solutions u_n obtained in the above theorem. As u_n satisfies the following equation

$$0 = \mathcal{M}^+_{\lambda,\Lambda}(D^2u_n) + \frac{\mu k(x) f(u_n)}{(u_n + \frac{1}{n})^{\alpha}} - \eta h(x) u_n^q \leq \mathcal{M}^+_{\lambda,\Lambda}(D^2u_n) + \frac{\mu k(x) f(u_n)}{\left(u_n^q + \frac{1}{n+1}\right)^{\alpha}} - \eta h(x) u_n^q.$$

So we have

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2u_{n+1}) + \frac{\mu k(x) f(u_{n+1})}{\left(u_{n+1} + \frac{1}{n+1}\right)^{\alpha}} - \eta h(x) u_{n+1}^q = 0 \leq \mathcal{M}^+_{\lambda,\Lambda}(D^2u_n) + \frac{\mu k(x) f(u_n)}{\left(u_n + \frac{1}{n+1}\right)^{\alpha}} - \eta h(x) u_n^q$$

in Ω. Therefore by Lemma 3.1, we find that u_n is a monotonic increasing sequence satisfying Inequality (3.2). Further, by using the standard diagonal argument and Theorem 2.3, we find a subsequence $\{u_{n_k}\}$ which converges uniformly to a function u which solves (1.1). Since the sequence is monotonic increasing, so $\{u_n\}$ converges to u. The uniqueness of the viscosity solution follows by the standard arguments. This completes the proof.

Below, we show the monotonicity properties of the solution of (1.1) with respect to η.

Proposition 3.4. *Let u_1 and u_2 be positive solutions of (1.1) corresponding to η_1 and η_2, respectively, where $0 < \eta_1 < \eta_2$. Then $u_2 \leq u_1$ in Ω.*

Proof. Let us consider a set $A = \{x \in \Omega \ : \ u_2(x) > u_1(x)\}$. If $A = \phi$, then we are done. So we suppose that $A \neq \phi$ and observe that

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2u_1) + \frac{\mu k(x) f(u_1)}{u_1^{\alpha}} - \eta_2 h(x) u_1^q \leq 0 = \mathcal{M}^+_{\lambda,\Lambda}(D^2u_2) + \frac{\mu k(x) f(u_2)}{u_2^{\alpha}} - \eta_2 h(x) u_2^q.$$

This implies that

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2u_1) - \mathcal{M}^+_{\lambda,\Lambda}(D^2u_2) + \mu k(x)\left[\frac{f(u_1)}{u_1^{\alpha}} - \frac{f(u_2)}{u_2^{\alpha}}\right] + \eta_2 h(x)[u_2^q - u_1^q] \leq 0.$$

Since f is nonincreasing function so the last two terms are nonnegative on A, we get

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2(u_1 - u_2)) \leq 0 \quad \text{in } A,$$

and $u_1 = u_2$ on ∂A. An application of maximum principle, see (Corollary 3.7(2) [8]) yields that $u_1 \geq u_2$ in A, which implies that $A = \phi$ and hence the proposition is proved. □

In the above theorem, we have shown the existence of a unique positive solution to (1.1) under the condition $f(0) > 0$. Now the question is what happens if $f(0) = 0$ and $f(0) < 0$. Next, we discuss these two cases.

4. Case $f(0) = 0$.

As we have assumed that f is nonnegative and nonincreasing in Theorem 3.3 and $f(0) = 0$, so this implies that $f \equiv 0$ and therefore (1.1) reduces to the following equation

$$\begin{cases} -\mathcal{M}^+_{\lambda,\Lambda}(D^2u) = -\eta h(x)u^q & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.1}$$

It is clear that $u \equiv 0$ is a solution of (4.1). Further, any nonnegative solution of (4.1), by maximum principle, is $u \equiv 0$. Thus this problem cannot have a positive solution.

Remark 4.1. We remark that all the above results also hold if we replace $\mathcal{M}^+_{\lambda,\Lambda}$ by fully nonlinear operator

$$F : \Omega \times \mathbb{R} \times \mathbb{R}^n \times S(n) \longrightarrow \mathbb{R}, \tag{4.2}$$

satisfying the following conditions (i)–(iii) for some $\gamma \geq 0$ and for all $u, v \in \mathbb{R}$, $p, q \in \mathbb{R}^n$ and $M, N \in S_{n\times n}$.

(i)

$$\begin{aligned} \mathcal{M}^-_{\lambda,\Lambda}(M-N) - \gamma|p-q| - \gamma|u-v| &\leq F(x,u,p,M) - F(x,v,q,N) \\ &\leq \mathcal{M}^+_{\lambda,\Lambda}(M-N) + \gamma|p-q| + \gamma|u-v|, \end{aligned}$$

(ii) $F(x, tu, tp, tM) = tF(x, u, p, M)$ for all $t \geq 0$,

(iii) F is convex in (u, p, M) and satisfies the following comparison principle.

Let $u, v \in C(\bar{\Omega})$ be L^n viscosity sub and supersolution of $F = 0$ and one of u or v is in $W^{2,n}_{loc}(\Omega)$ and $u \leq v$ on $\partial\Omega$, then $u \leq v$ in Ω.

Under the above assumptions on F, the existence of eigenfunction is available in [23], and consequently the maximum principle is given by Theorem 2.5.

Next, we discuss the case $f(0) < 0$.

5. Case $f(0) < 0$

As we are dealing with the case $f(0) < 0$, so we assume that $f : [0, \infty) \longrightarrow \mathbb{R}$ instead of $f : [0, \infty) \longrightarrow [0, \infty)$. We consider the following problem with singular nonlinearity

$$\begin{cases} -\mathcal{M}^+_{\lambda,\Lambda}(D^2u) = \frac{k(x)f(u)}{u^\alpha} - \eta h(x)u^q & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.1}$$

where $0 < \eta < 1$ and k, h are nonnegative continuous functions with $k \in L^\infty(\Omega)$ and positive on a subset of positive measure of Ω, $f : [0, \infty) \longrightarrow \mathbb{R}$ is a bounded continuous function such that $f(0) < 0$, $0 \leq h(x) \leq k(x)\ \forall\, x \in \Omega$. Let us consider

$$\Omega_\delta = \{x \in \Omega : d(x, \partial\Omega) > \delta\},$$

where $d(x, \partial\Omega)$ represents the distance of the point x from the boundary $\partial\Omega$. For each $\delta > 0$, let us consider the following problem

$$\begin{cases} -\mathcal{M}^+_{\lambda,\Lambda}(D^2 v_\delta) + c_1 v_\delta = (k(x) - c_2 h(x))\chi_{\Omega_\delta} & \text{in } \Omega \\ v_\delta = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.2}$$

where χ_{Ω_δ} denotes the characteristic function of the set Ω_δ, $c_1 = \eta\|h\|_{L^\infty(\Omega)}/\beta$ and $0 < c_2 = \eta/\beta < 1$ with $\beta = \frac{2}{1+\alpha}$. By Theorem 2.1 [17], there exists solution v_δ of (5.2) and also

$$\|v_\delta\|_{C^{1,\sigma}(\bar{\Omega})} \le C\|k(x) - c_2 h(x)\|_{L^\infty(\Omega)} \quad \text{for } \delta > 0.$$

Furthermore, since $(k(x) - c_2 h(x))\chi_{\Omega_\delta} \ge 0$ so $v_\delta \ge 0$. We also know that $(k - c_2 h)\chi_{\Omega_\delta} \to (k - c_2 h)\chi_\Omega$ in $L^p(\Omega)$ for all $p \in [1, \infty)$ as $\delta \to 0$ and $C^{1,\sigma}$ is compactly embedded in C^1. So there is a subsequence of v_δ which converges to v in C^1 as $\delta \to 0$ and by the stability result for viscosity solution, see Theorem 3.8 [13], v satisfies

$$\begin{cases} -\mathcal{M}^+_{\lambda,\Lambda}(D^2 v) + c_1 v = k(x) - c_2 h(x) & \text{in } \Omega. \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.3}$$

Since by assumption $k(x) - c_2 h(x) \ge 0$, so by maximum principle $v \ge 0$. Since $v = 0$ on $\partial\Omega$ and $v \ge 0$ so by the Höpf type lemma, see (Theorem 2.2 [17]), there exists a neighbourhood say $N \subset \bar{\Omega}$ of $\partial\Omega$ such that

$$\epsilon = \inf\{|Dv(x)| \mid x \in N\} > 0. \tag{5.4}$$

Furthermore, since $v_\delta \to v$ in $C^1(\bar{\Omega})$ as $\delta \to 0$, so we can find a δ_1 such that for $0 < \delta < \delta_1$, we have

$$\sup_N |\,|Dv_\delta(x)| - |Dv(x)|\,| \le \sup_N |Dv_\delta(x) - Dv(x)| \le \sup_{\bar{\Omega}} |Dv_\delta(x) - Dv(x)| < \frac{\epsilon}{2}.$$

In particular, for all $x \in N$, we have

$$-\frac{\epsilon}{2} < |Dv_\delta(x)| - |Dv(x)| < \frac{\epsilon}{2},$$

or

$$|Dv(x)| - \frac{\epsilon}{2} < |Dv_\delta(x)|.$$

So by Equation (5.4), we find that

$$\frac{\epsilon}{2} \le \inf\{|Dv_\delta(x)| \mid x \in N\} \quad \text{for } 0 < \delta < \delta_1.$$

Further, we can choose $\delta_2 > 0$ sufficiently small such that

$$\bar{\Omega}\backslash\Omega_\delta = \bar{\Omega} \cap \Omega_\delta^c \subset N, \quad \forall\, 0 < \delta < \delta_2. \tag{5.5}$$

Since we also have $v_\delta(x) = 0$ on $\partial\Omega$ for all δ so choose a $0 < \delta_3 < \min\{\delta_1, \delta_2\}$ such that

$$0 \le v_{\delta_3}(x) \le \sup_{\bar{\Omega}_{\delta_3}} v_{\delta_3} \quad \text{for } x \in \bar{\Omega}\backslash\Omega_{\delta_3}. \tag{5.6}$$

Let us fix δ_3 and denote it by δ, that is, $\delta = \delta_3$. Let us also set

$$a = \min_{\bar{\Omega}_\delta} v_\delta \quad \text{and} \quad A = \max_{\bar{\Omega}_\delta} v_\delta.$$

So we have following

$$\begin{array}{l}\min\{|Dv_\delta(x)| \ :\ x\in\bar{\Omega}\backslash\Omega_\delta\} > \frac{\epsilon}{2}.\\ 0\le v_\delta(x)\le A \quad \text{for } x\in\bar{\Omega}\backslash\Omega_\delta.\end{array} \tag{5.7}$$

Here we also have $v_\delta \ge \frac{\epsilon}{2}\delta$ on $\partial\Omega_\delta$, see [9]. So by maximum principle, we get $v_\delta \ge \frac{\epsilon}{2}\delta$ in $\bar{\Omega}_\delta$. Now we will present the following existence theorem.

Theorem 5.1. *Let $f : [0,\infty) \longrightarrow \mathbb{R}$ is a bounded continuous function and $f(0) < 0$, $0 < \alpha < 1$ and $\frac{1-\alpha}{2} < q \le 1$. Let us set $\beta = \frac{2}{\alpha+1}$. If $f(t^\beta) \ge -\frac{\lambda\beta(\beta-1)\epsilon^2}{4\|k\|_{L^\infty}} + \beta t := -M + \beta t$ on $[0,a]$, and if $f(t^\beta) \ge 2\beta t$ on $[a,A]$, then (5.1) has at least one positive solution.*

Proof. The idea of the proof is to truncate the function around origin. Let us set $g(t) = f(t)t^{-\alpha}$ and for each $n \in \mathbb{N}$, define

$$g_n(t) = \begin{cases} g(t), & \text{if } \ t \ge \rho_n,\\ \max\{g(\rho_n), g(t)\}, & 0 < t \le \rho_n,\end{cases} \tag{5.8}$$

where $\{\rho_n\}$ is a decreasing sequence of positive real numbers such that $\rho_n \to 0$ as $n\to\infty$. As $\lim_{t\to 0^+} g(t) = -\infty$, so each $g_n = g(\rho_n)$ on some interval $(0,\rho']$, and so g_n can be extended to have the value $g(\rho_n)$ at 0. Now we will consider the following truncated boundary value problem

$$\begin{cases} -\mathcal{M}^+_{\lambda,\Lambda}(D^2 w) = k(x)g_n(w) - \eta h(x)w^q & \text{in } \ \Omega\\ w \qquad\qquad\qquad = 0 & \text{on } \ \partial\Omega,\end{cases} \tag{5.9}$$

and show that it has a positive solution by constructing a positive subsolution and an appropriately ordered supersolution. Let us take $u = v_\delta^\beta$, where $\delta > 0$ fixed as in the discussion before the statement of the theorem, and after differentiating twice we get

$$D^2u = \beta(\beta-1)v_\delta^{\beta-2}Dv_\delta\otimes Dv_\delta + \beta v_\delta^{\beta-1}D^2v_\delta,$$

where again $x\otimes x$ is an $n\times n$ matrix with ij-entry (x_ix_j) and $x = (x_1, x_2, \ldots, x_n)$.

$$-\mathcal{M}^+_{\lambda,\Lambda}(D^2u) = -\mathcal{M}^+_{\lambda,\Lambda}(\beta(\beta-1)v_\delta^{\beta-2}Dv_\delta\otimes Dv_\delta + \beta v_\delta^{\beta-1}D^2v_\delta).$$

Now using (2.3), we obtain

$$-\mathcal{M}^+_{\lambda,\Lambda}(D^2u) \le -\mathcal{M}^-_{\lambda,\Lambda}(\beta(\beta-1)v_\delta^{\beta-2}Dv_\delta\otimes Dv_\delta) - \mathcal{M}^+_{\lambda,\Lambda}(\beta v_\delta^{\beta-1}D^2v_\delta).$$

Since v_δ is a solution of (5.2) and $Dv_\delta\otimes Dv_\delta$ has $|Dv_\delta|^2$ as the only nontrivial eigenvalue so we get

$$-\mathcal{M}^+_{\lambda,\Lambda}(D^2u) \le -\lambda\beta(\beta-1)v_\delta^{\beta-2}|Dv_\delta|^2 + \beta v_\delta^{\beta-1}\big[(k-c_2h)\chi_{\Omega_\delta} - c_1v_\delta\big]. \tag{5.10}$$

One can rewrite

$$\beta v_\delta^{\beta-1}c_2h\chi_{\Omega_\delta} = \beta v_\delta^{\beta-1}c_2h\chi_{\Omega_\delta} + \beta v_\delta^{\beta-1}c_2h\chi_{\Omega\backslash\Omega_\delta} - \beta v_\delta^{\beta-1}c_2h\chi_{\Omega\backslash\Omega_\delta}.$$

Now since $c_2 = \frac{\eta}{\beta}$ and $\chi_{\Omega_\delta}(x) + \chi_{\Omega\setminus\Omega_\delta}(x) = 1\ \forall\ x \in \Omega$, so we have

$$\begin{aligned}\beta v_\delta^{\beta-1} c_2 h \chi_{\Omega_\delta} &= \beta c_2 h v_\delta^{\beta-1}(\chi_{\Omega_\delta} + \chi_{\Omega\setminus\Omega_\delta}) - v_\delta^{\beta-1}\eta h \chi_{\Omega\setminus\Omega_\delta} \\ &= \eta h v_\delta^{\beta-1} - v_\delta^{\beta-1}\eta h \chi_{\Omega\setminus\Omega_\delta}.\end{aligned} \tag{5.11}$$

From (5.10), (5.11) and noting that $c_1 = \frac{\eta}{\beta}\|h\|_{L^\infty(\Omega)} \geq \frac{\eta}{\beta}h$, we find that

$$\begin{aligned}-\mathcal{M}^+_{\lambda,\Lambda}(D^2u) \leq &-\lambda\beta(\beta-1)v_\delta^{\beta-2}|Dv_\delta|^2 + \beta v_\delta^{\beta-1}k\chi_{\Omega_\delta} \\ &+ v_\delta^{\beta-1}\eta h \chi_{\Omega\setminus\Omega_\delta} - \eta h v_\delta^{\beta-1}\big[1+v_\delta\big].\end{aligned} \tag{5.12}$$

Further, sine $0 < \beta q - \beta + 1 < 1$ and $v_\delta \geq 0$, so

$$v_\delta^{\beta q - \beta + 1} < 1 + v_\delta.$$

So from (5.12) and noting that $\beta - 2 = -\beta\alpha$, $|Dv_\delta|^2 \geq \frac{\epsilon^2}{4}$ and $k \geq h$, we obtain

$$\begin{aligned}-\mathcal{M}^+_{\lambda,\Lambda}(D^2u) &\leq -\lambda\beta(\beta-1)v_\delta^{\beta-2}|Dv_\delta|^2 + \beta v_\delta^{\beta-1}k\chi_{\Omega_\delta} + v_\delta^{\beta-1}\eta h\chi_{\Omega\setminus\Omega_\delta} - \eta h (v_\delta^\beta)^q \\ &\leq \Big[-\tfrac{\lambda\beta(\beta-1)\epsilon^2}{4\|k\|_{L^\infty(\Omega)}}\chi_{\{x\in\Omega:\ 0\leq v_\delta(x)\leq a\}} \\ &\quad + \beta v_\delta \chi_{\{x\in\Omega\ :\ a\leq v_\delta(x)\leq A\}} + \beta v_\delta\chi_{\Omega\setminus\Omega_\delta}\Big]\, k(v_\delta)^{-\alpha} - \ \eta h(v_\delta^\beta)^q.\end{aligned} \tag{5.13}$$

By (5.7), we know that for $x \in \Omega\setminus\Omega_\delta$, we have $0 \leq v_\delta(x) \leq A$, so

$$\beta v_\delta \chi_{\Omega\setminus\Omega_\delta} \leq \beta v_\delta \chi_{\{x\in\Omega\ :0\leq v_\delta(x)\leq a\}} + \beta v_\delta\chi_{\{x\in\Omega\ :\ a\leq v_\delta(x)\leq A\}}.$$

Consequently, we have

$$\begin{aligned}-\mathcal{M}^+_{\lambda,\Lambda}(D^2u) &\leq \Big[\Big(-\tfrac{\lambda\beta(\beta-1)\epsilon^2}{4\|k\|_{L^\infty(\Omega)}} + \beta v_\delta\Big)\chi_{\{x\in\Omega:\ 0\leq v_\delta(x)\leq a\}} \\ &\quad + \Big(2\beta v_\delta\Big)\chi_{\{x\in\Omega\ :\ a\leq v_\delta(x)\leq A\}}\Big]\, k(v_\delta)^{-\alpha} \\ &\quad -\eta h(v_\delta^\beta)^q \\ &\leq \tfrac{k(x)f(v_\delta^\beta)}{(v_\delta^\beta)^\alpha} - \eta h(v_\delta^\beta)^q \\ &= \tfrac{k(x)f(u)}{u^\alpha} - \eta h(x)u^q.\end{aligned} \tag{5.14}$$

Thus $u = v_\delta^\beta$ is a subsolution of (5.9) for each n. Since f is a bounded continuous and $f(0) < 0$, so we can find a positive constant C satisfying $C \geq \max_{t\geq 0} f(t)t^{-\alpha}$. Let v be a solution of

$$\begin{cases} -\mathcal{M}^+_{\lambda,\Lambda}(D^2v) = k(x)C & \text{in } \ \Omega \\ v = 0 & \text{on } \ \partial\Omega, \end{cases} \tag{5.15}$$

which again exists by Lemma 3.1 [13]. This v works as a supersolution for (5.9) and therefore we have a supersolution v and a subsolution u of (5.9). Further, also note that

$$\begin{aligned}k(x)C &= k(x)\max\nolimits_{t\geq 0} f(t)t^{-\alpha} \\ &\geq \ k(x)\max\nolimits_{t\geq 0} f(t)t^{-\alpha} - h(x)t^q,\end{aligned}$$

for any $t \geq 0$. Since $u \geq 0$, so

$$k(x)C \geq k(x)f(u)u^{-\alpha} - h(x)u^q. \tag{5.16}$$

Thus, in view of (5.16) and (5.15) we get

$$\begin{aligned} -\mathcal{M}^{+}_{\lambda,\Lambda}(D^2 v) &= k(x)C \\ &\geq k(x)f(u)u^{-\alpha} - h(x)u^q \\ &\geq -\mathcal{M}^{+}_{\lambda,\Lambda}(D^2 u) \quad \text{(by (5.14))}, \end{aligned} \tag{5.17}$$

that is,

$$\begin{aligned} -\mathcal{M}^{+}_{\lambda,\Lambda}(D^2 u) \leq -\mathcal{M}^{+}_{\lambda,\Lambda}(D^2 v) &\quad \text{in } \Omega \\ \mathcal{M}^{+}_{\lambda,\Lambda}(D^2 v) - \mathcal{M}^{+}_{\lambda,\Lambda}(D^2 u) \leq 0 &\quad \text{in } \Omega \\ \mathcal{M}^{-}_{\lambda,\Lambda}(D^2 (v-u)) \leq 0 &\quad \text{in } \Omega. \end{aligned} \tag{5.18}$$

Again, since $v - u = 0$ on $\partial\Omega$, so by maximum principle, see (Corollary 3.7(2) [8]), we get $u \leq v$ in Ω. Now, for each $n \in \mathbb{N}$, let w_n be a solution of (5.9) satisfying $u \leq w_n \leq v$. Note that we have an L^∞ upper bound for w_n and strictly positive lower bound. Now using $W^{2,n}$-interior estimate, see [24] and compact embedding, we find a subsequence of w_n, which converges uniformly to a positive solution w of (5.1) in $C^1(\Omega) \cap C(\bar{\Omega})$. This completes the proof. □

In the next section, we deal with the nonexistence of the positive solution.

6. Nonexistence of the Positive Solution

In the previous section, we have shown the existence of a positive solution to (1.1) under the condition that $h(x) \geq 0$ for all $x \in \Omega$, $\inf_\Omega k > 0$ and $f(0) > 0$. Here we show the importance of conditions on h and k. If h and k do not satisfy the above-mentioned conditions, then there does not exist any positive solution to (1.1). The result states that under the assumptions $\sup_\Omega k \leq 0$ and $h(x) < 0$, (1.1) does not have a positive solution. Here for the convenience, we assume that h is some negative constant, say, $h(x) = -\gamma$, for $x \in \Omega$, where $\gamma > 0$. Under the above assumptions, (1.1) can be rewritten as follows:

$$\begin{cases} -\mathcal{M}^{+}_{\lambda,\Lambda}(D^2 u) = \frac{\mu k(x) f(u)}{u^\alpha} + \gamma\eta u^q & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{6.1}$$

Since $k(x) < 0$ in Ω, so we can find a constant $c(\gamma)$ such that if $\gamma \to 0$, then $c(\gamma) \to 0$ and

$$\frac{\mu k(x) f(u)}{u^\alpha} + \gamma\eta u^q < c(\gamma) u \quad \text{for } u > 0.$$

Thus if $u \in W^{2,n}_{loc} \cap C(\bar{\Omega})$ is a positive solution of (1.1), then u also satisfies the following equation

$$\begin{cases} -\mathcal{M}^{+}_{\lambda,\Lambda}(D^2 u) < c(\gamma) u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{6.2}$$

and by Theorem 2.5, $c(\gamma) > \mu_1^+$. Thus for $\gamma > 0$ sufficiently small, (6.1) does not have a positive solution.

In case when $\sup_\Omega k < 0$ and $h(x) \geq 0$ in Ω, one can see that (1.1) does not have any positive solution. Indeed, suppose if $u(x) > 0$ and satisfies

$$\begin{cases} -\mathcal{M}^+_{\lambda,\Lambda}(D^2u) = \frac{\mu k(x) f(u)}{u^\alpha} - \eta h(x) u^q & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{6.3}$$

then it is easy to observe that u also satisfies

$$\begin{cases} -\mathcal{M}^+_{\lambda,\Lambda}(D^2u) \leq -\eta h(x) u^q & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{6.4}$$

Since $u > 0$, so we get

$$\begin{cases} -\mathcal{M}^+_{\lambda,\Lambda}(D^2u) \leq 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{6.5}$$

Now by maximum principle, we get $u(x) \leq 0$, for all $x \in \Omega$, which is a contradiction and therefore (1.1) does not have a positive solution.

Remark 6.1. We remark that we have the nonexistence of the solution to (1.1) in the case when

(i) $\sup_\Omega k < 0$ and $h(x) < 0$;
(ii) $\sup_\Omega k < 0$ and $h(x) \geq 0$.

In case, when $\inf_\Omega k > 0$ and $h \leq 0$, it will be of interest to establish the existence or nonexistence of the solution of (1.1).

7. Examples

This section deals with two examples illustrating main theorems.

Example 7.1. Consider the following equation

$$\begin{cases} -\mathcal{M}^+_{\lambda,\Lambda}(D^2u) = \frac{e^{-|x|} f(u)}{u^\alpha} - \eta \frac{e^{-|x|}}{1+|x|^2} u^q & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{7.1}$$

where $0 < \alpha < 1$, $1 - \alpha/2 < q < 1$, $\eta > 0$, and

$$f(t) = \begin{cases} \left[e^{\frac{lt}{1+t}} + \beta t^{\frac{1}{\beta}} - (M+1)\right], & \text{if } t \in [0, \tilde{A}] \\ \left[e^{\frac{lt}{1+t}} + \beta \tilde{A}^{\frac{1}{\beta}} - (M+1)\right], & \text{if } t \in [\tilde{A}, \infty), \end{cases}$$

where $\tilde{A} = A^\beta + 1$, and A, M and β are same as in Theorem 5.1. For appropriate positive value of l, one can show that f satisfies the assumptions of Theorem 5.1 and thus by an application of Theorem 5.1, Equation (7.1) has a positive solution.

Let us take some example of f verifying the assumptions of Theorem 5.1.

Example 7.2.

$$f(t) = \begin{cases} \left[\log(1+lt) + \beta t^{\frac{1}{\beta}} - M\right], & \text{if } t \in [0, \tilde{A}] \\ \left[\log(1+l\tilde{A}) + \beta \tilde{A}^{\frac{1}{\beta}} - M\right], & \text{if } t \in [\tilde{A}, \infty), \end{cases}$$

where again $\tilde{A} = A^\beta + 1$ and A, M and β are from Theorem 5.1. Note that for $t \in [0, A]$, $t^\beta \in [0, \tilde{A}]$. Set $g(t) = f(t) - \beta t^{\frac{1}{\beta}} = \log(1 + lt) - M$, and notice that $g(0) = f(0) = -M$ and g is non-decreasing function so

$$g(t^\beta) \geq -M \quad \text{for } t \in [0, a].$$

Further, we can choose l large enough such that $\log(1 + la^\beta) \geq \beta A + M$. So

$$g(t^\beta) = \log(1 + lt^\beta) - M \geq \log(1 + la^\beta) - M \geq \beta A \geq \beta t \quad \text{for } \ t \in [a, A].$$

That is,

$$f(t^\beta) = g(t^\beta) + \beta t \geq 2\beta t \quad \text{in } \ t \in [a, A].$$

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